
Quark Flavor Interaction

The description of the low-energy physics of hadrons requires to model the interaction of the quark constituents. The observation that the (current) quark masses of the up (u) and down (d) quarks (and eventually the strange (s) quark) are significantly smaller than typical strong interaction energy scales serves as a major input. In a first approach it is thus reasonable to ignore the effects of the current quark masses. In that approximation axial transformations leave the QCD Lagrangian invariant and an additional global symmetry, the so-called *chiral symmetry* [1], emerges. This symmetry represents an important tool for model building because the models should be consistent with chiral symmetry. Any such model describes the interaction of light quarks with different flavors: *up*, *down* and eventually *strange*; hence the notion of quark flavor dynamics.

2.1 Chiral Symmetry

In the case of massless Dirac fermions that interact with boson fields via a vector interaction (as in QCD), left- and right-handed components

$$\Psi_{L,R} = \frac{1}{2} (1 \mp \gamma_5) \Psi \quad (2.1)$$

of the 4×4 spinors Ψ decouple. As a consequence the QCD Lagrangian with zero current masses decomposes into a sum of two Lagrangians that contain only right-(left-) handed fields, respectively. These two Lagrangians are invariant under global unitary flavor transformations of the corresponding right-(left-) handed fields, so that the QCD Lagrangian possesses an $U_L(N_f) \times U_R(N_f)$ symmetry. Here N_f is the number of quark flavors whose current quark masses are ignored. Depending on whether we consider the strange current quark mass as large or small we have $N_f = 2$ or $N_f = 3$, respectively. This symmetry group factorizes according to

$$U_L(N_f) \times U_R(N_f) \cong U_{L+R}(1) \times U_{L-R}(1) \times SU_L(N_f) \times SU_R(N_f) \quad (2.2)$$

and is called the *chiral group*. The invariance under $U_{L+R}(1)$ is responsible for the conservation of baryon number whereas $U_{L-R}(1)$ is subject to a quantum anomaly [2, 3]. This results in $2N_f - 1$ conserved flavor currents. The $2N_f$ flavor currents are most conveniently presented as linear combination of the left- and right-handed vector currents that are eigenstates of parity: the vector current J_μ^a and the axial vector current A_μ^a ,

$$\begin{aligned} J_\mu^a &= \bar{q}_L \gamma_\mu \frac{\lambda_a}{2} q_L + \bar{q}_R \gamma_\mu \frac{\lambda_a}{2} q_R = \bar{q} \gamma_\mu \frac{\lambda_a}{2} q \\ A_\mu^a &= -\bar{q}_L \gamma_\mu \frac{\lambda_a}{2} q_L + \bar{q}_R \gamma_\mu \frac{\lambda_a}{2} q_R = \bar{q} \gamma_\mu \gamma_5 \frac{\lambda_a}{2} q. \end{aligned} \quad (2.3)$$

Here λ_a ($a = 1, \dots, N_f^2 - 1$) are the Gell-Mann matrices of $SU(N_f)$ and $\lambda_0 = \sqrt{2/N_f} \mathbf{1}$ is proportional to the unit matrix in flavor space. The spinors are additionally column vectors with N_f entries of the quark flavors

$$q = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{N_f} \end{pmatrix} = \begin{pmatrix} \Psi_u \\ \Psi_d \\ \vdots \\ \Psi_{N_f} \end{pmatrix}. \quad (2.4)$$

The properties of these fermion fields under infinitesimal chiral transformations, (2.2), are summarized in Table A.1 of Appendix A. In the limit of vanishing current quark masses the above currents, apart from A_μ^0 , are conserved and that property should be reproduced within any model. In case these masses are non-zero but nevertheless identical for all flavors the vector current J_μ^a is still conserved for $a = 0, \dots, N_f$. The non-conservation of A_μ^0 can, e.g., be computed in a functional language [4] where it arises from the measure of the fermion fields not being invariant under chiral transformations. This non-conservation can be quantified,

$$\sqrt{\frac{N_f}{2}} \partial^\mu A_\mu^0 = -\frac{g^2}{8\pi^2} \text{tr} \left(\tilde{F}_{\mu\nu} F^{\mu\nu} \right) + \dots. \quad (2.5)$$

Here the trace goes over all discrete indices of the vector gauge fields to which the fermions, (2.4) couple, e.g., color, charge and/or flavor. Furthermore $F_{\mu\nu}$ is the field strength of the vector field that couples with coupling constant g to the fermions and $\tilde{F}_{\mu\nu}$ is dual to $F_{\mu\nu}$. The ellipsis refer to contributions that stem from finite current quark masses. A direct consequence of this anomaly is the neutral pion decay into two photons. That effect is discussed in C.4.

2.2 Dynamical Breaking of Chiral Symmetry

In general, symmetries like (2.2) can be realized by the particle content of the theory in two scenarios:

- Wigner–Weyl realization: the vacuum (lowest energy) configuration is invariant under the symmetry and the generators of the symmetry transform degenerate physical states into one another.
- Nambu–Goldstone realization (also called spontaneous symmetry breaking): the vacuum configuration is *not* invariant under the symmetry, rather the lowest energy state is degenerate. Operators that do not transform as singlets under the symmetry develop non-zero vacuum expectation values (VEV). Acting with the symmetry generators on such operators excites massless modes, the so-called Goldstone bosons (instead of transforming into other physical states). States that are related to one another by generators that do not commute with operators that possess non-zero VEVs are *not* degenerate.

Now it is for nature to decide which realization is put into effect. Since states of different chirality carry opposite parity, the question to be answered is whether or not the states of opposite parity are degenerate. For this purpose it is illuminating to consider the spectrum of the low-lying mesons as sketched in Fig. 2.1. Obviously the degeneracy expected in the Wigner–Weyl realization is not seen for the scalar and pseudoscalar mesons. While scalar meson masses are several hundred MeV (these states are quite broad, in addition) the pseudoscalar mesons start at a little above 100 MeV. As a matter of fact, and as we will recognize later, their small masses are solely due to non-zero current quark masses. Stated otherwise, the pseudoscalar mesons would have zero mass in the ideal world of massless current quarks. We conclude that chiral symmetry is realized in the Nambu–Goldstone phase with the pseudoscalar mesons being the (would-be) Goldstone modes. The only exception

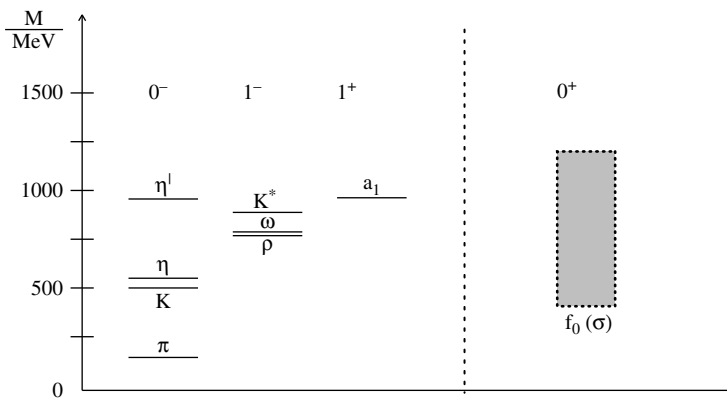


Fig. 2.1. Sketch of the spectrum of the low-lying mesons. *Left panel:* pseudoscalars (0^-), vectors (1^-) and axial vectors (1^+); *right panel:* scalars (0^+). Data taken from the particle data group [5]. This graphic is to illustrate the difference between the 0^- and 0^+ spectra. An updated account on the spectra in the scalar sector is given in the proceedings [6] and references therein

is the η' (or a linear combination of octet and singlet η s) that remains massive even when the current quark masses are sent to zero. QCD does not have a $U_A(1)$ symmetry because of the anomaly (2.5) and the fact that there are field configurations in QCD (instantons) for which the spatial integral of the right hand side of (2.5) does not vanish even though this four dimensional integral can be transformed into a three dimensional surface integral. Instantons will be discussed in Sect. 2.6; here it suffices to remark that they induce interactions [7] that upon bosonization (to be described in the following section) provide a mass term for pseudoscalar flavor singlet meson [8, 9]. This explains the absence of a (would-be) Goldstone boson for the spontaneously broken $U_A(1)$ symmetry.

