Contents

1. Introduction 1

1.1 What Is Chance, and Why Study It? 3
1.1.1 Chance vs Determinism 3
1.1.2 Probability Problems in Optics 4
1.1.3 Statistical Problems in Optics 5

2. The Axiomatic Approach 7

2.1 Notion of an Experiment; Events 7
2.1.1 Event Space; The Space Event 8
2.1.2 Disjoint Events 9
2.1.3 The Certain Event 9
Exercise 2.1 9
2.2 Definition of Probability 9
2.3 Relation to Frequency of Occurence 10
2.4 Some Elementary Consequences 10
2.4.1 Additivity Property 11
2.4.2 Normalization Property 11
2.5 Marginal Probability 12
2.6 The "Traditional" Definition of Probability 12
2.7 Illustrative Problem: A Dice Game 13
2.8 Illustrative Problem: Let's (try to) Take a Trip 14
2.9 Law of Large Numbers 15
2.10 Optical Objects and Images as Probability Laws 16
2.11 Conditional Probability 17
Exercise 2.2 18
2.12 The Quantity of Information 19
2.13 Statistical Independence 21
2.13.1 Illustrative Problem: Let's (try to) Take a Trip (Continued) 22
2.14 Informationless Messages 23
2.15 A Definition of Noise 23
2.16 "Additivity" Property of Information 24
2.17 Partition Law 25
2.18 Illustrative Problem: Transmittance Through a Film 25
2.19 How to Correct a Success Rate for Guesses 26
Exercise 2.3 27
2.20 Bayes' Rule 27
2.21 Some Optical Applications 28

2.22 Information Theory Application 30
2.23 Application to Markov Events 30
2.24 Complex Number Events 32
Exercise 2.4 32

3. Continuous Random Variables 37

3.1 Definition of Random Variable 37
3.2 Probability Density Function, Basic Properties 37
3.3 Information Theory Application: Continuous Limit 39
3.4 Optical Application: Continuous Form of Imaging Law . . 40
3.5 Expected Values, Moments 40
3.6 Optical Application: Moments of the Slit Diffraction Pattern 41
3.7 Information Theory Application 43
3.8 Case of Statistical Independence 44
3.9 Mean of a Sum 44
3.10 Optical Application 45
3.11 Deterministic Limit; Representations of the Dirac δ-Function 46
3.12 Correspondence Between Discrete and Continuous Cases . 47
3.13 Cumulative Probability 48
3.14 The Means of an Algebraic Expression: A Simplified Approach 48
3.15 A Potpourri of Probability Laws 50
3.15.1 Poisson 50
3.15.2 Binomial 51
3.15.3 Uniform 51
3.15.4 Exponential 52
3.15.5 Normal (One-Dimensional) 53
3.15.6 Normal (Two-Dimensional) 53
3.15.7 Normal (Multi-Dimensional) 55
3.15.8 Skewed Gaussian Case; Gram-Charlier Expansion . 56
3.15.9 Optical Application 57
3.15.10 Geometric Law 58
3.15.11 Cauchy Law 59
3.15.12 Sinc^2 Law 59
Exercise 3.1 60

4. Fourier Methods in Probability 70

4.1 Characteristic Function Defined 70
4.2 Use in Generating Moments 71
4.3 An Alternative to Describing RV x 71
4.4 On Optical Applications 71
4.5 Shift Theorem 72
4.6 Poisson Case 72
4.7 Binomial Case 73

