

Contents

Preface	xi
1 Introduction	
1.1 Brief history	2
1.1.1 Meissner effect – diamagnetism	3
1.1.2 The London equation and the penetration depth	4
1.1.3 The coherence length	6
1.1.4 Classification of superconductors	6
1.1.5 Vortices	7
1.1.6 Summary	9
1.2 The G-L phenomenological theory	10
1.2.1 The free energy and the G-L equations	10
1.2.2 Rescaling and the values of the constants	13
1.2.3 Gauge invariance	15
1.3 Some considerations arising from scaling	16
1.3.1 The two characteristic lengths $\xi(T)$ and $\lambda(T)$	16
1.3.2 The validity of the G-L theory	17
1.4 The evolutionary G-L system – 2-d case	21
1.4.1 The system	21
1.4.2 Mathematical scaling	21
1.4.3 The G-L functional as a Lyapunov functional	23
1.4.4 Gauge invariance	24
1.4.5 A uniform bound on $ \psi $	25
1.5 Exterior evolutionary Maxwell system	26
1.5.1 Review of the Maxwell system	27
1.5.2 The G-L superconductivity model	28
1.5.3 The setting of the problem	28
1.6 Exterior steady-state Maxwell system	31
1.7 Surface energy, superconductor classification	32
1.7.1 The sign of σ_{ns} when $\kappa \ll 1$	34
1.7.2 The sign of σ_{ns} when $\kappa \gg 1$	35

1.7.3	The case $\kappa = 1/\sqrt{2}$	35
1.7.4	Conclusion	36
1.8	Difference between 2-d and 3-d models	36
1.9	Bibliographical remarks	38
2	Mathematical Foundation	
2.1	Co-dimension one phase transition problems	40
2.1.1	Steady state problems	40
2.1.2	Evolutionary problems	42
2.1.3	Long time behaviour	44
2.2	Co-dimension two phase transition problems	46
2.2.1	Steady state problems on bounded domains	46
2.2.2	Steady state problems on \mathbb{R}^2	47
2.2.3	Evolutionary problems	48
2.2.4	Long time behaviour	49
2.3	Mathematical description of vortices in \mathbb{R}^2	49
2.4	Asymptotic methods for describing vortices in \mathbb{R}^2	53
2.4.1	Steady state case in \mathbb{R}^2 :	53
2.4.2	Evolutionary case in \mathbb{R}^2 – Introduction:	54
2.4.3	Evolutionary case in \mathbb{R}^2 – far field expansion:	56
2.4.4	Evolutionary case in \mathbb{R}^2 – local structure of the far field solution near a vortex	56
2.4.5	Evolutionary case in \mathbb{R}^2 – Core expansion	58
2.4.6	Evolutionary case in \mathbb{R}^2 – Matching of the core and far field expansions	58
2.4.7	Vortex motion equation	59
2.5	Asymptotic methods for describing vortices in \mathbb{R}^3	61
2.5.1	Steady state case in \mathbb{R}^3	61
2.5.2	Evolutionary case in \mathbb{R}^3	64
2.6	Bibliographical remarks	65
3	Asymptotics Involving Magnetic Potential	
3.1	Basic facts concerning fluid vortices	67
3.2	Asymptotic analysis	70
3.2.1	2-D steady state case	70
3.2.2	Evolutionary case	71
3.2.3	Far field	72
3.2.4	Core region	75
3.3	Asymptotic analysis of densely packed vortices	80
3.3.1	Outer region - a mean field model	81
3.3.2	Intermediate region	82
3.3.3	Core region	83
3.4	Bibliographical remarks	85

4 Steady State Solutions

4.1	Existence of steady state solutions	87
4.1.1	The outside field is a given function, 2-d case	87
4.1.2	The outside field is governed by the Maxwell system, 3-d case	90
4.2	Stability and mapping properties of solutions	91
4.2.1	Non-existence of local maxima	91
4.2.2	Boundedness of the order parameter	91
4.2.3	Constant solutions and mixed state solutions	92
4.3	Co-dimension two vortex domain	93
4.4	Breakdown of superconductivity	97
4.5	A linearized problem	104
4.6	Bibliographical remarks	115

5 Evolutionary Solutions

5.1	2-d solutions with given external field	118
5.1.1	Mathematical setting	119
5.1.2	Existence and uniqueness of solutions	123
5.1.3	Proof of Theorem 1.2	123
5.1.4	Proof of Theorem 1.1	129
5.2	Existence of 3-d evolutionary solutions	133
5.3	The existence of an ω -limit set as $t \rightarrow \infty$	141
5.4	An abstract theorem on global attractors	147
5.5	Global attractor for the G-L sstem	149
5.6	Physical bounds on the global attractor	152
5.7	The uniqueness of the long time limit of the evolutionary G-L solutions	158
5.8	Bibliographical remarks	159

6 Complex G-L Type Phase Transition Theory

6.1	Existence and basic properties of solutions	162
6.2	BBH type upper bound for energy of minimizers	164
6.3	Global estimates	166
6.4	Local estimates	171
6.5	The behaviour of solutions near vortices	174
6.6	Global ε -independent estimates	185
6.7	Convergence of the solutions as $\varepsilon \rightarrow 0$	192
6.8	Main results on the limit functions	196
6.9	Renormalized energies	205
6.10	Bibliographical remarks	219

7 The Slow Motion of Vortices

7.1	Introduction	221
7.2	Preliminaries	224
7.3	Estimates from below for the mobilities	234
7.4	Estimates from above for the mobilities	244
7.5	Bibliographical remarks	250

8 Thin Plate/Film G-L Models

8.1	The outside Maxwell system – steady state case	252
8.1.1	The energy bound	253
8.1.2	Convergence properties of the rescaled variables	255
8.1.3	Passing to the limit	257
8.2	The outside field is given – evolutionary case	259
8.2.1	Existence and uniqueness of solutions	259
8.2.2	The limit when $\varepsilon \rightarrow 0$	265
8.2.3	Some estimates	268
8.2.4	The convergence	273
8.3	The outside field is given – formal analysis	278
8.3.1	Variational formulation	279
8.3.2	Formal asymptotic analysis when $\varepsilon \rightarrow 0$	279
8.4	Bibliographical remarks	281

9 Pinning Theory

9.1	Local Pohozaev-type identity	284
9.2	Estimate the energy of minimizers	288
9.3	Local estimates	290
9.4	Global Estimates	294
9.5	Convergence of solutions and the term $\frac{1}{\varepsilon^2} \int_{\Omega} (\psi_{\varepsilon} ^2 - 1)^2$	294
9.6	Properties of (ψ_*, \mathbf{A}_*)	301
9.7	Renormalized energy	305
9.8	Pinning of vortices in other circumstances	321
9.8.1	G-L model subject to thermo-perturbation or large horizontal field	321
9.8.2	An anisotropic G-L model	323
9.8.3	A thin film G-L model	324
9.9	Bibliographical remarks	325

10 Numerical Analysis

10.1	Introduction	327
10.2	Discretization	331
10.2.1	Weak formulation	331
10.2.2	Discretization	332
10.3	Stability estimates	335
10.4	Error estimates	339

10.5	A numerical example	350
10.6	Discretization using variable step length	351
10.7	A dual problem	353
10.7.1	Stability estimates	354
10.7.2	Error representation formula	359
10.8	A posteriori error analysis	361
10.8.1	Residuals	362
10.8.2	Proof of Theorem 4.1	369
10.9	Numerical implementation	370
10.9.1	Comparison of the schemes	370
10.10	Bibliographical remarks	374
References		375
Index		384