Hence the meson spectrum suggests that in the limit of massless quarks only the vector symmetry is realized in the spectrum while the axial symmetry is spontaneously broken. That is, there is an operator that is invariant under vector transformations but not under axial transformations. Noting that vector transformations do alter left- and right-handed spinors equally while axial transformations do not and that

$$\bar{\Psi}\Psi = \bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L, \quad (2.6)$$

it is perspicuous that the simplest such operator is $\bar{q}q$ and that the dynamics of QCD imply a non-zero quark condensate,

$$\langle\bar{q}q\rangle \neq 0. \quad (2.7)$$

Model building therefore requires to

- (i) Find a simple mechanism that yields such a VEV or
- (ii) Start from a formulation that has (2.7) built in.

In the next subsection we will discuss the Nambu–Jona–Lasinio model [10, 11] as a (simple) example to follow path (i). On the other hand, treatments like chiral perturbation theory [12, 13, 14, 15, 16] are designed according to (ii).

2.3 The Nambu–Jona–Lasinio Model

To be specific we will consider the Nambu–Jona–Lasinio (NJL) model described by the Lagrangian (For reviews see, e.g., [17, 18, 19].)

$$\begin{aligned} \mathcal{L}_{\text{NJL}} = & \bar{q}(i\not{\partial} - \hat{m}_0)q + 2G_1 \sum_{a=0}^{N_f^2-1} \left(\left(\bar{q} \frac{\lambda_a}{2} q \right)^2 + \left(\bar{q} \frac{\lambda_a}{2} i\gamma_5 q \right)^2 \right) \\ & - 2G_2 \sum_{a=0}^{N_f^2-1} \left(\left(\bar{q} \frac{\lambda_a}{2} \gamma_\mu q \right)^2 + \left(\bar{q} \frac{\lambda_a}{2} \gamma_5 \gamma_\mu q \right)^2 \right), \end{aligned} \quad (2.8)$$

where $\hat{m}_0 = \text{diag}(m_u, m_d, \dots, m_{N_f})$ is the *current* quark mass matrix and $G_{1,2}$ are two so-far undetermined coupling constants. The discussion and results of Appendix A immediately show that the interaction terms in (2.8) are

invariant under chiral transformations, (2.2). Hence chiral symmetry is only broken by the small current quark masses.

This type of model can, e.g., be motivated from QCD by Fierz-transformation of the *color* current–current interaction that emerges after integrating out the gluon fields [19] and omitting $1/N_C$ -suppressed diquark-correlations. An easily traceable calculation is that of [20]. Here we only mention that this scenario yields $G_i \propto g_{\text{QCD}}^2 \times \mathcal{O}(N_C^0)$, where g_{QCD} is the QCD gauge coupling. Later we will argue that a sensible generalization of QCD to arbitrary N_C requires $g_{\text{QCD}} = \mathcal{O}(1/\sqrt{N_C})$, cf. (4.1), and thus $G_i = \mathcal{O}(1/N_C)$.

We want to express the quark (fermion) theory in (2.8) as an effective meson (boson) theory. Since the interaction is quartic in the quark spinors, this is actually straightforward and merely requires the completion of squares [21]. Consider, e.g.,

$$\frac{1}{8G_1} S_a^2 - S_a \bar{q} \frac{\lambda_a}{2} q = \frac{1}{8G_1} \left(S_a + 4G_1 \bar{q} \frac{\lambda_a}{2} q \right)^2 - 2G_1 \left(\bar{q} \frac{\lambda_a}{2} q \right)^2 \quad (2.9)$$

and functionally integrate over the auxiliary (meson) field S_a ,

$$\begin{aligned} & \exp \left[i \int d^4x \, 2G_1 \left(\bar{q} \frac{\lambda_a}{2} q \right)^2 \right] = \\ & \int [DS_a] \exp \left[-i \int d^4x \left(\frac{1}{8G_1} S_a^2 + S_a \bar{q} \frac{\lambda_a}{2} q \right) \right] \end{aligned} \quad (2.10)$$

up to a normalization constant. Of course, analogous relations hold for the remaining interaction terms. Combining all auxiliary fields into a single matrix valued meson field Φ allows us to formally write the generating functional as

$$\begin{aligned} Z_{\text{NJL}} &= \int [Dq] [D\bar{q}] \exp \left(i \int d^4x \, \mathcal{L}_{\text{NJL}} \right) \\ &= \int [D\Phi] \exp \left(-\frac{i}{2} \int d^4x \, \text{tr}_F [(\Phi - \hat{m}_0) Q^{-1} (\Phi - \hat{m}_0)] \right) \\ &\quad \times \int [Dq] [D\bar{q}] \exp \left(i \int d^4x \, \bar{q} (i\not{\partial} - \Phi \cdot \Lambda) q \right). \end{aligned} \quad (2.11)$$

Here “ tr_F ” denotes the trace in flavor space and the current quark mass matrix has been absorbed into a constant shift of Φ . Furthermore shorthand (matrix) notations are utilized for the flavor–Dirac structure (Λ) and the coupling constants (Q) that occur in the interaction terms of (2.8). Rather than giving them explicitly, we decompose the generic meson field Φ into irreducible Lorentz tensors

$$\Phi \cdot \Lambda = S + i\gamma_5 P - \not{V} - \not{A}\gamma_5. \quad (2.12)$$

Here S , P , V and A are scalar, pseudoscalar, vector and axial-vector fields, respectively. They are all hermitian matrices in flavor space. Furthermore the decomposition, (2.12), provides a transparent expression

$$\begin{aligned} \frac{1}{2}\text{tr}(\Phi - \hat{m}_0)Q^{-1}(\Phi - \hat{m}_0) &= \frac{1}{4G_1}\text{tr}((S - \hat{m}_0)^2 + P^2) \\ &\quad - \frac{1}{4G_2}\text{tr}(V_\mu V^\mu + A_\mu A^\mu), \end{aligned} \quad (2.13)$$

for the argument that appears in the exponential of the mesonic part of the generating functional. When convenient, we will use Φ as a short-hand notation for all fields S , P , V and A and combine scalar and pseudoscalar fields to $M = S + iP$ and $M^\dagger = S - iP$.

