CONTENTS

PREFACE v

SYMBOLS xv

Part I Algebraic Systems

CHAPTER 1 THE FIVE-ELEMENT SYSTEM: ADDITION 2

1.1 Recalling the Natural Numbers, *2*
1.2 The Five-Element System, *3*
1.3 Addition, *3*
1.4 Closure, *5*
1.5 Commutativity, *7*
1.6 Addition of Three Elements, *9*
1.7 Law of Cancellation, *13*
1.8 Identity Element, *15*
1.9 The Inverse, *16*
1.10 Properties of the Inverse, *17*
1.11 Uses of the Inverse, *19*
1.12 Two Generalizations, *22*
1.13 The Integers, *23*

CHAPTER 2 THE FIVE-ELEMENT SYSTEM: MULTIPLICATION 26

2.1 Multiplication of Natural Numbers, *26*
2.2 Multiplication of Elements, *28*
2.3 Closure and Commutativity, *29*
2.4 Associativity, *30*
2.5 Identity and Inverse, *31*
2.6 Properties of the Inverse, *32*
2.7 Uses of the Inverse, *33*
2.8 The Reduced Five-Element System, *35*
2.9 Some Generalizations, *35*
2.10 Groups, *37*
2.11 Comment on Group Criteria, *40*
2.12 The Multiplicative Property of the Element 5, *41*
2.13 Combining Operations, *43*
2.14 Applications of the Distributive Law, *45*
2.15 The Rational Numbers, *46*
2.16 The Real Numbers, *48*

CHAPTER 3 THE FIVE-ELEMENT SYSTEM: FURTHER PROPERTIES 50

3.1 On the Solution of Equations, *50*
3.2 Subtraction, *53*
3.3 Properties of Subtraction, *55*
3.4 Comment on Subtraction, *56*
3.5 Introduction to Quotients, *57*
3.6 Division, *58*
3.7 Properties of Division, *60*
3.8 Comment on Division, *61*
3.9 A Note on Symbolism, *63*
3.10 Cryptography, *63*
3.11 Retrospect and Prospect, *66*

CHAPTER 4 A SIX-ELEMENT SYSTEM 67

4.1 Formation of the System, *67*
4.2 Addition, *67*
4.3 Properties of Addition, *69*
4.4 Multiplication, *70*
4.5 Properties of Multiplication, *71*
4.6 Divisors of the Additive Identity, *73*

4.7 Structure of the System, *75*
4.8 Occurrence of Divisors of the Additive Identity, *75*
4.9 Retrospect and Prospect, *77*

CHAPTER 5 A SYSTEM OF PAIRS OF ELEMENTS 78

5.1 Definition and Notation, *78*
5.2 Equality, *79*
5.3 Addition, *80*
5.4 Commutativity of Addition, *81*
5.5 Associativity of Addition, *83*
5.6 The Additive Group of Pairs, *85*
5.7 Associated Pairs-and-Elements Relations, *86*
5.8 Multiplication, *87*
5.9 Properties of Multiplication, *88*
5.10 Divisors of the Additive Identity, *89*
5.11 Multiplicative Identity and Inverse, *90*
5.12 Comment, *92*

CHAPTER 6 A SECOND SYSTEM OF PAIRS OF ELEMENTS 93

6.1 Equivalence of Pairs, *93*
6.2 A Second System of Pairs, *94*
6.3 Equivalence Sets, *95*
6.4 Addition, *97*
6.5 A System of Equivalence Sets, *99*
6.6 An Isomorphism, *100*
6.7 Multiplication, *101*
6.8 Multiplication of Equivalence Sets, *102*
6.9 Structure of the System of Equivalence Sets, *103*
6.10 Pairs of Natural Numbers (Optional), *104*
6.11 Equivalence Sets of Pairs of Natural Numbers (Optional), *105*
6.12 Pairs of Pairs, *108*

