

Contents

Preface 7

0. Introduction 9

 0.1 Background 9

 0.2 Difference methods – Finite element methods 10

 0.3 Scope of the book 12

1. Introduction to FEM for elliptic problems 14

 1.1 Variational formulation of a one-dimensional model problem 14

 1.2 FEM for the model problem with piecewise linear functions 18

 1.3 An error estimate for FEM for the model problem 23

 1.4 FEM for the Poisson equation 26

 1.5 The Hilbert spaces $L_2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ 33

 1.6 A geometric interpretation of FEM 38

 1.7 A Neumann problem. Natural and essential boundary conditions 40

 1.8 Remarks on programming 43

 1.9 Remarks on finite element software 48

2. Abstract formulation of the finite element method for elliptic problems 50

 2.1 Introduction. The continuous problem 50

 2.2 Discretization. An error estimate 52

 2.3 The energy norm 55

 2.4 Some examples 55

3. Some finite element spaces 67

 3.1 Introduction. Regularity requirements 67

 3.2 Some examples of finite elements 68

 3.3 Summary 79

4. Approximation theory for FEM. Error estimates for elliptic problems 84

 4.1 Introduction 84

 4.2 Interpolation with piecewise linear functions in two dimensions 84

4.3	Interpolation with polynomials of higher degree	90
4.4	Error estimates for FEM for elliptic problems	91
4.5	On the regularity of the exact solution	92
4.6	Adaptive methods	94
4.7	An error estimate in the $L_2(\Omega)$ -nom	97

5. Some applications to elliptic problems 101

5.1	The elasticity problem	101
5.2	Stokes problem	106
5.3	A plate problem	108

6. Direct methods for solving linear systems of equations 112

6.1	Introduction	112
6.2	Gaussian elimination. Cholesky's method	112
6.3	Operation counts. Band matrices	114
6.4	Fill-in	116
6.5	The frontal method	117
6.6	Nested dissection	120

7. Minimization algorithms. Iterative methods 123

7.1	Introduction	123
7.2	The gradient method	128
7.3	The conjugate gradient method	131
7.4	Preconditioning	136
7.5	Multigrid methods	137
7.6	Work estimates for direct and iterative methods	139
7.7	The condition number of the stiffness matrix	141

8. FEM for parabolic problems 146

8.1	Introduction	146
8.2	A one-dimensional model problem	147
8.3	Semi-discretization in space	149
8.4	Discretization in space and time	152
8.4.1	Background	152
8.4.2	The backward Euler and Crank-Nicolson methods	153
8.4.3	The discontinuous Galerkin method	157
8.4.4	Error estimates for fully discrete approximations and automatic time and space step control	158

9. Hyperbolic problems	167
9.1 Introduction	167
9.2 A convection-diffusion problem	168
9.3 General remarks on numerical methods for hyperbolic equations	171
9.4 Outline and preliminaries	173
9.5 Standard Galerkin	176
9.6 Classical artificial diffusion	181
9.7 The streamline diffusion method	181
9.7.1 The streamline diffusion method with $\epsilon=0$	182
9.7.2 The streamline diffusion method with $\epsilon>0$	185
9.8 The discontinuous Galerkin method	189
9.9 The streamline diffusion method for time-dependent convection-diffusion problems	199
9.10 Friedrichs' systems	205
9.10.1 The continuous problem	205
9.10.2 The standard Galerkin method	207
9.10.3 The streamline diffusion method	207
9.10.4 The discontinuous Galerkin method	207
9.11 Second order hyperbolic problems	210
10. Boundary element methods	214
10.1 Introduction	214
10.2 Some integral equations	216
10.2.1 An integral equation for an exterior Dirichlet problem using a single layer potential	219
10.2.2 An exterior Dirichlet problem with double layer potential	220
10.2.3 An exterior Neumann problem with single layer potential	222
10.2.4 Alternative integral equation formulations	223
10.3 Finite element methods	224
10.3.1 FEM for a Fredholm equation of the first kind	224
10.3.2 FEM for a Fredholm equation of the second kind	227
11. Mixed finite element methods	232
11.1 Introduction	232
11.2 Some examples	234
12. Curved elements and numerical integration	239
12.1 Curved elements	239
12.2 Numerical integration (quadrature)	245

13. Some non-linear problems	248
13.1 Introduction	248
13.2 Convex minimization problems	248
13.2.1 The continuous problem	248
13.2.2 Discretizations	254
13.2.3 Numerical methods for convex minimization problems	255
13.3 A non-linear parabolic problem	257
13.4 The incompressible Euler equations	258
13.4.1 The continuous problem	258
13.4.2 The streamline diffusion method in (ω, ψ) -formulation	259
13.4.3 The discontinuous Galerkin method in (ω, ψ) -formulation	260
13.4.4 The streamline diffusion method in (u, p) -formulation	261
13.5 The incompressible Navier-Stokes equations	262
13.6 Compressible flow: Burgers' equation	263

References 269

Index 275