The generating functional factorizes into mesonic and fermionic parts with a generalized Yukawa interaction between mesons and fermions. In (2.11) the quark field appears bilinearly in the exponent and can now be integrated out. This integration yields the determinant of the operator in between the spinors. Using the identity $\log \text{Det}(A) = \text{Tr} \log(A)$ finally leads to a purely mesonic theory $\mathcal{A}[\Phi]$ that is given by

$$\begin{aligned} Z_{\text{NJL}} &= \int [D\Phi] \exp(i\mathcal{A}[\Phi]) \quad \text{with} \\ \mathcal{A}[\Phi] &= -\frac{1}{2} \int d^4x (\Phi - \hat{m}_0)Q^{-1}(\Phi - \hat{m}_0) - i \text{Tr} \log(i\cancel{\partial} - \Phi). \end{aligned} \quad (2.14)$$

Here Tr denotes the functional trace that also includes space–time integration on top of summing over discrete indices. The quarks carry color degrees of freedom. Yet the NJL-model interaction is color neutral, so the associated trace merely causes multiplication by N_C , the number of color degrees of freedom, i.e., $\text{Tr} \rightarrow N_C \text{Tr}_{\text{DF}}$, with the latter trace involving only Dirac and flavor discrete indices (together with space–time integration).

In (2.14) we have essentially met our goal to bosonize the fermion model. Of course, the interaction in (2.8) has been “invented” to exactly facilitate that goal. In (2.14) we sum up all one fermion loop diagrams, i.e., $\mathcal{A}[\Phi]$ is complete at $\mathcal{O}(\hbar)$. The full quantum action of the NJL-model also contains higher order contributions. In the present treatment they do not occur because we have treated the meson fields Φ classically. Higher order terms arise from their quantum properties. For that reason the action, (2.14) is sometimes called the semi-bosonized NJL-model.

Note that the action $\mathcal{A}[\Phi]$ is a non-linear, even non-polynomial function of the meson field Φ ; even more, $\text{Tr} \log(i\cancel{\partial} - \Phi)$ is non-local. The quantum theory defined by (2.14) is, however, equivalent to the underlying NJL model defined by the Lagrangian (2.8). On the other hand, the generating functional (2.14) has the advantage that it may be treated semiclassically. In particular, according to (2.10) a stationary point for Φ is to be identified with a VEV of a quark bilinear. This paves the way toward the second goal at which we aim, a microscopic quark model with a non-zero translationally invariant stationary point $S_0 \sim G_1 \langle \bar{q}q \rangle$ to parameterize spontaneous chiral symmetry breaking. Unfortunately the action, (2.14) is not yet suited for actual calculations because of ultraviolet divergences in $\text{Tr} \log(i\cancel{\partial} - \Phi)$. A regularization prescription is needed, and as the model (2.8) is not renormalizable

(the coupling constants G_i have dimension $1/(\text{mass})^2$), the model itself is only completely defined when a regularization scheme is provided. For definiteness we will use Schwinger’s proper time regularization [22] which introduces an $O(4)$ -invariant cut-off Λ after continuation to Euclidean space that is enforced by the Wick rotation.¹ Even though other regularization schemes give similar results [25], its choice is part of the model building. In Euclidean space it is necessary to consider the real and imaginary parts of the non-local piece separately

$$\mathcal{A}_F := -i \text{Tr} \log(i\cancel{\partial} - \Phi \cdot \Lambda) \xrightarrow[\text{rotation}]{\text{Wick}} \mathcal{A}_R + \mathcal{A}_I, \quad (2.15)$$

with

$$\mathcal{A}_R = \frac{1}{2} \text{Tr} \log(\cancel{D}_E^\dagger \cancel{D}_E) \quad \text{and} \quad \mathcal{A}_I = \frac{1}{2} \text{Tr} \log \left((\cancel{D}_E^\dagger)^{-1} \cancel{D}_E \right). \quad (2.16)$$

Here $\cancel{D}_E = \sum_{\mu=1}^4 D_{E\mu} \gamma_\mu$ is the argument of the logarithm in (2.15) analytically continued to Euclidean space (\mathcal{A}_F is real in Minkowski space). The real part \mathcal{A}_R diverges for large momenta p whereas the imaginary part \mathcal{A}_I does not contain ultraviolet divergencies, i.e., it is finite without regularization. Therefore one has the option of keeping \mathcal{A}_I unregularized, or to regularize it in a way consistent with the regularization of \mathcal{A}_R . Note that this defines two different models.² For the real part of the action the proper time regularization consists in replacing the logarithm by a parameter integral

$$\mathcal{A}_R \rightarrow -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} \exp \left(-s \cancel{D}_E^\dagger \cancel{D}_E \right), \quad (2.17)$$

which for $\Lambda \rightarrow \infty$ reproduces the logarithm up to an irrelevant additive constant, cf. (B.18). For finite Λ the contributions from small s in the integral are left out. On the other hand, only the small s values are sensible to the regime where the expectation value of $\cancel{D}_E^\dagger \cancel{D}_E$ is large. That is, the contributions from large momenta in the functional trace are suppressed. Hence Λ is an ultraviolet cut-off. (The notation Λ for this cut-off should not be confused with the abbreviation for the flavor–Dirac structure in (2.11) and (2.12).)

To discuss chiral symmetry breaking it suffices to consider \mathcal{A}_R as regularized in (2.17) and omit (axial)vector interactions for the time being, i.e., $V_\mu = A_\mu = 0$. Variation with respect to the scalar and pseudoscalar fields yields the Dyson–Schwinger or gap equations. By symmetry, their translationally invariant solutions must be Lorentz scalar and of neutral flavor. However,

¹ Analogous formulations in Minkowski space are reported in [23, 24], see also Sect. 7.7.

² The proper reproduction of the anomaly, (2.5) seems to prohibit regularization of \mathcal{A}_I , [26]; see, however, [27]. As will be discussed in Chap. 3, the requirement that soliton configurations possess integer baryon number corroborates non-regularization of \mathcal{A}_I .

different current quark masses prevent the solution from being proportional to the unit matrix in flavor space. We therefore parameterize $\langle M_{ij}(x) \rangle = \delta_{ij} m_i$ for $i, j = 1, \dots, N_f$. From (2.15) it is obvious that m_i acts as a mass for the quark of flavor i . Therefore a non-zero value is called *constituent* quark mass. In the proper time regularization the explicit form of the gap equation is (The calculation may be traced from Appendix B.)

$$m_i = m_{0,i} - 2G_1 \langle \bar{q}q \rangle_i \quad \text{where} \quad \langle \bar{q}q \rangle_i = -m_i^3 \frac{N_C}{4\pi^2} \Gamma \left(-1, \frac{m_i^2}{\Lambda^2} \right). \quad (2.18)$$

The notation already identifies the quark condensate $\bar{q}q$ as it is obtained from the stationary point of the scalar field, (2.10). The interpretation in terms of the quark loop is apparent from (B.15). This leads to the graphical representation

$$m_i = m_{0,i} + \text{---} \overset{\text{---} \bigcirc \text{---}}{\bullet} \text{---} \text{---}$$

(The diagram shows a horizontal line with a dot in the middle. Above the dot is a circle. The line continues to the right of the dot. The label 'i' is placed above the line on both sides of the dot.)

with the mass of the quark in the loop being m_i , the dynamically generated constituent quark mass.

