
Table of Contents

*The chapter on the Fall of the Rupee you may omit.
It is somewhat too sensational.*

OSCAR WILDE
The Importance of Being Ernest (1895)

1. INTRODUCTION	1
2. LINEAR ALGEBRA BACKGROUND	3
2.1. Definitions and Operations	3
2.1.1. Vectors	3
2.1.2. Matrices	6
2.1.3. Matrices with Special Structure	11
2.2. Vector Spaces	14
2.2.1. Linear Dependence and Independence	14
2.2.2. Range and Null Spaces	17
2.2.3. Singular and Nonsingular Matrices	20
2.3. Eigenvalues and Singular Values	22
2.3.1. Eigenvalues and Eigenvectors	22
2.3.2. The Singular Value Decomposition	24
2.4. Norms	24
2.4.1. Vector Norms	25
2.4.2. Matrix Norms	26
Further Reading	27
Exercises	28
3. COMPUTATION AND CONDITION	33
3.1. Measures of Closeness	33
3.1.1. Error Measurements	33
3.1.2. Order Notation	35
3.2. The Effects of Finite Precision	36
3.2.1. Floating-Point Representation	36
3.2.2. Arithmetic Operations	39
3.2.3. Cancellation Error	40
3.2.4. Solving a Quadratic Equation	42
3.3. Condition of a Mathematical Problem	43
3.4. Computable Measures of Goodness	45

3.5. Error Analysis	47
3.5.1. Forward Error Analysis	47
3.5.2. Backward Error Analysis	48
3.6. Condition of a Linear System	50
3.6.1. Condition of a Nonsingular Matrix	50
3.6.2. Revelations of the Singular Value Decomposition	54
Further Reading	58
Exercises	59
4. LINEAR EQUATIONS	71
4.1. Overview	71
4.2. Triangular Linear Systems	72
4.2.1. The Meaning of Triangular Form	72
4.2.1.1. Permutation matrices	73
4.2.1.2. Triangular forms	74
4.2.2. Solving a Lower-Triangular System	74
4.3. Gaussian Elimination	76
4.3.1. Reduction to Upper-Triangular Form	76
4.3.2. The LU Factorization	80
4.3.2.1. Calculation of the LU factors	81
4.3.2.2. Connection with Gaussian elimination	83
4.3.3. Using Triangular Factors to Solve $Ax = b$	84
4.4. Pivoting	85
4.4.1. The Need for Pivoting	85
4.4.1.1. Row interchanges	85
4.4.1.2. Accumulation of row interchanges	86
4.4.2. Pivoting Strategies	88
4.4.2.1. The effect of small pivots	88
4.4.2.2. Partial pivoting	89
4.5. Applications of Error Analysis	92
4.5.1. Forming the Inner Product	92
4.5.2. Computed Matrix-Vector Products	94
4.5.3. Solving a Triangular System	95
4.5.4. Error Analysis of Gaussian Elimination	96
4.6. Properties of the Computed Solution	97
4.7. Using the Inverse Matrix	99
4.7.1. Computing the Inverse	99
4.7.2. Solving a Linear System with the Computed Inverse	100
4.8. Block Elimination and Factorization	104
4.9. Symmetric Matrices	106
4.9.1. Taking Advantage of Symmetry	106
4.9.2. The Cholesky Factorization	108
4.9.3. Factorization of Symmetric Indefinite Matrices	112
4.9.3.1. The LBL^T factorization	112
4.9.3.2. Block elimination	113

