

# Contents

Preface	v
<b>Panoramic Overview</b>	
Part 1. The Base Theory of Pro-Lie Groups . . . . .	6
Part 2. The Algebra of Pro-Lie Algebras . . . . .	19
Part 3. The Fine Lie Theory of Pro-Lie Groups . . . . .	28
Part 4. Global Structure Theory of Connected Pro-Lie Groups . . . . .	35
Part 5. The Role of Compactness on the Pro-Lie Algebra Level . . . . .	44
Part 6. The Role of Compact Subgroups of Pro-Lie Groups . . . . .	52
Part 7. Local Splitting According to Iwasawa . . . . .	61
<b>1 Limits of Topological Groups</b> 63	
Limits . . . . .	63
The External Approach to Projective Limits . . . . .	77
Projective Limits and Local Compactness . . . . .	82
The Fundamental Theorem on Projective Limits . . . . .	88
The Internal Approach to Projective Limits . . . . .	91
Projective Limits and Completeness . . . . .	93
The Closed Subgroup Theorem . . . . .	96
The Role of Local Compactness . . . . .	100
The Role of Closed Full Subcategories in Complete Categories . . . . .	102
Postscript . . . . .	104
<b>2 Lie Groups and the Lie Theory of Topological Groups</b> 107	
The General Definition of a Lie Group . . . . .	107
The Exponential Function of Topological Groups . . . . .	110
The Lie Algebra of a Topological Group . . . . .	115
The Category of Topological Groups with Lie Algebras . . . . .	119
The Lie Algebra Functor Has a Left Adjoint Functor . . . . .	126
Sophus Lie's Third Fundamental Theorem . . . . .	130
The Adjoint Representation of a Topological Group with a Lie Algebra . . . . .	131
Postscript . . . . .	133
<b>3 Pro-Lie Groups</b> 135	
Projective Limits of Lie Groups . . . . .	135
The Lie Algebras of Projective Limits of Lie Groups . . . . .	137
Pro-Lie Algebras . . . . .	138
Weakly Complete Topological Vector Spaces and Lie Algebras . . . . .	143
Pro-Lie Groups . . . . .	148

Some Examples . . . . .	160
An Overview of the Definitions of a Pro-Lie Group . . . . .	160
Postscript . . . . .	164
<b>4 Quotients of Pro-Lie Groups</b>	168
Quotient Objects in Categories . . . . .	169
Quotient Groups of Pro-Lie Groups . . . . .	170
The Exponential Function of Compact Abelian Groups and Quotient Morphisms . . . . .	173
The One Parameter Subgroup Lifting Theorem . . . . .	182
Sufficient Conditions for Quotients to be Complete . . . . .	195
Quotients and Quotient Maps between Pro-Lie Groups . . . . .	208
Postscript . . . . .	210
<b>5 Abelian Pro-Lie Groups</b>	212
Examples of Abelian Pro-Lie Groups . . . . .	212
Weil's Lemma . . . . .	215
Vector Group Splitting Theorems . . . . .	219
Compactly Generated Abelian Pro-Lie Groups . . . . .	233
Weakly Complete Topological Vector Spaces Revisited . . . . .	235
The Duality Theory of Abelian Pro-Lie Groups . . . . .	237
The Toral Homomorphic Images of an Abelian Pro-Lie Group . . . . .	241
Postscript . . . . .	246
<b>6 Lie's Third Fundamental Theorem</b>	249
Lie's Third Fundamental Theorem for Pro-Lie Groups . . . . .	249
Semidirect Products . . . . .	264
Postscript . . . . .	266
<b>7 Profinite-Dimensional Modules and Lie Algebras</b>	269
Modules over a Lie Algebra . . . . .	269
Duality of Modules . . . . .	272
Semisimple and Reductive Modules . . . . .	277
Reductive Pro-Lie Algebras . . . . .	281
Transfinitely Solvable Lie Algebras . . . . .	285
The Radical and Levi–Mal'cev: Existence . . . . .	291
Transfinitely Nilpotent Lie Algebras . . . . .	296
The Nilpotent Radicals . . . . .	300
Special Endomorphisms of Pro-Lie Algebras . . . . .	305
Levi–Mal'cev: Uniqueness . . . . .	309
Direct and Semidirect Sums Revisited . . . . .	313
Cartan Subalgebras of Pro-Lie Algebras . . . . .	315
Theorem of Ado . . . . .	331
Postscript . . . . .	332