We see from Fig. 2.2 that in the chiral limit ($m_0 = 0$) the quark condensate and therefore also the quark constituent mass is zero when the coupling constant G_1 stays below a critical value whose precise datum depends on the cut-off, Λ . Above this critical value the trivial solution coexists with a

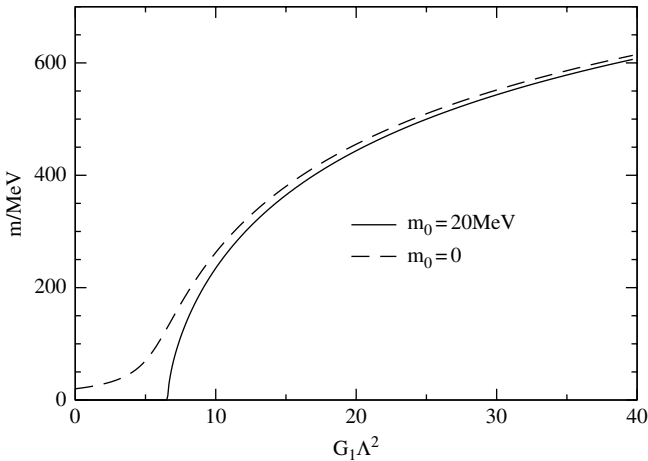


Fig. 2.2. The solution of the gap equation (2.18) for vanishing current mass $m_0 = 20$ MeV (solid line) and $m_0 = 0$ (dashed line) as function of the coupling constant G_1 . In this specific computation $\Lambda = 630$ MeV has been chosen

non-trivial one. The effective potential of a constant scalar field in the chiral limit [20]

$$V_{\text{eff}}(M) = \frac{1}{2G_1}\Sigma^2 + \frac{N_C}{16\pi^2} \left[\Sigma^4 \Gamma\left(0, \frac{\Sigma^2}{\Lambda^2}\right) - (\Sigma^2 - \Lambda^2) \Lambda^2 e^{-\Sigma^2/\Lambda^2} \right], \quad (2.19)$$

where $\Sigma = \sqrt{\text{tr}(MM^\dagger)}/N_f$ shows that the solution with $\Sigma \neq 0$ is energetically favored. Commonly the so-defined Σ is called the chiral radius.

Having established the existence of a non-trivial VEV $\langle \bar{q}q \rangle$, we still have to verify that massless pions emerge, at least for $m_{0,i} = 0$. Fortunately the examination of pion properties also allows us to assign a physical meaning to the above introduced and so far undetermined ultraviolet cut-off Λ . For simplicity we will omit flavor symmetry breaking in this context and call m the solution to the flavor symmetric gap equation, (2.18) with $m_{0,i} \equiv m_0$. The effects of $m_{0,i} \neq m_{0,j \neq i}$ may be traced from the literature [28, 29]. The Goldstone modes are expected to be orthogonal to the mode carrying the VEV. We therefore parameterize

$$M = mU(x) \quad \text{with} \quad U(x) = \exp \left[i \sum_{a=1}^{N_f^2-1} \phi_a(x) \frac{\lambda_a}{2} \right], \quad (2.20)$$

which also defines the chiral field, $U(x)$. Substituting M and $U = 1 + i\phi_a(x)\lambda_a/2$ into (2.12) shows that the real fields $\phi_a(x)$ couple to the quarks via γ_5 , as pseudoscalars should. In addition, this ensures that the modes ϕ_a are indeed orthogonal to the scalar modes that contain the VEV. The main task is to expand the (regularized) action up to quadratic order in $\phi_a(x)$. The techniques for this calculation are provided in Appendix B. The result is most conveniently presented in (Euclidean) momentum space

$$\mathcal{A}^{(2)} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \sum_a \tilde{\phi}_a(q) D^{-1}(q^2) \tilde{\phi}_a(-q), \quad (2.21)$$

where the superscript indicates the expansion up to second order and $\tilde{\phi}_a(q)$ is the Fourier transformation of $\phi_a(x)$. The first term in the inverse propagator

$$D^{-1}(q^2) = -\frac{m_0 m}{G_1} - q^2 f(q^2) \quad \text{with} \\ f^2(q^2) = m^2 \frac{N_C}{4\pi^2} \int_0^1 dx \Gamma\left(0, \frac{m^2 + x(1-x)q^2}{\Lambda^2}\right) \quad (2.22)$$

originates from the local part of the action \mathcal{A} , (2.14). The mass of $\phi_a(x)$ as extracted from the pole condition, $D^{-1}(-m_\phi^2) = 0$ obviously vanishes in the chiral limit $m_0 = 0$. Identifying the modes $\phi_a(x)$ as pions and comparison with the non-linear σ model (or coupling an external axial current to \mathcal{A} , cf. Sect. 2.5 and (B.25)) furthermore shows that the pion decay constant is

$$f_\pi^2 = f^2(-m_\pi^2). \quad (2.23)$$

Imposing the empirical values $f_\pi = 93 \text{ MeV}$ and $m_\pi = 138 \text{ MeV}$ thus yields a further relation between the cut-off Λ and the constituent quark mass m . In practice a value for the constituent quark mass $m \sim 400 \text{ MeV}$ is chosen for the reason discussed later in Sect. 5.6. Equation (2.23) then provides the corresponding value $\Lambda \sim 630 \text{ MeV}$. Subsequently the gap equation, (2.18) yields the coupling constant G_1 and finally the current quark mass m_0 is determined from $D^{-1}(-m_\pi^2) = 0$,

$$m_\pi^2 f_\pi^2 = \frac{m_0 m}{G_1}. \quad (2.24)$$

Stated otherwise, the constituent quark mass is considered as the only adjustable model parameter.

We also confirm an important statement of Sect. 2.2: The pion would indeed be a massless Goldstone boson if the current quark mass, m_0 , were zero.

Similar computations have been applied to the vector interactions. These investigations determine the coupling constant G_2 from the empirical value of the ρ -meson mass $m_\rho = 770 \text{ MeV}$. In particular the vector interactions contain π - A_1 mixing and the model yields the estimate [30]

$$m_{A_1}^2 = m_\rho^2 + 6m^2 + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \quad (2.25)$$

according to which the axial-vector mesons are significantly heavier than the vector mesons. Approximate results for the (axial)vector mesons can also be obtained in the gradient expansion that will be subject of the next section. Furthermore exhaustive studies of flavor symmetry breaking effects in the meson sector have been performed. Concerning the three-flavor model it should be noted that there is only one additional parameter, the strange quark current mass.³ Hence the kaon decay constant, f_K is a prediction. This prediction comes out a bit on the low side: $f_K/f_\pi \approx 1.11$ vs. 1.21 empirically [5], nevertheless it is in the right ball park. For further details we refer the interested reader to original studies [29, 30, 31] and review articles [17, 18, 19]. In any event, the above discussion is sufficient as a set-up of the model to discuss baryons as solitons in Chap. 3.