4.8 Uniform Case 74
4.9 Exponential Case 74
4.10 Normal Case (One Dimension) 74
4.11 Multidimensional Cases 74
4.12 Normal Case (Two Dimensions) 75
4.13 Convolution Theorem, Transfer Theorem 75
4.14 Probability Law for the Sum of Two Independent RV's . . . 75
4.15 Optical Applications 77
4.15.1 Imaging Equation as the Sum of Two Random Displacements 77
4.15.2 Unsharp Masking 77
4.16 Sum of n Independent RV's; the "Random Walk" Phenomenon 79
Exercise 4.1 79
4.17 Resulting Mean and Variance: Normal, Poisson, and General Cases 81
4.18 Sum of n Dependent RV's 81
4.19 Case of Two Gaussian Bivariate RV's 82
4.20 Sampling Theorems for Probability 83
4.21 Case of Limited Range of x, Derivation 83
4.22 Discussion 84
4.23 Optical Application 85
4.24 Case of Limited Range of ω 86
4.25 Central Limit Theorem 87
4.26 Derivation 87
Exercise 4.2 89
4.27 How Large Does n Have to be? 89
4.28 Optical Applications 90
4.28.1 Cascaded Optical Systems 90
4.28.2 Laser Resonator 91
4.28.3 Atmospheric Turbulence 92
4.29 Generating Normally Distributed Numbers from Uniformly Random Numbers 93
4.30 The Error Function 95
Exercise 4.3 96

5. Functions of Random Variables 99

5.1 Case of a Single Random Variable 99
5.2 Unique Root 100
5.3 Application from Geometrical Optics 101
5.4 Multiple Roots 103
5.5 Illustrative Example 103
5.6 Case of n Random Variables, r Roots 104
5.7 Optical Applications 105
5.8 Statistical Modeling 105

5.9 Application of Transformation Theory to Laser Speckle . . 105
5.9.1 Physical Layout 106
5.9.2 Plan 106
5.9.3 Statistical Model 107
5.9.4 Marginal Probabilities for Light Amplitudes U_{re}, U_{im} 108
5.9.5 Correlation Between U_{re} and U_{im} 109
5.9.6 Joint Probability Law for U_{re}, U_{im} 110
5.9.7 Probability Laws for Intensity and Phase; Transformation of the RV's 110
5.9.8 Marginal Law for Intensity and Phase 111
5.9.9 Signal-to-Noise (S/N) Ratio in the Speckle Image . 111
5.10 Speckle Reduction by Use of a Scanning Aperture 112
5.10.1 Statistical Model 113
5.10.2 Probability Density for Output Intensity $p_I(v)$ . . . 113
5.10.3 Moments and S/N Ratio 115
5.10.4 Standard Form for the Chi-Square Distribution . . 116
5.11 Calculation of Spot Intensity Profiles Using Transformation Theory 117
5.11.1 Illustrative Example 118
5.11.2 Implementation by Ray-Trace 120
5.12 Application of Transformation Theory to a Satellite-Ground Communication Problem 120
Exercise 5.1 124

6. Bernoulli Trials and its Limiting Cases 134

6.1 Analysis 134
6.2 Illustrative Problems 136
6.2.1 Illustrative Problem: Let's (try to) Take a Trip: The Last Word 136
6.2.2 Illustrative Problem: Mental Telepathy as a Communication Link? 137
6.3 Characteristic Function and Moments 139
6.4 Optical Application: Checkerboard Model of Granularity . . 139
6.5 The Poisson Limit 142
6.5.1 Analysis 142
6.5.2 Example of Degree of Approximation 143
6.6 Optical Application: The Shot Effect 144
6.7 Optical Application: Combined Sources 145
6.8 Poisson Joint Count for Two Detectors - Intensity Interferometry 146
6.9 The Normal Limit (DeMoivre-Laplace Law) 150
6.9.1 Derivation 150
6.9.2 Conditions of Use 151
6.9.3 Use of the Error Function 152
Exercise 6.1 154

7. The Monte Carlo Calculation 162

7.1 Producing Random Numbers that Obey a Prescribed Probability Law 163
7.1.1 Illustrative Case 164
7.1.2 Normal Case 165
7.2 Analysis of the Photographic Emulsion by Monte Carlo Calculation 165
7.3 Application of the Monte Carlo Calculation to Remote Sensing 167
7.4 Monte Carlo Formation of Optical Images 169
7.4.1 An Example 170
7.5 Monte Carlo Simulation of Speckle Patterns 171
Exercise 7.1 172

