

## Table of Contents

<b>Chapter 1 General Discretization Methods . . . . .</b>	<b>1</b>
<b>1.1. Basic Definitions . . . . .</b>	<b>1</b>
1.1.1 Discretization Methods . . . . .	1
1.1.2 Consistency . . . . .	5
1.1.3 Convergence . . . . .	7
1.1.4 Stability . . . . .	9
<b>1.2 Results Concerning Stability . . . . .</b>	<b>11</b>
1.2.1 Existence of the Solution of the Discretization . . . . .	11
1.2.2 The Basic Convergence Theorem . . . . .	13
1.2.3 Linearization . . . . .	15
1.2.4 Stability of Neighboring Discretizations . . . . .	18
<b>1.3 Asymptotic Expansions of the Discretization Errors . . . . .</b>	<b>21</b>
1.3.1 Asymptotic Expansion of the Local Discretization Error . . . . .	21
1.3.2 Asymptotic Expansion of the Global Discretization Error . . . . .	25
1.3.3 Asymptotic Expansions in Even Powers of $n$ . . . . .	29
1.3.4 The Principal Error Terms . . . . .	30
<b>1.4 Applications of Asymptotic Expansions . . . . .</b>	<b>33</b>
1.4.1 Richardson Extrapolation . . . . .	33
1.4.2 Linear Extrapolation . . . . .	37
1.4.3 Rational Extrapolation . . . . .	42
1.4.4 Difference Correction . . . . .	44
<b>1.5 Error Analysis . . . . .</b>	<b>49</b>
1.5.1 Computing Error . . . . .	49
1.5.2 Error Estimates . . . . .	51
1.5.3 Strong Stability . . . . .	54
1.5.4 Richardson-extrapolation and Error Estimation . . . . .	56
1.5.5 Statistical Analysis of Round-off Errors . . . . .	58
<b>1.6 Practical Aspects . . . . .</b>	<b>61</b>

<b>Chapter 2 Forward Step Methods</b>	63
2.1 Preliminaries . . . . .	63
2.1.1 Initial Value Problems for Ordinary Differential Equations . . . . .	63
2.1.2 Grids . . . . .	65
2.1.3 Characterization of Forward Step Methods . . . . .	67
2.1.4 Restricting the Interval . . . . .	69
2.1.5 Notation . . . . .	72
2.2 The Meaning of Consistency, Convergence, and Stability with Forward Step Methods . . . . .	74
2.2.1 Our Choice of Norms in $E_n$ and $E_n^0$ . . . . .	74
2.2.2 Other Definitions of Consistency and Convergence . . . . .	76
2.2.3 Other Definitions of Stability . . . . .	79
2.2.4 Spijker's Norm for $E_n^0$ . . . . .	81
2.2.5 Stability of Neighboring Discretizations . . . . .	84
2.3 Strong Stability of f.s.m. . . . .	87
2.3.1 Perturbation of IVP 1 . . . . .	87
2.3.2 Discretizations of {IVP 1} . . . . .	92
2.3.3 Exponential Stability for Difference Equations on $[0, \infty)$ . . . . .	94
2.3.4 Exponential Stability of Neighboring Discretizations . . . . .	98
2.3.5 Strong Exponential Stability . . . . .	101
2.3.6 Stability Regions . . . . .	103
2.3.7 Stiff Systems of Differential Equations . . . . .	104
<b>Chapter 3 Runge-Kutta Methods</b>	107
3.1 RK-procedures . . . . .	107
3.1.1 Characterization . . . . .	107
3.1.2 Local Solution and Increment Function . . . . .	110
3.1.3 Elementary Differentials . . . . .	111
3.1.4 The Expansion of the Local Solution . . . . .	115
3.1.5 The Exact Increment Function . . . . .	117
3.2 The Group of RK-schemes . . . . .	120
3.2.1 RK-schemes . . . . .	120
3.2.2 Inverses of RK-schemes . . . . .	125
3.2.3 Equivalent Generating Matrices . . . . .	127
3.2.4 Explicit and Implicit RK-schemes . . . . .	131
3.2.5 Symmetric RK-procedures . . . . .	134
3.3 RK-methods and Their Orders . . . . .	135
3.3.1 RK-methods . . . . .	135
3.3.2 The Order of Consistency . . . . .	137

