Table of Contents

Chapter I. Mathematical Foundation of the Stokes Problem 1

§ 1. Generalities on Some Elliptic Boundary Value Problems 1
1.1. Basic Concepts on Sobolev Spaces 1
1.2. Abstract Elliptic Theory 10
1.3. Example 1: Dirichlet's Problem for the Laplace Operator 11
1.4. Example 2: Neumann's Problem for the Laplace Operator 13
1.5. Example 3: Dirichlet's Problem for the Biharmonic Operator 15

§ 2. Function Spaces for the Stokes Problem 18
2.1. Preliminary Results 18
2.2. Some Properties of Spaces Related to the Divergence Operator 22
2.3. Some Properties of Spaces Related to the Curl Operator 30

§ 3. A Decomposition of Vector Fields 36
3.1. Decomposition of Two-Dimensional Vector Fields 37
3.2. Application to the Regularity of Functions of $H(\mathrm{div};\Omega)\cap H(\mathbf{curl};\Omega)$ 43
3.3. Decomposition of Three-Dimensional Vector Fields 45
3.4. The Imbedding of $H(\mathrm{div};\Omega)\cap H_0(\mathbf{curl};\Omega)$ into $H^1(\Omega)^3$ 51
3.5. The Imbedding of $H_0(\mathrm{div};\Omega)\cap H(\mathbf{curl};\Omega)$ into $H^1(\Omega)^3$ 53

§ 4. Analysis of an Abstract Variational Problem 56
4.1. A General Result 57
4.2. A Saddle-Point Approach 61
4.3. Approximation by Regularization or Penalty 65
4.4. Iterative Methods of Gradient Type 68

§ 5. The Stokes Equations 78
5.1. The Dirichlet Problem in the Velocity-Pressure Formulation 80
5.2. The Stream Function Formulation of the Dirichlet Problem in Two Dimensions 88
5.3. The Three-Dimensional Case 90

Appendix A. Results of Standard Finite Element Approximation 95

A.1. Triangular Finite Elements 95
A.2. Quadrilateral Finite Elements 104
A.3. Interpolation of Discontinuous Functions 109

Chapter II. Numerical Solution of the Stokes Problem in the Primitive Variables 112

§ 1. General Approximation 112
1.1. An Abstract Approximation Result 112
1.2. Decoupling the Computation of u_h and λ_h 120
1.3. Application to the Homogeneous Stokes Problem 123
1.4. Checking the inf-sup Condition 129

§ 2. Simplicial Finite Element Methods Using Discontinuous Pressures 132
2.1. A First Order Approximation on Triangular Elements 133
2.2. Higher-Order Approximation on Triangular Elements 139
2.3. The Three-Dimensional case: First and Higher-Order Schemes 144

§ 3. Quadrilateral Finite Element Methods Using Discontinuous Pressures 152
3.1. A quadrilateral Finite Element of Order One 152
3.2. Higher-Order Quadrilateral Elements 156
3.3. An Example of Checkerboard Instability: the $Q_1 - P_0$ Element 160
3.4. Error Estimates for the $Q_1 - P_0$ Element 170

§ 4. Continuous Approximation of the Pressure 173
4.1. A First Order Method: the "Mini" Finite Element 174
4.2. The "Hood-Taylor" Finite Element Method 176
4.3. The "Glowinski-Pironneau" Finite Element Method 183
4.4. Implementation of the Glowinski-Pironneau Scheme 190

Chapter III. Incompressible Mixed Finite Element Methods for Solving the Stokes Problem 193

§ 1. Mixed Approximation of an Abstract Problem 193
1.1. A Mixed Variational Problem 193
1.2. Abstract Mixed Approximation 196

§ 2. The "Stream Function-Vorticity-Pressure" Method for the Stokes Problem in Two Dimensions 198
2.1. A Mixed Formulation 199

2.2. Mixed Approximation and Application to Finite Elements of Degree l 204
2.3. The Technique of Mesh-Dependent Norms 213

§ 3. Further Topics on the "Stream Function-Vorticity-Pressure" Scheme 222
3.1. Refinement of the Error Analysis 222
3.2. Super Convergence Using Quadrilateral Finite Elements of Degree l 231

§ 4. A "Stream Function-Gradient of Velocity Tensor" Method in Two Dimensions 237
4.1. The Hellan-Herrmann-Johnson Formulation 237
4.2. Approximation with Triangular Finite Elements of Degree l 242
4.3. Additional Results for the Hellan-Herrmann-Johnson Scheme 251
4.4. Discontinuous Approximation of the Pressure 252

§ 5. A "Vector Potential-Vorticity" Scheme in Three Dimensions 257
5.1. A Mixed Formulation of the Three-Dimensional Stokes Problem ... 257
5.2. Mixed Approximation in $H(\mathbf{curl}; \Omega)$ 260
5.3. A Family of Conforming Finite Elements in $H(\mathbf{curl}; \Omega)$ 262
5.4. Error Analysis for Finite Elements of Degree l 270
5.5. Discontinuous Approximation of the Pressure 273

Chapter IV. Theory and Approximation of the Navier-Stokes Problem .. 278

§ 1. A Class of Nonlinear Problems 278

§ 2. Theory of the Steady-State Navier-Stokes Equations 284
2.1. The Dirichlet Problem in the Velocity-Pressure Formulation 284
2.2. The Stream Function Formulation of the Homogeneous Problem . 293

§ 3. Approximation of Branches of Nonsingular Solutions 297
3.1. An Abstract Framework 297
3.2. Approximation of Branches of Nonsingular Solutions 301
3.3. Application to a Class of Nonlinear Problems 306
3.4. Non-Differentiable Approximation of Branches of Nonsingular Solutions 311

§ 4. Numerical Analysis of Centered Finite Element Schemes 316
4.1. Formulation in Primitive Variables: Methods Using Discontinuous Pressures 316
4.2. Formulation in Primitive Variables: the Case of Continuous Pressures 322
4.3. Mixed Incompressible Methods: the "Stream Function-Vorticity" Formulation 327
4.4. Remarks on the "Stream Function-Gradient of Velocity Tensor" Scheme 334

§5. Numerical Analysis of Upwind Schemes 336
5.1. Upwinding in the Stream Function-Vorticity Scheme 336
5.2. Error Analysis of the Upwind Scheme 340
5.3. Approximating the Pressure with the Upwind Scheme 350

§6. Numerical Algorithms 352
6.1. General Methods of Descent and Application to Gradient Methods 352
6.2. Least-Squares and Gradient Methods to Solve the Navier-Stokes Equations 357
6.3. Newton's Method and the Continuation Method 362

References 368

Index of Mathematical Symbols 372

Subject Index 373