preface xxv
introduction xxvii

part a number systems: sets, relations, operations, uses 1

unit 1: preliminary whole number ideas 1

1.1 Introduction 3
1.2 Problem Set: Uses of Whole Numbers 4
1.2.1 Collecting uses of whole numbers 4
1.2.2 Whole numbers used in gathering and recording data 5
1.2.3 The natural order of whole numbers 8
1.2.4 Whole number lines 10
1.2.5 Whole numbers for identification or coding 12
1.2.6 Summary 14
1.3 Activity: Basic Set Ideas 14
1.4 Activity: Relations for Whole Numbers: "Is Equal To," "Is Less Than," "Is Greater Than" 16
1.4.1 Learning to count 18
1.4.2 Comparing sets without counting 19
1.4.3 Comparing measures 21
1.4.4 Properties of relations 24
1.5 Summary and Pedagogical Remarks 27

unit 2: addition of whole numbers 32

2.1 Introduction 32
2.2 Activity: The Meaning of Addition of Whole Numbers 33
2.2.1 Addition of whole number counts 33
2.2.2 Addition of whole number measures using rods 35
2.2.3 Addition of whole number measures using number lines 36

2.2.4 Counts or hops on the number line 37
2.2.5 Addition with slide rule made from number lines 37

2.3 Activity: Properties of Addition of Whole Numbers 39
2.3.1 Closure 39
2.3.2 Uniqueness of sums; equivalence classes of sums 39
2.3.3 Zero as the identity for addition of whole numbers 41
2.3.4 Whole numbers have no additive inverses 43
2.3.5 The commutative property of addition of whole numbers 43
2.3.6 The associative property of addition of whole numbers 44
2.3.7 A "do as you please" property for addition of whole numbers 45

2.4 Summary and Pedagogical Remarks 46

unit 3: multiplication of whole numbers 49

3.1 Introduction 49

3.2 Activity: The Meaning of Multiplication of Whole Numbers 50
3.2.1 Multiplication of counts using repeated addition 50
3.2.2 Multiplication with measures using repeated addition 50
3.2.3 Multiplication using arrays 52
3.2.4 Cartesian products; combinations 55
3.2.5 Connections among arrays, tree diagrams, Cartesian products 59

3.3 Activity: Properties of Multiplication of Whole Numbers 61
3.3.1 Closure, uniqueness, equivalence classes of products 62
3.3.2 The identity element for multiplication 63

3.3.3 Whole numbers have no multiplicative inverse 63
3.3.4 Multiplication property of zero 64
3.3.5 Commutative property of multiplication of whole numbers 65
3.3.6 Associative property of multiplication of whole numbers 65
3.3.7 The "do as you please" property of multiplication 66
3.3.8 The distributive property of multiplication over addition 67

3.4 Summary and Pedagogical Remarks 69

unit 4: subtraction of whole numbers 73

4.1 Introduction 73
4.2 Classroom Notes: The Meaning of Subtraction of Whole Numbers 74
4.2.1 "Take away" situations with counts or with measures 74
4.2.2 "Comparison" situations with counts and measures 76
4.2.3 Number lines in subtraction 78
4.2.4 Addition-Subtraction Links 79
4.3 Classroom Notes: Properties of Whole Number Subtraction 80
4.3.1 Subtraction of whole numbers is *not* closed 80
4.3.2 Uniqueness; equivalence classes of differences 80
4.3.3 Zero is a right identity, but not a left identity, for subtraction 81
4.3.4 For subtraction, every whole number is its own inverse 81
4.3.5 Subtraction of whole numbers is not commutative 81
4.3.6 Subtraction of whole numbers is *not* associative 81

4.3.7 There is *not* a "do as you please" property for subtraction of whole numbers 81

4.4 Work sheets on Subtraction of Whole Numbers 82
4.4.1 Take-away situations with counts 82
4.4.2 Take-away situations with measures 83
4.4.3 Comparison (how much more) with counts 83
4.4.4 Comparison (how much more) with measures 84
4.4.5 Number lines and rods used for subtraction 85
4.4.6 Addition-subtraction links 85