8. Stochastic Processes 177

8.1 Definition of Stochastic Process 177
8.2 Definition of Power Spectrum 178
8.2.1 Some Examples of Power Spectra 180
8.3 Definition of Autocorrelation Function; Kinds of Stationarity 181
8.4 Fourier Transform Theorem 182
8.5 Case of a "White" Power Spectrum 182
8.6 Application: Average Transfer Function Through Atmospheric Turbulence 183
8.6.1 Statistical Model for Phase Fluctuations 185
8.6.2 A Transfer Function for Turbulence 186
8.7 Transfer Theorems for Power Spectra 188
8.7.1 Determining the MTF Using Random Objects . . 188
8.7.2 Speckle Interferometry of Labeyrie 190
8.7.3 Resolution Limits of Speckle Interferometry . . . 192
Exercise 8.1 194
8.8 Transfer Theorem for Autocorrelation: The Knox-Thompson Method. 197
8.9 Additive Noise 199
8.10 Random Noise 201
8.11 Ergodic Property 202
Exercise 8.2 203
8.12 Optimum Restoring Filter 206
8.12.1 Definition of Restoring Filter 206
8.12.2 Model. 207
8.12.3 Solution 208
Exercise 8.3 210
8.13 Information Content in the Optical Image 211
8.13.1 Statistical Model 212
8.13.2 Analysis 212

8.13.3 Noise Entropy 213
8.13.4 Data Entropy 214
8.13.5 The Answer 215
8.13.6 Interpretation 215
8.14 Data Information and Its Ability to be Restored 217
8.15 Superposition Processes; the Shot Noise Process 218
8.15.1 Probability Law for i 219
8.15.2 Some Important Averages 219
8.15.3 Mean Value $\langle i(x_0) \rangle$ 221
8.15.4 Shot Noise Case 221
8.15.5 Second Moment $\langle i^2(x_0) \rangle$ 222
8.15.6 Variance $\sigma^2(x_0)$ 222
8.15.7 Shot Noise Case 223
8.15.8 Signal-to-Noise (S/N) Ratio 223
Exercise 8.4 224
8.15.9 Autocorrelation Function 225
8.15.10 Shot Noise Case 226
8.15.11 Application: An Overlapping Circular Grain Model for the Emulsion 226
8.15.12 Application: Light Fluctuations due to Randomly Tilted Waves, the "Swimming pool" Effect 228
Exercise 8.5 231

9. Introduction to Statistical Methods: Estimating the Mean, Median, Variance, S/N, and Simple Probability 233

9.1 Estimating a Mean from a Finite Sample 234
9.2 Statistical Model 234
9.3 Analysis 235
9.4 Discussion 236
9.5 Error in a Discrete, Linear Processor: Why Linear Methods Often Fail 237
9.6 Estimating a Probability: Derivation of the Law of Large Numbers 239
9.7 Variance of Error 240
9.8 Illustrative Uses of Error Expression 241
9.8.1 Estimating Probabilities from Empirical Rates . . . 241
9.8.2 Aperture Size for Required Accuracy in Transmittance Readings 242
9.9 Probability Law for the Estimated Probability; Confidence Limits 243
9.10 Calculation of the Sample Variance 245
9.10.1 Unbiased Estimate of the Variance 245
9.10.2 Expected Error in the Sample Variance 247
9.10.3 Illustrative Problems 249

9.11 Estimating the Signal-to-Noise Ratio; Student's Probability Law 250
9.11.1 Probability Law for SNR 251
9.11.2 Moments of SNR 252
9.11.3 Limit $c \to 0$; a Student Probability Law 254
9.12 Properties of a Median Window 254
9.13 Statistics of the Median 257
9.13.1 Probability Law for the Median 257
9.13.2 Laser Speckle Case: Exponential Probability Law . 258
Exercise 9.1 260