<b>8 The Structure of Simply Connected Pro-Lie Groups</b>	335
The Adjoint Action . . . . .	335
Simply Connected Pronilpotent Pro-Lie Groups . . . . .	336
The Topological Splitting Technique . . . . .	340
Simple Connectivity . . . . .	343
Universal Morphism versus Universal Covering Morphism . . . . .	352
Postscript . . . . .	354
<b>9 Analytic Subgroups and the Lie Theory of Pro-Lie Groups</b>	356
The Exponential Function on the Inner Derivation Algebra . . . . .	356
Analytic Subgroups . . . . .	360
Automorphisms and Invariant Analytic Subgroups . . . . .	369
Centralizers . . . . .	370
Normalizers . . . . .	373
Subalgebras and Subgroups . . . . .	374
The Center . . . . .	376
The Commutator Subgroup . . . . .	376
Finite-Dimensional Connected Pro-Lie Groups . . . . .	385
Compact Central Subgroups . . . . .	402
Divisibility of Groups and Connected Pro-Lie Groups . . . . .	404
The Open Mapping Theorem . . . . .	409
Completing Proto-Lie Groups . . . . .	413
Unitary Representations . . . . .	415
Postscript . . . . .	416
<b>10 The Global Structure of Connected Pro-Lie Groups</b>	419
Solvability of Pro-Lie Groups . . . . .	420
The Radical . . . . .	431
Semisimple and Reductive Groups . . . . .	434
Nilpotency of Pro-Lie Groups . . . . .	443
The Nilradical and the Coreductive Radical . . . . .	447
The Structure of Reductive Pro-Lie Groups . . . . .	452
Postscript . . . . .	458
<b>11 Splitting Theorems for Pro-Lie Groups</b>	461
Splitting Reductive Groups Semidirectly . . . . .	462
Vector Group Splitting in Noncommutative Groups . . . . .	473
The Structure of Pronilpotent and Prosolvable Groups . . . . .	478
Conjugacy Theorems . . . . .	487
Postscript . . . . .	490

<b>12 Compact Subgroups of Pro-Lie Groups</b>	493
Procompact Modules and Lie Algebras . . . . .	493
Procompact Lie Algebras and Compactly Embedded Lie Subalgebras of Pro-Lie Algebras . . . . .	500
Maximal Compactly Embedded Subalgebras of Pro-Lie Algebras . . . . .	504
Conjugacy of Maximal Compactly Embedded Subalgebras . . . . .	507
Compact Connected Groups . . . . .	517
Compact Subgroups . . . . .	519
Potentially Compact Pro-Lie Groups . . . . .	521
The Conjugacy of Maximal Compact Connected Subgroups . . . . .	524
The Analytic Subgroups Having a Full Lie Algebra . . . . .	532
Maximal Compact Subgroups of Connected Pro-Lie Groups . . . . .	544
An Alternative Open Mapping Theorem . . . . .	556
On the Center of a Connected Pro-Lie Group . . . . .	558
Postscript . . . . .	561
<b>13 Iwasawa's Local Splitting Theorem</b>	566
Locally Splitting Lie Group Quotients of Pro-Lie Groups . . . . .	566
The Lie Algebra Theory of the Local Splitting . . . . .	571
Splitting on the Group Level . . . . .	579
Some Comments on Connectedness . . . . .	584
Postscript . . . . .	584
<b>14 Catalog of Examples</b>	587
Classification of the Examples in the Catalog . . . . .	587
Abelian Pro-Lie Groups . . . . .	588
A Simple Construction . . . . .	595
Pronilpotent Pro-Lie Groups . . . . .	598
Prosolvable Pro-Lie Groups . . . . .	602
Semisimple and Reductive Pro-Lie Groups . . . . .	608
Mixed Groups . . . . .	615
Examples Concerning the Definition of Lie and Pro-Lie Groups . . . . .	616
Analytic Subgroups of Pro-Lie Groups . . . . .	620
Examples Concerning Simple Connectivity . . . . .	622
Example Concerning $\mathfrak{g}$ -Module Theory . . . . .	622
Postscript . . . . .	623
<b>1 Appendix 1 The Campbell–Hausdorff Formalism</b>	624
<b>2 Appendix 2 Weakly Complete Topological Vector Spaces</b>	629
<b>3 Appendix 3 Various Pieces of Information on Semisimple Lie Algebras</b>	651
Postscript . . . . .	655

Bibliography	657
List of Symbols	667
Index	669