We want to conclude this section by discussing the transformation properties of the (pseudo)scalar field M under global chiral transformations. After all, the model for the quark flavor dynamics was built to reflect the chiral properties of QCD. According to the bosonization prescription we explicitly write the flavor indices (color and spin degrees are summed),

³ We stress that the current quark masses in (2.8) are those of the model. They are model parameters and should not be confused with those in QCD.

$$M_{ij}(x) = \sum_{a=0}^{N_f^2-1} M_a(x) \lambda_a^{(ij)} \propto \sum_{a=0}^{N_f^2-1} \lambda_a^{(ij)} \left[\bar{q}_m \frac{\lambda_a^{(mn)}}{2} q_n - \bar{q}_m \gamma_5 \frac{\lambda_a^{(mn)}}{2} q_n \right]. \quad (2.26)$$

In general one would expect a bilocal expression on the right hand side. However, here we are only interested in *global* aspects and we may ignore that complication. For the same reason we omit constants of proportionality. From the completeness relation for $SU(N)$ generators ($T_a = \lambda_a/2$)

$$\sum_{a=1}^{N^2-1} (T_a)_{ij} (T_a)_{kl} = \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2N} \delta_{ij} \delta_{kl} \quad (2.27)$$

we find

$$M_{ij}(x) \propto \frac{1}{2} [\bar{q}_j q_i - \bar{q}_j \gamma_5 q_i] = \bar{q}_{Rj} q_{Li}, \quad (2.28)$$

with explicit reference to chirality. Under global chiral rotations

$$q_L \rightarrow L q_L \quad \text{and} \quad q_R \rightarrow R q_R \quad (2.29)$$

with L and R constant $SU(N_f)$ matrices, we thus induce

$$M(x) \longrightarrow LM(x)R^\dagger. \quad (2.30)$$

This, of course, is consistent with the requirement that

$$\bar{q} (i\cancel{\partial} - MP_R - M^\dagger P_L) q \quad (2.31)$$

is chirally invariant. Seemingly trivial, the coexistence of (2.29) and (2.30) is a very important result: We have just learned how to translate the chiral transformation properties of QCD to the meson fields M . Hence we may identify the internal symmetries of a model for M with those of QCD! Since most of the QCD hadron matrix elements are to be computed from symmetry currents we are allowed to identify the QCD matrix elements with those computed in the model. This is indeed the only venue which permits model calculations of QCD observables. Conversely, model calculations that are based on identifying model degrees of freedom with those of QCD (rather than just identifying currents) are less trustworthy.

2.4 Gradient Expansion

The bosonized action, (2.14), and its regularized version, (2.17), are non-local meson theories. Many of the technical problems in the above-described calculations emerge from this non-locality. It transforms into an infinite series of local derivative terms by a Taylor expansion in the separation. The coefficients of these derivative terms are determined from the non-local action

via the gradient expansion [32, 33, 34]. Assuming that the meson fields Φ vary only slowly in space and thereby mitigating non-local effects, we may approximate this Taylor series by truncating it to a low, say next-to-leading, order. Here we will briefly gather the main results of this approximation for the NJL model, in particular because it serves to construct effective meson theories in which the construction of soliton solutions and their quantization are significantly more perspicuous than in bosonized NJL-type models. In this discussion of the gradient expansion we will mostly follow the treatment of [30]. That calculation counts (axial)vector meson fields at the same order as a single derivative which is suggested by covariant derivatives. For the sum of the local part of the action, (2.14), and the regularized real part of the determinant, (2.17), it yields,

$$\begin{aligned} \mathcal{A}_R = \int d^4x \left\{ -V_{\text{eff}}(M) + \frac{1}{4G_2} \text{tr} (V_\mu V^\mu + A_\mu A^\mu) \right. \\ \left. + \frac{1}{2g_V^2} \text{tr} [3\nabla_\nu M^\dagger \nabla^\nu M - F_{\mu\nu}^V F^{V\mu\nu} - F_{\mu\nu}^A F^{A\mu\nu}] \right\} + \dots \end{aligned} \quad (2.32)$$

where

$$\nabla^\nu M = \partial^\nu M + i[V^\nu, M] - i\gamma_5\{A^\nu, M\} \quad (2.33)$$

denotes the covariant derivative of the scalar–pseudoscalar field M . The vector and axial-vector parts of the field strength tensor are

$$\begin{aligned} F_{\mu\nu}^V &= \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu] + i[A_\mu, A_\nu], \\ F_{\mu\nu}^A &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, V_\nu] + i[V_\mu, A_\nu], \end{aligned} \quad (2.34)$$

and the effective potential V_{eff} is given in (2.19). The resulting coefficient

$$\frac{1}{g_V^2} = \frac{N_C}{24\pi^2} \Gamma\left(0, \frac{m^2}{\Lambda^2}\right) = \frac{f^2(0)}{6m^2} \quad (2.35)$$

exemplifies the role of the derivative expansion: In momentum space it is a Taylor series about $q^2 = 0$. The substitution of the parameterization, (2.20) into (2.32), suggests $f_\pi = f(0)$ in the absence of (axial)vector mesons. This differs from the exact result, (2.23) by $\mathcal{O}(m_\pi^2/\Lambda^2)$.

Let us now look a bit closer at the vector meson fields. As a consequence of spontaneous chiral symmetry breaking the anti-commutator in (2.33) has a piece that is purely proportional to the axial-vector field. Thus a term of the form $A_\mu \partial^\mu (M - M^\dagger)$ emerges in the action, (2.32). Essentially this is π - A_1 mixing and requires a redefinition of the axial-vector field to account for the physical particle content. Then the (axial)vector meson masses are identified as

$$m_V^2 = \frac{g_V^2}{4G_2} \quad \text{and} \quad m_A^2 = m_V^2 + 6m^2. \quad (2.36)$$

The redefinition of the axial-vector field furthermore induces an additional quadratic derivative term for the pion field from the local part of the action. This requires a renormalization to identify the physical pion field. In total one reads off the pion decay constant

$$f_\pi^2 = \frac{f^2(0)}{1 + 4G_1 f^2(0)}. \quad (2.37)$$

In terms of the two flavors with $M = m(1 + i\boldsymbol{\pi} \cdot \boldsymbol{\tau}/f_\pi + \dots)$ and $V_\mu = (g_V/2)\boldsymbol{\rho}_\mu \cdot \boldsymbol{\tau}$ the commutator in (2.33) generates the vertex

$$\mathcal{L}_{\rho\pi\pi} = g_V \boldsymbol{\rho}_\mu \cdot (\boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi}) \quad (2.38)$$

that describes the ρ -meson decay into two pions and identifies g_V as the associated coupling constant, $g_{\rho\pi\pi} = g_V$. Putting (2.35)–(2.38) together relates observable quantities,

$$\frac{m_V^2}{m_A^2} = \frac{g_{\rho\pi\pi}^2 f_\pi^2}{m_A^2 - m_V^2}. \quad (2.39)$$

As a consequence, the particular choice for G_2 that ensures Weinberg's relation between vector and axial-vector masses $m_A = \sqrt{2}m_V$ [35] also gives the KSRF relation $m_V = \sqrt{2}g_{\rho\pi\pi}f_\pi$ for the ρ -meson decay [36, 37]. The interaction Lagrangian, (2.38) results in the ρ -meson width $\Gamma(\rho \rightarrow \pi\pi) = \frac{g_{\rho\pi\pi}^2}{6\pi m_\rho^2} |\mathbf{q}_\pi|^3$, where $|\mathbf{q}_\pi| \approx 360$ MeV is the pion momentum in the ρ -meson rest frame. The empirical value $\Gamma(\rho \rightarrow \pi\pi) \approx 150$ MeV [5] gives $g_{\rho\pi\pi} \approx 6$ vs. 5.85 from KSRF.