4.9.3.3.	Direct factorization	114
4.9.3.4.	Pivoting strategies	116
4.9.3.5.	Complete and partial pivoting	117
4.9.3.6.	Computing the factorization	119
4.9.3.7.	Calculating the inertia of A	120
4.10.	Orthogonal Triangularization	120
4.10.1.	Householder Transformations	121
4.10.2.	The QR Factorization	122
4.10.3.	Solving the Linear System	125
4.11.	Introduction to Updating	125
4.11.1.	The Meaning of Updating	126
4.11.2.	Rank-One Modification; the Sherman-Morrison Formula	126
4.11.3.	Selective Annihilation	127
4.11.3.1.	Plane rotations	128
4.11.3.2.	Stabilized elementary transformations	129
4.11.3.3.	Sweeps	132
4.11.4.	The QR Factorization	133
4.11.5.	The LU Factorization	139
4.11.5.1.	Updating the elimination form	140
4.11.5.2.	Updating the LU factorization	146
4.11.6.	A Numerical Example	147
	Further Reading	150
	Exercises	151
 5. COMPATIBLE SYSTEMS		161
5.1.	Beyond Nonsingular Systems	161
5.2.	Matrices with Full Rank	162
5.2.1.	Full Row Rank	162
5.2.2.	Full Column Rank	164
5.3.	Background	165
5.3.1.	Characterization of a Solution	165
5.3.2.	Transformation to Rank-Revealing Form	167
5.3.3.	Condition of a General Matrix	169
5.3.3.1.	The pseudoinverse	170
5.3.3.2.	The generalized condition number	171
5.4.	Triangular Rank-Revealing Form	173
5.4.1.	The LU Factorization	173
5.4.1.1.	Reduction to upper-triangular form	173
5.4.1.2.	A condensed LU factorization	176
5.4.1.3.	Normalized triangular matrices	177
5.4.2.	The QR Factorization	178
5.4.3.	Solving a Triangular Rank-Revealing System	181
5.4.3.1.	A generic approach	181
5.4.3.2.	Using the LU factorization	183
5.4.3.3.	Using the QR factorization	185

5.5. The Minimum-Length Solution	186
5.5.1. A Formal Representation	187
5.5.2. The LU Factorization	188
5.5.3. The QR Factorization	189
5.5.3.1. Using the QR factors of A	190
5.5.3.2. Using the QR factors of A^T	190
5.5.4. The Complete Orthogonal Factorization	192
5.6. The Singular Value Decomposition	195
5.7. Condition of a Compatible System	197
5.8. Estimation of Numerical Rank	199
5.8.1. The Difficulty of Defining Numerical Rank	199
5.8.2. The Singular Value Decomposition	200
5.8.3. Triangular Factorizations	201
5.8.4. The Effect of Rank-Estimation Strategies	203
Further Reading	205
Exercises	206
6. LINEAR LEAST SQUARES	215
6.1. Background	215
6.2. Formulation of a Least-Squares Problem	216
6.2.1. A Model of Happiness	216
6.2.2. Definition of a Least-Squares Problem	217
6.3. Properties of the Least-Squares Solution	218
6.3.1. Characterization of the Optimal Residual	218
6.3.2. Changes in the Right-Hand Side	221
6.4. The Normal Equations	223
6.5. Orthogonal Transformation	226
6.5.1. Motivation	226
6.5.2. The QR Factorization	227
6.6. The Minimum-Length Least-Squares Solution	230
6.6.1. A Formal Representation	230
6.6.2. The LU Factorization	231
6.6.3. The QR Factorization	232
6.6.4. The Complete Orthogonal Factorization	233
6.6.5. The Singular Value Decomposition	233
6.7. Condition of the Least-Squares Problem	234
6.7.1. Perturbations in the Right-Hand Side	235
6.7.2. Perturbations in the Matrix	237
6.8. Range and Null Spaces	240
6.8.1. Rank-Retaining Factorizations	240
6.8.2. The Singular Value Decomposition	241
6.8.3. The QR and Complete Orthogonal Factorizations	242
6.8.4. The LU Factorization	243
6.9. Projections	244
6.9.1. Projection onto a Subspace	244

