

# Contents

<b>Introduction</b>	<b>1</b>
<b>Chapter 1</b>	
<b>Fundamentals of the Theory of Difference Schemes</b>	<b>8</b>
1.1. Basic Equations and Their Adjoints	8
1.1.1. Norm Estimates of Certain Matrices	12
1.1.2. Computing the Spectral Bounds of a Positive Matrix	13
1.1.3. Eigenvalues and Eigenfunctions of the Laplace Operator	21
1.1.4. Eigenvalues and Eigenvectors of the Finite-Difference Analog of the Laplace Operator	24
1.2. Approximation	27
1.3. Countable Stability	36
1.4. The Convergence Theorem	44
<b>Chapter 2</b>	
<b>Methods of Constructing Difference Schemes for Differential Equations</b>	<b>47</b>
2.1. Variational Methods in Mathematical Physics	48
2.1.1. Some Problems of Variational Calculation	48
2.1.1. The Ritz Method	55
2.1.3. The Galërkin Method	60
2.1.4. The Method of Least Squares	63
2.2. The Method of Integral Identities	64
2.2.1. Method of Constructing Difference Equations for Problems with Discontinuous Coefficients on the Basis of an Integral Identity	64
2.2.2. The Variational Form of an Integral Identity	72

<b>2.3. Difference Schemes for Equations with Discontinuous Coefficients Based on Variational Principles</b>	<b>84</b>
2.3.1. Simple Difference Equations for a Diffusion Based on the Ritz Method	85
2.3.2. Constructions of Simple Difference Schemes Based on the Galérkin (Finite Elements) Method	88
<b>2.4. Principles for the Construction of Subspaces for the Solution of One-Dimensional Problems by Variational Methods</b>	<b>90</b>
2.4.1. A General Approach to the Construction of Subspaces of Piecewise-Polynomial Functions	91
2.4.2. Constructing a Basis Using Trigonometric Functions and Applying It in Variational Methods	94
<b>2.5. Variational-Difference Schemes for Two-Dimensional Equations of Elliptic Type</b>	<b>100</b>
2.5.1. The Ritz Method	100
2.5.2. The Galérkin Method	106
2.5.3. Methods for Constructing Subspaces	109
<b>2.6. Variational Methods for Multi-Dimensional Problems</b>	<b>112</b>
2.6.1. Methods of Choosing the Subspaces	112
2.6.2. Coordinate-by-Coordinate Methods for Multi-Dimensional Problems	114
<b>2.7. The Method of Fictive Domains</b>	<b>116</b>
<b>Chapter 3</b>	
<b>Interpolation of Net Functions</b>	<b>122</b>
<b>3.1. Interpolation of Functions of One Variable</b>	<b>123</b>
3.1.1. Interpolation of Functions of One Variable by Cubic Splines	123
3.1.2. Piecewise-Cubic Interpolation with Smoothing	127
3.1.3. Smooth Construction	129
3.1.4. The Convergence of Spline Functions	131
<b>3.2. Interpolation of Functions of Two or More Variables</b>	<b>133</b>
<b>3.3. An <math>r</math>-Smooth Approximation to a Function of Several Variables</b>	<b>135</b>
<b>3.4. Elements of the General Theory of Splines</b>	<b>141</b>
<b>Chapter 4</b>	
<b>Methods for Solving Stationary Problems of Mathematical Physics</b>	<b>147</b>
<b>4.1. General Concepts of Iteration Theory</b>	<b>149</b>
<b>4.2. Some Iterative Methods and Their Optimization</b>	<b>150</b>
4.2.1. The Simplest Iteration Method	150
4.2.2. Convergence and Optimization of Stationary Iterative Methods	153
4.2.3. The Successive Over-Relaxation Method	156
4.2.4. The Chebyshev Iteration Method	161
4.2.5. Comparison of the Convergence Rates of Various Iteration Methods for a System of Finite-Difference Equations	169
<b>4.3. Nonstationary Iteration Methods</b>	<b>171</b>
4.3.1. Convergence Theorems	171
4.3.2. The Method of Minimizing the Residuals	173
4.3.3. The Conjugate Gradient Method	175

<b>4.4. The Splitting-Up Method</b>	<b>180</b>
4.4.1. The Commutative Case	183
4.4.2. The Noncommutative Case	188
4.4.3. Variational and Chebyshevian Optimization of Splitting-Up Methods	192
<b>4.5. Iteration Methods for Systems with Singular Matrices</b>	<b>194</b>
4.5.1. Consistent Systems	195
4.5.2. Inconsistent Systems	197
4.5.3. The Matrix Analog of the Method of Fictive Regions	199
<b>4.6. Iterative Methods for Inaccurate Input Data</b>	<b>203</b>
<b>4.7. Direct Methods for Solving Finite-Difference Systems</b>	<b>205</b>
4.7.1. The Fast Fourier Transform	205
4.7.2. The Cyclic Reduction Method	210
4.7.3. Factorization of Difference Equations	212
<b>Chapter 5</b>	
<b>Methods for Solving Nonstationary Problems</b>	<b>224</b>
<b>5.1. Second-Order Approximation Difference Schemes with Time-Varying Operators</b>	<b>224</b>
<b>5.2. Nonhomogeneous Equations of the Evolution Type</b>	<b>227</b>
<b>5.3. Splitting-Up Methods for Nonstationary Problems</b>	<b>228</b>
5.3.1. The Stabilization Method	229
5.3.2. The Predictor–Corrector Method	233
5.3.3. The Component-by-Component Splitting-Up Method	237
5.3.4. Some General Remarks	242
<b>5.4. Multi-Component Splitting</b>	<b>243</b>
5.4.1. The Stabilization Method	244
5.4.2. The Predictor–Corrector Method	245
5.4.3. The Component-by-Component Splitting-Up Method Based on the Elementary Schemes	247
5.4.4. Splitting-Up of Quasi-Linear Problems	252
<b>5.5. General Approach to Component-by-Component Splitting</b>	<b>253</b>
<b>5.6. Methods of Solving Equations of the Hyperbolic Type</b>	<b>257</b>
5.6.1. The Stabilization Method	257
5.6.2. Reduction of the Wave Equation to an Evolution Problem	261
<b>Chapter 6</b>	
<b>Richardson’s Method for Increasing the Accuracy of Approximate Solutions</b>	<b>267</b>
<b>6.1. Ordinary First-Order Differential Equations</b>	<b>268</b>
<b>6.2. General Results</b>	<b>273</b>
6.2.1. The Decomposition Theorem	273
6.2.2. Acceleration of Convergence	279
<b>6.3. Simple Integral Equations</b>	<b>285</b>
6.3.1. The Fredholm Equation of the Second Kind	285
6.3.2. The Volterra Equation of the First Kind	287