10. Estimating a Probability Law 264

10.1 Estimating Probability Densities Using Orthogonal Expansions 265
10.2 Karhunen-Loeve Expansion 268
10.3 The Multinomial Probability Law 269
10.3.1 Derivation 269
10.3.2 Illustrative Example 270
10.4 Estimating a Probability Law Using Maximum Likelihood 270
10.4.1 Principle of Maximum Likelihood 271
10.4.2 Maximum Entropy Estimate 272
10.4.3 The Search for "Maximum Prior Ignorance" . . . 273
10.4.4 Other Types of Estimates (Summary) 275
10.4.5 Return to Maximum Entropy Estimation, Discrete Case 275
10.4.6 Transition to a Continuous Random Variable . . . 276
10.4.7 Solution 277
10.4.8 Maximized H 278
10.4.9 Illustrative Example: Significance of the Normal Law 278
10.4.10 The Smoothness Property; Least Biased Aspect . . 279
10.4.11 A Well Known Distribution Derived 279
10.4.12 When Does the Maximum Entropy Estimate Equal the True Law? 281
10.4.13 Maximum Likelihood Estimation of Optical Objects 282
10.4.14 Case of Nearly Featureless Objects 284
Exercise 10.1 286

11. The Chi-Square Test of Significance 294

11.1 Forming the χ^2 Statistic 295
11.2 Probability Law for χ^2 Statistic 297
11.3 When is a Coin Fixed? 299
11.4 Equivalence of Chi-Square to Other Statistics; Sufficient Statistics 299
11.5 When is a Vote Decisive? 301

11.6 Generalization to N Voters 302
11.7 Use as an Image Detector 303
Exercise 11.1 305

12. The Student t-Test on the Mean 307

12.1 Cases Where Data Accuracy is Unknown 308
12.2 Philosophy of the Approach: Statistical Inference 308
12.3 Forming the Statistic 310
12.4 Student's t-Distribution; Derivation 311
12.5 Some Properties of Student's t-Distribution 312
12.6 Application to the Problem; Student's t-Test 314
12.7 Illustrative Example 314
12.8 Other Applications 316
Exercise 12.1 317

13. The F-Test on Variance 320

13.1 Snedecor's F-Distribution; Derivation 320
13.2 Some Properties of Snedecor's F-Distribution 321
13.3 The F-Test 322
13.4 Illustrative Example 323
13.5 Application to Image Detection 324
Exercise 13.1 325

14. Least-Squares Curve Fitting – Regression Analysis 327

14.1 Summation Model for the Physical Effect 327
14.2 Linear Regression Model for the Noise 330
14.3 Equivalence of ML and Least-Squares Solutions 331
14.4 Solution 333
14.5 Return to Film Problem 333
14.6 "Significant" Factors; the R-statistic 334
14.7 Example: Was T^2 an Insignificant Factor? 336
14.8 Accuracy of the Estimated Coefficients 337
14.8.1 Absorptance of an Optical Fiber 337
14.8.2 Variance of Error in the General Case 339
14.8.3 Error in the Estimated Absorptance of an Optical Fiber 341
Exercise 14.1 342

15. Principal Components Analysis 350

15.1 A Photographic Problem 350
15.2 Equivalent Eigenvalue Problem 351

15.3 The Eigenvalues as Sample Variances 353
15.4 The Data in Terms of Principal Components 353
15.5 Reduction in Data Dimensionality 354
15.6 Return to the H–D Problem 355
15.7 Application to Multispectral Imagery 356
15.8 Error Analysis 358
Exercise 15.1 361

16. The Controversy Between Bayesians and Classicists 363

16.1 Bayesian Approach to Confidence Limits for an Estimated Probability 364
16.1.1 Probability Law for the Unknown Probability 365
16.1.2 Assumption of a Uniform Prior 366
16.1.3 Irrelevance of Choice of Prior Statistic $p_0(x)$ if N is Large 367
16.1.4 Limiting Form for N Large 367
16.1.5 Illustrative Problem 368
16.2 Laplace's Rule of Succession 368
16.2.1 Derivation 369
Exercise 16.1 371
16.2.2 Role of the Prior 372
Exercise 16.2 372

Appendix A. Error Function and its Derivative [4.12] 374
Appendix B. Percentage Points of the χ^2-Distribution; Values of χ^2 in Terms of α and ν 376
Appendix C. Percentage Points of the t-Distribution; Values of t in Terms of A and ν 378
Appendix D. Percentage Points of the F-Distribution; Values of F in Terms of Q, ν_1, ν_2 379
Appendix E. A Crib Sheet of Statistical Parameters and their Errors 382
Appendix F. Synopsis of Statistical Tests 384

References 387

Subject Index 393