6.9.2.	Projections Associated with A	245
6.9.3.	Computation of Projections	246
6.9.3.1.	Projections of a vector	246
6.9.3.2.	Projections obtained from orthonormal bases	247
Further Reading	247
Exercises	248
7. LINEAR CONSTRAINTS I: LINEAR PROGRAMMING		259
7.1.	Introduction	259
7.2.	Formulating a Linear Program	260
7.2.1.	The Portfolio Problem	260
7.2.2.	Formulation as a Linear Program	261
7.3.	Linear Functions	264
7.3.1.	The Normal Vector	265
7.3.2.	Contour Plots	266
7.3.3.	One-Dimensional Variation	267
7.3.4.	Boundedness	268
7.4.	Linear Equality Constraints	270
7.4.1.	Feasible Points	270
7.4.2.	Feasible Directions	271
7.5.	Linear Inequality Constraints	272
7.5.1.	Feasible Points	273
7.5.2.	Active and Inactive Constraints	276
7.5.3.	Feasible Directions	277
7.5.4.	The Step to the Nearest Constraint	281
7.5.5.	Vertices	282
7.6.	Optimality for Equality Constraints	288
7.6.1.	Feasible Descent Directions	289
7.6.2.	Derivation of Optimality Conditions	289
7.7.	Optimality for Inequality Constraints	293
7.7.1.	Feasible Descent Directions	294
7.7.2.	Farkas' Lemma	295
7.7.2.1.	The easy parts	296
7.7.2.2.	Complications of linear dependence	297
7.7.2.3.	The separating hyperplane theorem	297
7.7.2.4.	Completion of the proof of Farkas' Lemma	300
7.7.3.	Summary of Optimality Conditions	301
7.7.4.	Vertex Minimizers	303
7.7.4.1.	Existence of a vertex minimizer	303
7.7.4.2.	Results about uniqueness	304
7.7.4.3.	A small example	306
7.8.	Linear Programs with Mixed Constraints	307
7.8.1.	Form of Mixed Constraints	308
7.8.2.	Optimality Conditions for Mixed Constraints	308
7.8.3.	Definition of Standard Form	309

7.8.3.1. Transformation to standard form	309
7.8.3.2. Properties of standard form	310
7.8.3.3. Optimality conditions for standard form	311
7.9. Finding a Feasible Point	311
7.9.1. Formulation of a Phase-1 Linear Program	312
7.9.2. Adding a Single Artificial Variable to Inequality Form	313
7.9.3. Minimizing the Sum of Infeasibilities	314
7.9.4. Phase 1 for Standard Form	316
7.10. Duality	317
7.10.1. Equality Constraints	317
7.10.2. Standard Form	318
7.10.3. All-Inequality Constraints	319
7.10.4. Relations between the Primal and Dual	320
7.10.4.1. Dual degeneracy	320
7.10.4.2. The duality gap	321
7.10.4.3. Feasibility and boundedness	321
Further Reading	324
Exercises	325
8. THE SIMPLEX METHOD	337
8.1. Background	337
8.2. Linear Programs in Inequality Form	338
8.2.1. Motivation for a Simplex Step	339
8.2.2. Definition of the Simplex Method	343
8.2.3. An Example of the Simplex Method	345
8.2.4. Termination of the Simplex Method	348
8.3. Degeneracy	348
8.3.1. Degeneracy and the Simplex Method	348
8.3.2. Cycling	350
8.3.3. Anti-Cycling Techniques	352
8.4. Linear Programs in Standard Form	355
8.4.1. Basic and Nonbasic Variables	356
8.4.2. Motivation for the Standard-Form Simplex Method	359
8.4.2.1. Specialization of the generic algorithm	359
8.4.2.2. Derivation in terms of basic and nonbasic variables	359
8.4.2.3. Steps of the standard-form simplex method	361
8.4.3. Summary of the Standard-Form Simplex Method	364
8.4.4. An Example of the Standard-Form Simplex Method	366
8.4.5. Standard-Form Simplex Tableaus	368
8.4.5.1. Dictionaries	368
8.4.5.2. Constructing a full tableau	369
8.4.5.3. Updating a full tableau	371
8.4.5.4. Finding an initial tableau	374
8.4.6. Solving the Dual Problem	375
8.5. Non-Simplex Active-Set Methods	375

8.5.1. Motivation for an Active-Set Strategy	375
8.5.2. Finding a Null-Space Descent Direction	377
8.5.3. Steepest-Descent Null-Space Directions	377
8.5.4. Definition of a Non-Simplex Method	379
8.5.5. A Non-Simplex Example	381
8.6. Solving a Phase-1 Linear Program	382
8.6.1. Using a Single Artificial Variable in Inequality Form	383
8.6.2. Minimizing the Sum of Infeasibilities in Inequality Form	386
8.6.3. Temporary Bounds	387
8.6.3.1. Creation of temporary bounds	388
8.6.3.2. Deletion of temporary bounds	388
8.6.3.3. An example of temporary bounds	389
8.6.4. Standard-Form Phase-1 Problems	390
8.6.4.1. Creating an all-artificial basis	390
8.6.4.2. Specifying an initial basis	391
8.7. Complexity of the Simplex Method	395
8.7.1. Measurements of Performance	395
8.7.2. Behavior of the Simplex Method	396
Further Reading	398
Exercises	399
BIBLIOGRAPHY	409
INDEX	411