The special case that only pseudoscalar fields are considered, i.e., $M = mU$ as in (2.20) has been thoroughly investigated in the context of the gradient expansion. For $U \in SU(N_f)$ the leading term must have at least two derivatives. From (2.32) and (2.35) it is obvious that this term is the non-linear σ -model,

$$\mathcal{L}_{\text{nl}\sigma} = -\frac{f_\pi^2}{4} \text{tr}(\alpha_\mu \alpha^\mu) = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger), \quad (2.40)$$

with $\alpha_\mu = U^\dagger \partial_\mu U$. In the two-flavor case we consider the chiral field $U = \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}/f_\pi)$ as the non-linear representation for the pion fields and expand,

$$\mathcal{L}_{\text{nl}\sigma} = \frac{1}{2} (\partial_\mu \boldsymbol{\pi}) \cdot (\partial^\mu \boldsymbol{\pi}) + \frac{1}{6f_\pi^2} \left[(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi})^2 - \boldsymbol{\pi}^2 (\partial_\mu \boldsymbol{\pi}) \cdot (\partial^\mu \boldsymbol{\pi}) \right] + \dots, \quad (2.41)$$

which determines the four pion coupling constant to be proportional to $1/f_\pi^2$. From (2.22) we infer that this effective four pion coupling constant scales as $1/N_C$. The expansion, (2.41), is the principal starting point of chiral perturbation theory [12, 13, 14, 15, 16]. Higher order derivative terms are also known for the pseudoscalar case. The contribution with four derivatives acting on U is [30, 38]

$$\begin{aligned} \mathcal{L}^{(4)} = & \frac{N_C}{96\pi^2} \Gamma\left(1, \frac{m^2}{\Lambda^2}\right) \text{tr} \left\{ (\alpha_\mu \alpha^\mu)^2 - (\partial_\mu \alpha^\mu)^2 \right\} \\ & + \frac{N_C}{192\pi^2} \Gamma\left(2, \frac{m^2}{\Lambda^2}\right) \text{tr} \left\{ \alpha_\mu \alpha_\nu \alpha^\mu \alpha^\nu - 2(\alpha_\mu \alpha^\mu)^2 \right\}. \end{aligned} \quad (2.42)$$

The coefficients are finite in the limit $\Lambda \rightarrow \infty$,

$$\lim_{\Lambda \rightarrow \infty} \mathcal{L}^{(4)} = \frac{N_C}{384\pi^2} \text{tr} \left\{ [\alpha_\mu, \alpha_\nu]^2 - 4(\partial_\mu \alpha^\mu)^2 + 2(\alpha_\mu \alpha^\mu)^2 \right\}. \quad (2.43)$$

The commutator term will later play a decisive role in the framework of the Skyrme model.

So far we have only considered the modulus of the fermion determinant in Euclidean space, (2.15). But also the phase is non-zero, even in the absence of vector meson fields. It is related to the anomaly as it arises from the fact that the fermion determinant is not invariant under (local) axial transformations. In leading order of the derivative expansion this phase results in the Wess–Zumino–Witten action [39] when rotated back to Minkowski space. The calculation is somewhat involved and we will only sketch it here.

Starting point is the parameterization

$$M = S + iP = \xi_L^\dagger \Sigma \xi_R, \quad (2.44)$$

and the observation is that the local chiral transformation

$$i\tilde{\mathcal{D}} = \mathcal{T} i \mathcal{D} \mathcal{T}^\dagger, \quad \text{with} \quad \mathcal{T} = \xi_L + \xi_R - (\xi_L - \xi_R) \gamma_5 \quad (2.45)$$

removes the pseudoscalar fields from the Dirac operator

$$i\tilde{\mathcal{D}} = i(\not{\partial} + \tilde{V} + \tilde{A} \gamma_5) - \Sigma, \quad (2.46)$$

in favor of the induced (axial) vector fields \tilde{V}_μ and \tilde{A}_μ . The idea now is to compute the fermion determinant for \tilde{D} in the proper time scheme

$$\mathcal{A}_F = \frac{1}{2} \text{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-s\tilde{D}\tilde{D}} \quad (2.47)$$

and perform the transformation inverse to (2.45) in order to incorporate the chiral field $U(x) = \xi_L^\dagger(x) \xi_R(x)$. Under an infinitesimal local chiral transformation the (axial)vector fields vary as

$$\delta \left(\tilde{V}_\mu + i\gamma_5 \tilde{A}_\mu \right) = [\tilde{D}_\mu, \delta\alpha(x)] + i[\tilde{D}_\mu, \delta\beta(x)] \gamma_5, \quad (2.48)$$

where $\delta\alpha = \Omega^\dagger \delta\Omega$ and $\delta\beta = \omega^\dagger \delta\omega$ are Cartan matrix fields in flavor space (i.e., Ω and ω are unitary). The regularized fermion determinant transforms as

$$\delta \mathcal{A}_F = 2i \text{Tr} \left[e^{-\tilde{D}\tilde{D}/\Lambda^2} \omega^\dagger \delta\omega \gamma_5 \right]. \quad (2.49)$$

This is actually nothing but Fujikawa's formulation of the chiral anomaly [40]. The regularization has been chosen such that only axial transformations contribute. This guarantees that the vector current is conserved. We consider (2.49) as a differential equation in functional space that must be integrated from $\omega(x) = 1$ to $\omega(x) = U(x)$. This is a complicated calculation that involves various aspects of differential geometry. Details are given in [41], see also [42]. Heat kernel methods [43] may be employed to verify that the right hand side of (2.49) is finite as $\Lambda \rightarrow \infty$. In the absence of vector mesons and in leading order of the gradient expansion the calculation yields the Wess–Zumino term

$$\Gamma_{\text{WZ}} = -\frac{iN_C}{240\pi^2} \int_{M_5} d^5x \epsilon^{\mu\nu\rho\sigma\tau} \text{tr}(\alpha_\mu \alpha_\nu \alpha_\rho \alpha_\sigma \alpha_\tau). \quad (2.50)$$

The integral is over a five-dimensional manifold whose boundary is Minkowski space, $\partial M_5 = M_4$. This fifth dimension reflects the auxiliary variable which is introduced to formally integrate the anomaly equation (2.49) in functional space [44]: The additional variable $\tau \in \mathbb{R}$ generalizes $U(x) \rightarrow U_\tau(x) = [U(x)]^\tau$. Then also the induced (axial)vector fields \tilde{V}_μ and \tilde{A}_μ that are contained in \tilde{D} parametrically depend on τ . The Wess–Zumino term, (2.50) finally arises by integrating the anomaly equation (2.49) from $\tau = 0$ to $\tau = 1$.

Obviously the Wess–Zumino term is non-local. In practice its contribution to an observable in four-dimensional space is computed with the help of Stoke's theorem. Due to the anti-symmetric structure this term vanishes in the two-flavor reduction. However, in the three-flavor version it describes processes like kaon scattering into three pions. Also, when gauged with electromagnetic fields, it properly describes the anomalous π^0 decay into two photons [39], see Appendix C where we repeat that calculation. In the discussion of the soliton picture (Chap. 6) for baryons we will recognize that the Wess–Zumino term has very decisive consequences for the quantization of the soliton: The baryon number one soliton is forced to possess half-integer spin when N_C is odd.