3.3.3 Construction of High-order RK-procedures . . . . .	141
3.3.4 Attainable Order of $m$ -stage RK-procedures . . . . .	144
3.3.5 Effective Order of RK-schemes . . . . .	148
 3.4 Analysis of the Discretization Error . . . . .	150
3.4.1 The Principal Error Function . . . . .	150
3.4.2 Asymptotic Expansion of the Discretization Error . . . . .	153
3.4.3 The Principal Term of the Global Discretization Error . . . . .	157
3.4.4 Estimation of the Local Discretization Error . . . . .	161
 3.5 Strong Stability of RK-methods . . . . .	165
3.5.1 Strong Stability for Sufficiently Large $n$ . . . . .	165
3.5.2 Strong Stability for Arbitrary $n$ . . . . .	170
3.5.3 Stability Regions of RK-methods . . . . .	174
3.5.4 Use of Stability Regions for General {IVP 1}, . . . . .	178
3.5.5 Suggestion for a General Approach . . . . .	182
 <b>Chapter 4 Linear Multistep Methods</b> . . . . .	185
 4.1 Linear $k$ -step Schemes . . . . .	185
4.1.1 Characterization . . . . .	185
4.1.2 The Order of Linear $k$ -step Schemes . . . . .	191
4.1.3 Construction of Linear $k$ -step Schemes of High Order . . . . .	195
 4.2 Uniform Linear $k$ -step Methods . . . . .	199
4.2.1 Characterization, Consistency . . . . .	199
4.2.2 Auxiliary Results . . . . .	203
4.2.3 Stability of Uniform Linear $k$ -step Methods . . . . .	206
4.2.4 Convergence . . . . .	210
4.2.5 Highest Obtainable Orders of Convergence . . . . .	214
 4.3 Cyclic Linear $k$ -step Methods . . . . .	216
4.3.1 Stability of Cyclic Linear $k$ -step Methods . . . . .	216
4.3.2 The Auxiliary Method . . . . .	221
4.3.3 Attainable Order of Cyclic Linear Multistep Methods . . . . .	225
 4.4 Asymptotic Expansions . . . . .	228
4.4.1 The Local Discretization Error . . . . .	228
4.4.2 Asymptotic Expansion of the Global Discretization Error, Preparations . . . . .	232
4.4.3 The Case of No Extraneous Essential Zeros . . . . .	234
4.4.4 The Case of Extraneous Essential Zeros . . . . .	240
 4.5 Further Analysis of the Discretization Error . . . . .	245
4.5.1 Weak Stability . . . . .	245
4.5.2 Smoothing . . . . .	248

4.5.3 Symmetric Linear $k$ -step Schemes . . . . .	250
4.5.4 Asymptotic Expansions in Powers of $h^2$ . . . . .	256
4.5.5 Estimation of the Discretization Error . . . . .	260
4.6 Strong Stability of Linear Multistep Methods . . . . .	263
4.6.1 Strong Stability for Sufficiently Large $n$ . . . . .	263
4.6.2 Stability Regions of Linear Multistep Methods . . . . .	266
4.6.3 Strong Stability for Arbitrary $n$ . . . . .	270
<b>Chapter 5 Multistage Multistep Methods</b> . . . . .	272
5.1 General Analysis . . . . .	272
5.1.1 A General Class of Multistage Multistep Procedures . . . . .	272
5.1.2 Simple $m$ -stage $k$ -step Methods . . . . .	275
5.1.3 Stability and Convergence of Simple $m$ -stage $k$ -step Methods . . . . .	278
5.2 Predictor-corrector Methods . . . . .	282
5.2.1 Characterization, Subclasses . . . . .	282
5.2.2 Stability and Order of Predictor-corrector Methods . . . . .	284
5.2.3 Analysis of the Discretization Error . . . . .	290
5.2.4 Estimation of the Local Discretization Error . . . . .	294
5.2.5 Estimation of the Global Discretization Error . . . . .	297
5.3 Predictor-corrector Methods with Off-step Points . . . . .	300
5.3.1 Characterization . . . . .	300
5.3.2 Determination of the Coefficients and Attainable Order . . . . .	302
5.3.3 Stability of High Order PC-methods with Off-step Points . . . . .	306
5.4 Cyclic Forward Step Methods . . . . .	308
5.4.1 Characterization . . . . .	308
5.4.2 Stability and Error Propagation . . . . .	311
5.4.3 Primitive $m$ -cyclic $k$ -step Methods . . . . .	316
5.4.4 General Straight $m$ -cyclic $k$ -step Methods . . . . .	322
5.5 Strong Stability . . . . .	324
5.5.1 Characteristic Polynomial, Stability Regions . . . . .	324
5.5.2 Stability Regions of PC-methods . . . . .	326
5.5.3 Stability Regions of Cyclic Methods . . . . .	329
<b>Chapter 6 Other Discretization Methods for IVP 1</b> . . . . .	332
6.1 Discretization Methods with Derivatives of $f$ . . . . .	332
6.1.1 Recursive Computation of Higher Derivatives of the Local Solution . . . . .	332
6.1.2 Power Series Methods . . . . .	335

6.1.3 The Perturbation Theory of Groebner-Knapp-Wanner . . . . .	336
6.1.4 Groebner-Knapp-Wanner Methods . . . . .	339
6.1.5 Runge-Kutta-Fehlberg Methods . . . . .	343
6.1.6 Multistep Methods with Higher Derivatives . . . . .	347
<b>6.2 General Multi-value Methods . . . . .</b>	<b>349</b>
6.2.1 Nordsieck's Approach . . . . .	349
6.2.2 Nordsieck Predictor-corrector Methods . . . . .	354
6.2.3 Equivalence of Generalized Nordsieck Methods . . . . .	357
6.2.4 Appraisal of Nordsieck Methods . . . . .	361
<b>6.3 Extrapolation Methods . . . . .</b>	<b>362</b>
6.3.1 The Structure of an Extrapolation Method . . . . .	362
6.3.2 Gragg's Method . . . . .	364
6.3.3 Strong Stability of $\mathfrak{M}_G$ . . . . .	369
6.3.4 The Gragg-Bulirsch-Stoer Extrapolation Method . . . . .	372
6.3.5 Extrapolation Methods for Stiff Systems . . . . .	375
<b>Bibliography . . . . .</b>	<b>380</b>
<b>Subject Index . . . . .</b>	<b>385</b>