<b>6.4. The One-Dimensional Diffusion Equation</b>	<b>290</b>
6.4.1. The Difference Method	291
6.4.2. The Galérkin Method	293
<b>6.5. Nonstationary Problems</b>	<b>299</b>
6.5.1. The Heat Equation	299
6.5.2. The Splitting-Up Method for the Evolutionary Equation	305
<b>6.6. Richardson's Extrapolation for Multi-Dimensional Problems</b>	<b>306</b>
 <b>Chapter 7</b>	
<b>Numerical Methods for Some Inverse Problems</b>	<b>312</b>
<b>7.1. Fundamental Definitions and Examples</b>	<b>313</b>
<b>7.2. Solution of the Inverse Evolution Problem with a Constant Operator</b>	<b>321</b>
7.2.1. The Fourier Method	322
7.2.2. Reduction to the Solution of a Direct Equation	324
<b>7.3. Inverse Evolution Problems with Time-Varying Operators</b>	<b>327</b>
<b>7.4. Methods of Perturbation Theory for Inverse Problems</b>	<b>333</b>
7.4.1. Some Problems of the Linear Theory of Measurements	333
7.4.2. Conjugate Functions and the Notion of Value	335
7.4.3. Perturbation Theory for Linear Functionals	337
7.4.4. Numerical Methods for Inverse Problems and Design of Experiment	339
<b>7.5. Perturbation Theory for Complex Nonlinear Models</b>	<b>344</b>
7.5.1. Fundamental and Adjoint Equations	345
7.5.2. The Adjoint Equation in Perturbation Theory	347
7.5.3. Perturbation Theory for Nonstationary Problems	348
7.5.4. Spectral Methods in Perturbation Theory	350
 <b>Chapter 8</b>	
<b>Methods of Optimization</b>	<b>352</b>
<b>8.1. Convex Programming</b>	<b>352</b>
<b>8.2. Linear Programming</b>	<b>357</b>
<b>8.3. Quadratic Programming</b>	<b>362</b>
<b>8.4. Numerical Methods in Convex Programming Problems</b>	<b>366</b>
<b>8.5. Dynamic Programming</b>	<b>371</b>
<b>8.6. Pontrjagin's Maximum Principle</b>	<b>375</b>
<b>8.7. Extremal Problems with Constraints and Variational Inequalities</b>	<b>381</b>
8.7.1. Elements of the General Theory	382
8.7.2. Examples of Extremal Problems	384
8.7.3. Numerical Methods in Extremal Problems	390

## Chapter 9

<b>Some Problems of Mathematical Physics</b>	<b>396</b>
9.1. The Poisson Equation	396
9.1.1. The Dirichlet Problem for the One-Dimensional Poisson Equation	396
9.1.2. The One-Dimensional von Neumann Problem	398
9.1.3. The Two-Dimensional Poisson Equation	401
9.1.4. A Problem of Boundary Conditions	409
9.2. The Heat Equation	411
9.2.1. The One-Dimensional Problem of Heat Conduction	412
9.2.2. The Two-Dimensional Problem of Heat Conduction	416
9.3. The Wave Equation	417
9.4. The Equation of Motion	421
9.4.1. The Simplest Equations of Motion	422
9.4.2. The Two-Dimensional Equation of Motion with Variable Coefficients	428
9.4.3. The Multi-Dimensional Equation of Motion	434
9.5. The Neutron Transport Equation	439
9.5.1. The Nonstationary Equation	439
9.5.2. The Transport Equation in Self-Adjoint Form	451

## Chapter 10

<b>A Review of the Methods of Numerical Mathematics</b>	<b>456</b>
10.1. The Theory of Approximation, Stability, and Convergence of Difference Schemes	456
10.2. Numerical Methods for Problems of Mathematical Physics	459
10.3. Conditionally Well-Posed Problems	465
10.4. Numerical Methods in Linear Algebra	466
10.5. Optimization Problems in Numerical Methods	470
10.6. Optimization Methods	472
10.7. Some Trends in Numerical Mathematics	473
<b>References</b>	<b>476</b>
<b>Index of Notation</b>	<b>505</b>
<b>Index</b>	<b>507</b>