2.5 PCAC

In (2.23) we have identified the residuum of the propagator for the pseudoscalar fields ϕ_a as the (square of the) pion decay constant, f_π . We have then normalized the argument of the chiral field in the non-linear σ model accordingly, cf. (2.40). Actually f_π is not a pure strong interaction quantity. Hence that identification appears a bit premature and we will now argue in its favor. The pion decay constant is measured from the pion decay into muon and muon–neutrino. This is an electroweak process and the corresponding (low-energy) interaction is prescribed as a current–current Lagrangian,

$$\mathcal{L}_{\pi \rightarrow \mu\nu_\mu} = cA_\mu^{(\text{hadr})} A^{(\text{lept}),\mu} \quad (2.51)$$

where the coupling constant, c , is found from the Weinberg–Salam model for the electroweak interactions. Furthermore $A_\mu^{(\text{hadr})}$ and $A_\mu^{(\text{lept})}$ are the axial current operators for the hadrons (i.e., $\boldsymbol{\pi}$) and leptons (μ, ν_μ), respectively. For simplicity the sum over flavor indices is not made explicit in (2.51). It is now obvious that we have to compute the hadronic matrix element $\langle 0|A_\mu|\boldsymbol{\pi}(p)\rangle$ to investigate the decay of a pion with momentum p . We have dropped the superscript because it is unambiguous that we are concerned with hadronic axial current from now on. Also, eventual vector interactions are not shown in (2.51), because the analog pion matrix element vanishes by parity.

The pion decay constant is simply defined as the pion matrix of the axial current

$$\langle 0|A_\mu^a(x)|\pi^b(q)\rangle = if_\pi\delta_{ab}q_\mu e^{-iqx}. \quad (2.52)$$

In principle, Lorentz covariance allows f_π to depend on q^2 , but the pion is on-shell so $q^2 = m_\pi^2$ is fixed. Of course, it is possible to compute this matrix element in the above-discussed model for the quark flavor dynamics. As sketched at the end of Appendix B this indeed yields (2.23). It is more illuminating to consider this matrix element in the local effective meson theory. In this model the axial current is obtained as the Noether current for the chiral transformation $L = R^\dagger$ in (2.30). The resulting infinitesimal variation of the chiral field is proportional to the anti-commutator $\{U, \boldsymbol{\tau}\}$. To compute the matrix element, (2.52) we only need the part of the axial current that is linear in the pion field. From the non-linear σ model, (2.40) it is straightforwardly found to be

$$\mathbf{A}_\mu^{(\text{nl}\sigma)}(x) = i\frac{f_\pi^2}{2}\text{tr}\left[\frac{\boldsymbol{\tau}}{2}(\alpha_\mu + U\alpha_\mu U^\dagger)\right] = -f_\pi\partial_\mu\boldsymbol{\pi}(x) + \mathcal{O}(\boldsymbol{\pi}^2). \quad (2.53)$$

This shows that the previous identification of the pion decay constant is indeed equivalent to its actual definition, (2.52).

We may differentiate (2.52) to find

$$\langle 0|\partial^\mu A_\mu^a(x)|\pi^b(q)\rangle = f_\pi m_\pi^2\delta_{ab}e^{-iqx}. \quad (2.54)$$

This clearly demonstrates the role of the pion as a would-be Goldstone boson: if the axial current were conserved, the pion would indeed be massless. We reexpress the right hand side of (2.54) as the matrix element $f_\pi m_\pi^2\langle 0|\hat{\pi}^a(x)|\pi^b(q)\rangle$ where $\hat{\pi}^a(x)$ is the pion field operator. The resulting generalization of (2.54) into an operator identity

$$\partial^\mu \mathbf{A}_\mu(x) = f_\pi m_\pi^2 \hat{\boldsymbol{\pi}}(x) \quad (2.55)$$

is called the *partially conserved axial vector current* (PCAC) hypothesis. PCAC relates a current whose matrix elements are measured in the weak interaction to an operator in strong interactions and one is tempted to assume that numerous predictions follow from it. However, in practice more assumptions must often be made to arrive at definite results. Direct relations from

PCAC concern matrix elements at zero momentum transfer and smoothness of the form factors must be taken for granted to extrapolate to the physically relevant regime.

In soliton physics PCAC is sometimes interpreted in the opposite way. While the axial current can be computed as Noether current, the identification of the interpolating pion field operator is not as straightforward because in these models the asymptotic pion field is a superposition of a classical field and fluctuations about it. Then (2.55) is unprejudicedly utilized as a definition of the pion field operator in terms of \mathbf{A}_μ . As stated PCAC is a hypothesis, and we will discuss an example in Chap. 9 where that definition (or at least its generalization to the three-flavor case) appears to be inconsistent.

2.6 Relation to Instanton Effects

Often instanton effects are utilized to motivate or even to form the basis for a derivation of the model Lagrangian, (2.8), cf. [45, 46]. This certainly is an overemphasis of such effects. However, instanton effects can be argued to induce dynamical chiral symmetry breaking in a way similar to the quartic quark interaction in (2.8).

To illuminate that point let us briefly recall the nature of instantons and their relevance to QCD. Instantons are localized field configurations that minimize the Euclidean Yang–Mills action

$$S_E[A] = \frac{1}{2g^2} \int d^4x_E \operatorname{tr} (F_{\mu\nu} F_{\mu\nu}) \quad (2.56)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength tensor. The gauge field, A_μ itself is matrix valued in color space. In the standard realization instantons are embedded in the $SU(2)$ subgroup whose generators are proportional to the Pauli matrices $\boldsymbol{\tau}$,

$$A_\mu^{(\text{inst})} = \frac{x^2}{x^2 + \rho^2} V(x) \partial_\mu V^\dagger(x) \quad \text{with} \quad V(x) = \frac{1}{\rho} (x_4 + i\boldsymbol{\tau} \cdot \mathbf{x}). \quad (2.57)$$

Here x_4 is the Euclidean time, $x^2 = x_4^2 + \mathbf{x}^2$ and ρ is a constant scale parameter that measures the extension (size) of the instanton. Rescaling immediately shows that the action, (2.56) does not depend on ρ . Anti-instanton configurations are simply constructed by substituting $V(x) \rightarrow V^\dagger(x)$. Vacuum configurations are characterized by $F_{\mu\nu} \equiv 0$ for which it suffices that $A_\mu^{(\text{vac})} = W(x) \partial_\mu W^\dagger(x)$ is pure gauge. Any of these vacuum configurations is characterized by a topological charge n . The instanton configuration, (2.57) mediates between vacua with charges n at $x_4 = -\infty$ and $n + 1$ at $x_4 = \infty$. Even more, the semiclassical analysis reveals that the transition amplitude between such two vacua is exactly $e^{-S_E[A^{(\text{inst})}]} = e^{-8\pi^2/g^2} [1]$.