CHAPTER 7 A SYSTEM OF MATRICES 109

7.1 Definition and Notation, *109*
7.2 Equality, *110*

7.3 Addition, *111*
7.4 The Additive Group of Matrices, *113*
7.5 Scalar Matrices, *115*
7.6 Further Subgroups of the Additive Group of Matrices (Optional), *117*
7.7 Multiplication, *119*
7.8 An Example, *121*
7.9 Properties of Multiplication, *123*
7.10 Scalar Matrices Under Multiplication, *125*
7.11 Determinants of Matrices, *128*
7.12 The Adjoint Matrix, *130*
7.13 Solution of Matrix Equations, *132*
7.14 The Solution of Simultaneous Linear Equations, *134*
7.15 A Practical Problem, *137*
7.16 Matrices in Cryptography, *139*
7.17 Retrospect and Prospect, *141*

Part II Logic and Sets

CHAPTER 8 THE ALGEBRA OF LOGIC 144

8.1 Statements and Connectives, *145*
8.2 Punctuation, *148*
8.3 Examples, *150*
8.4 Truth Values, *151*
8.5 Truth Tables, *155*
8.6 Relations Between Patterns: Logical Implication, *159*
8.7 Logical Equivalence, *162*
8.8 Logical Negation, *164*
8.9 The Logical System as an Algebraic System (Optional), *165*
8.10 Open Sentences, *167*
8.11 Open-Sentence Patterns, *168*
8.12 Logical Relations Between Open-Sentence Patterns, *170*
8.13 Quantified Sentences, *171*
8.14 Translation, *173*
8.15 Truth Values of Quantified Sentences, *175*
8.16 Relations Between Quantified-Sentence Patterns, *176*
8.17 Retrospect and Prospect, *177*

CHAPTER 9 THE ALGEBRA OF SETS 178

9.1 Definition and Notation, *178*
9.2 Relations, *180*
9.3 Equality, *180*
9.4 Inclusion, *182*
9.5 Properties of Inclusion of Sets, *184*
9.6 Diagrams, *184*
9.7 Meeting and Disjointness, *185*
9.8 Properties of Meeting and Disjointness of Sets, *187*
9.9 Quantified Sentences and Relations Between Sets (Optional), *188*
9.10 Logical Argument, *189*
9.11 Operations, *190*
9.12 Intersection, *191*
9.13 Diagramming Intersection, *193*
9.14 Properties of Intersection, *195*
9.15 Union, *196*
9.16 Properties of Union of Sets, *198*
9.17 Logical Operations and Set Operations (Optional), *199*
9.18 Complementation, *202*
9.19 Properties of Complementation, *203*
9.20 Logical Operations and Complementation of Sets (Optional), *204*
9.21 Problems on Sets (Optional), *205*
9.22 Boolean Algebra (Optional), *206*
9.23 The Design of Electric Circuits (Optional), *208*
9.24 The Algebra of Sets and the Algebra of Logic, *211*
9.25 Retrospect and Prospect, *212*

Part III Geometry

CHAPTER 10 INTRODUCTION TO GEOMETRY 216

10.1 Euclidean Geometry, *216*
10.2 Three Noneuclidean Geometers: Lobachevski, Bolyai, Riemann, *220*
10.3 The First Portion of *The Elements*, *222*

10.4 Three Important Polygons, *224*
10.5 Theorems on Euclidean Parallels, *228*
10.6 Comment, *231*

CHAPTER 11 HYPERBOLIC GEOMETRY 233

11.1 Beginning Concepts, *233*
11.2 On Nonintersecting Lines, *236*
11.3 Further Properties of Nonintersecting Lines, *239*
11.4 Properties of Parallels, *241*
11.5 The Three Polygons, *245*
11.6 Representations, *249*
11.7 Comment, *252*

CHAPTER 12 ELLIPTIC GEOMETRY 253

12.1 Modification of Axioms, *253*
12.2 Beginning Concepts, *254*
12.3 Lines, Poles, and Distances, *256*
12.4 Further Properties of Lines, *259*
12.5 Right Triangles, *262*
12.6 The Three Polygons, *263*
12.7 Representation, *265*
12.8 The Three Geometries, *266*
12.9 Retrospect, *267*

APPENDIX: GEOMETRY 268

REFERENCES 271

ANSWERS AND HINTS 273

INDEX 307