In the next step fermions are coupled to the instanton. Consider the eigenvalues, λ_n of the Dirac operator with the instanton background and label their spectral density by $\nu(\lambda_n)$. Then the fermion determinant in the instanton background can formally⁴ be written as (in Euclidean space)

$$\log \text{Det} \left(i\not{\partial} + \not{A}^{(\text{inst})} + im \right) = \frac{1}{2} \int_{-\infty}^{\infty} d\lambda \nu(\lambda) \log (\lambda^2 + m^2) . \quad (2.58)$$

As in (B.15) the quark condensate $\langle \bar{q}(x)q(x) \rangle$ is obtained from the derivative of the left hand side with respect to the quark mass, m . Utilizing the δ -function representation $\delta(x) = \lim_{\epsilon \rightarrow 0^+} \pi\epsilon/(x^2 + \epsilon^2)$ yields the famous Casher–Banks relation [47]

$$\langle \bar{q}(x)q(x) \rangle \xrightarrow{m \rightarrow 0} -\pi\nu(0) \quad (2.59)$$

that relates the quark condensate in the chiral limit to the zero-mode density. The crucial observation is that, as a result of the Atiah–Singer index theorem, an instanton background generates a zero-mode fermion that is right-handed in the limit $m \rightarrow 0$. Choosing as above to embed the instanton in the $SU(2)$ color subgroup, the zero-mode spinor reads

$$q_{\mathbf{1}}(x) = \frac{\rho}{\pi\sqrt{2x^2}(x^2 + \rho^2)^{3/2}} \boldsymbol{\gamma} \cdot \mathbf{x} \Omega . \quad (2.60)$$

In the chiral representation of the Dirac matrices Ω is a 4×2 matrix with the upper 2×2 block being zero and the lower one equal to $i\tau_2$. The right index of Ω denotes color and eventually couples to the instanton, (2.57). An anti-instanton also generates a zero-mode fermion as in (2.60), however, the 2×2 blocks in Ω exchanged. Thus the zero-mode of an anti-instanton is left-handed.

To set up a model (sometimes called the instanton liquid model [48]), assume that the vacuum is filled by an ensemble of (well-separated) instantons and anti-instantons. Such an ensemble produces $\nu(0) \neq 0$ and thus a non-zero quark condensate. It is beyond the scope of this monograph to repeat the actual model calculations,⁵ however, the lesson to be learned is that instanton effects in QCD indeed may cause chiral symmetry to be spontaneously broken.

When the fermion propagator S in the single instanton background

$$S(x, y) \approx S_0(x, y) + \frac{q_{\mathbf{1}}(x)q_{\mathbf{1}}^{\dagger}(y)}{-im} \quad (2.61)$$

is approximated by the sum of the free propagator S_0 and the zero-mode component, an action functional that reproduces this fermion propagator can be constructed. This functional contains a non-local interaction of the fermion fields with the spinor of the zero-mode. Essentially $2N_f$ fermions couple to the single instanton. In the instanton liquid model the above approximation

⁴ Note that we treat ultraviolet divergent objects as if they were finite.

⁵ See, e.g., the review articles [46, 48, 49, 50] and references therein.

scheme is generalized to multi-instanton and anti-instanton configurations in the first step. Subsequently their positions and orientations are integrated over. This induces quark correlations and at the same time defines an effective model for the quark flavor dynamics. For $N_f = 2$ and in leading order $1/N_C$ the flavor structure of this effective model relates to the G_1 term in (2.8) [46]. Even though instantons couple only to right-handed fermions and anti-instantons only to left-handed ones, it is not surprising that a chiral invariant quark interaction emerges because instanton and anti-instanton ensembles are independently averaged and in this process neither is favored over the other. Since the (anti)instantons interact with $2N_f$ fermions, it is more or less obvious that the spin flavor structure of (2.8) results. However, there is one small piece of information that can be gained from the instanton model: Since in that model the interaction is mediated by (the Fourier transformation of) the instanton, the inverse of the average instanton size ($\bar{\rho}$) provides a natural energy cut-off for this four-fermion interaction. Early studies of phenomenological applications of the instanton liquid model yielded $\bar{\rho} \approx \frac{1}{3}$ fm [51, 52] for this a priori free parameter. This result was reproduced within a variational approach to stabilize the (anti)instanton ensemble [53] utilizing the classical instanton anti-instanton interaction. This leads to an energy cut-off in the order of 600 MeV, a value consistent with the NJL model estimate in Sect. 2.3. Rather than from QCD, the instanton liquid estimate of the energy cut-off arises from fits to empirical data. Hence the agreement with NJL model result is not surprising.

Nevertheless there are some conceptual differences between the NJL model action and those instanton-induced interactions. We will briefly consider them for the case $N_f = 2$. We merely display the result and in doing so it is useful to introduce *non-local* fermion fields [54]

$$\psi(x) = \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} r(k) q(y), \quad (2.62)$$

where $r(k)$ is extracted from the Fourier transformation of the fermion zero-mode in the instanton background. The non-local transformation, (2.62) suppresses the high-frequency modes and thereby introduces the above-mentioned cut-off. In these non-local fields the instanton-induced (effective) potential seems actually local [46, 50],

$$V_{\text{inst.ind.}} = -g_1^2 \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 - (\bar{\psi}\tau\psi)^2 - (\bar{\psi}i\gamma_5\psi)^2 \right], \quad (2.63)$$

where the interaction strength, g_1^2 , is proportional to the inverse (anti)instanton density in the liquid. While $V_{\text{inst.ind.}}$ is invariant under $SU_L(2) \times SU_R(2)$ chiral transformations, it varies under $U_A(1)$. In the NJL model interaction, (2.8) this is still a (classical) symmetry and all spin-flavor structures in the G_1 term are attractive (as suggested by the gluon exchange approach [20]), while in (2.63) $(\bar{\psi}\tau\psi)$ and $(\bar{\psi}i\gamma_5\psi)$ are repulsive. Thus the instanton liquid model

would in particular suggest that isovector partners of the scalar isoscalar meson were not bound. This seems at variance with recent empirical studies that suggest the existence of even a nonet of scalar mesons [55].

There is some evidence from lattice measurements for the existence of an (anti)instanton ensemble in the vacuum [56, 57, 58]. An arbitrary gluon configuration on the lattice is dominated by the quantum noise of high-frequency modes. They can be eliminated in a smoothing procedure, so-called *cooling*, which leaves over isolated structures that may indeed be interpreted as an (anti)instanton ensemble. From the cooled configuration average instanton sizes of the order of 0.3 fm are estimated in agreement (and partial support) with the above-mentioned studies. However, it should be mentioned that the cooling procedure is not really converging because after many arbitrary iterations the instantons and anti-instantons will annihilate each other more or less completely. For example, the results displayed in Table 1 of [57] suggest that the number of instantons does not saturate as the number of cooling steps increases. Stated otherwise, the extracted (anti)instanton properties depend on the number of conducted cooling steps. As a way out, it has been proposed to extrapolate this functional behavior to zero cooling steps. This procedure does not seem very conclusive as Table 1 in [58] indicates.

2.7 Final Note on Chiral Quark Models

As a brief summary on this chapter about the quark flavor dynamics it seems fair to say that there many ways to motivate an NJL-type interaction as in (2.8), even from QCD. Most of the considerations contain approximations whose validity is difficult to judge. Probably any bona fide argumentation that respects chiral symmetry and stops one step before completely omitting any interaction will result in such a model. Above we have argued that the current–current approximation and the instanton liquid model do so, the field strength (re)formulation [59] of QCD is yet another example.

The NJL-type models focus on chiral symmetry and a dynamical description of its spontaneous breaking. However, these models lack the important feature of color confinement. This should not be forgotten even though it is not a serious drawback for most applications because they are not affected by unphysical quark anti-quark thresholds. This is particularly the case for static solitons that we will exhaustively discuss in the following chapter.

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