4.5 Summary and Pedagogical Remarks 87

unit 5: division of whole numbers 89

5.1 Introduction 89
5.2 Activity: The Meaning and Sources of Division of Whole Numbers 89
5.2.1 Exploration of division 90
5.2.2 Sources of division 90
5.2.3 Sources of division 91
5.2.4 Sources of division with measures 92
5.2.5 Sources of division with measures 93
5.2.6 "Remainders" in division 93

5.3 Problem Set: Links between Division and Multiplication of Whole Numbers 95
5.3.1 "Checking" division by multiplying 95
5.3.2 "Checking" when there are remainders $p = (n \times q) + r$ 96
5.3.3 Multiplication-division links: arrays, repeated addition, and repeated subtraction 97
5.3.4 Using a multiplication table to do division 98

5.3.5 Division by a whole number versus multiplication by a fraction 99

5.4 Problem Set: Properties of Division of Whole Numbers 100
5.4.1 Closure property 100
5.4.2 A substitute for "closure" of division of whole numbers 101
5.4.3 Uniqueness; equivalence classes of quotients 101
5.4.4 The number 1 as a right identity 102
5.4.5 Inverses 102
5.4.6 Commutativity 103
5.4.7 Associativity 103
5.4.8 Division *of* zero by a nonzero whole number 104
5.4.9 Division *by* zero: Impossible! 104

5.5 Summary and Pedagogical Remarks 105

unit 6: extension of the whole numbers to the integers 108

6.1 Introduction 108
6.2 Activity: Blue and White Numbers 109
6.2.1 The basic rule for equivalence 109
6.2.2 Addition of BW numbers 111
6.2.3 Subtraction of BW numbers 113
6.2.4 Multiplication of BW numbers 114
6.2.5 Division of RW numbers 116

6.3 Activity: A Measure Embodiment of Integers With Directed Rods 117
6.3.1 Making the embodiment 117
6.3.2 Addition 117
6.3.3 Subtraction 119
6.3.4 Multiplication 120
6.3.5 Division 121

6.4 Problem Set: Extending the Set of Whole Numbers to the Integers 122
6.4.1 Uses of integers in common life 122
6.4.2 Uses of integers in mathematics 124

6.4.3 Extension of the whole numbers to integers on the number line 125

6.5 Problem Set: Relations and Operations with the Integers 127
6.5.1 The "is less than relation" 128
6.5.2 The "opposite of" operation 128
6.5.3 The "absolute value of" operation 129
6.5.4 Addition of integers 131
6.5.5 Subtraction of integers 131
6.5.6 The multiplication operation 132
6.5.7 The division operation 136

6.6 Problem Set: The Properties of Integers 137
6.6.1 Introductory remarks 137
6.6.2 Some useful "theorems" about the system of integers 139

6.7 Pedagogical Remarks 141

unit 7: extension of the integers to the rational numbers
144

7.1 Introduction 144

7.2 Problem Set: Why Bother with Fractions? Sources and Uses 148
7.2.1 Various motivations for fractions 148
7.2.2 Fractions as used in the common world 151

7.3 Activity: Paper Strips as Embodiments of (Rational) Fractions 155
7.3.1 The set of rational fractions embodied by paper folding 155
7.3.2 Relations: "is equal to," "is less than," "is greater than" 156
7.3.3 Multiplication of fractions using paper strips 157
7.3.4 Multiplicative inverse: reciprocals 158
7.3.5 Division of fractions using paper strips 159

7.4 Activity: Fractions with Number Line and Rod Embodiments 159
7.4.1 Making and using the number line 160

7.4.2 Equivalence classes of fractions 161
7.4.3 Using the number line to explore the "is less than," "is greater than," and "is equal to" relations 164
7.4.4 Addition of fractions with rods on the number line 165
7.4.5 Subtraction of fractions with rods on the number line 169
7.4.6 Division of fractions with rods on the number line 171

7.5 Problem Set: A Fraction Potpourri 173
7.5.1 Fractions as "operators" on sets of objects or on measures 173
7.5.2 Other uses of rods to embody fraction work 174
7.5.3 "Addition-only" methods for getting equivalent fractions 175
7.5.4 Equivalent fractions and multiplication of fractions by marking rectangular units 176
7.5.5 Justifying the "multiply by the reciprocal of the divisor" rule for division of fractions 177
7.5.6 More on why zero denominators or divisions by zero are *never* allowed 178

7.6 Summary of Properties of the System of Rational Fractions 179
7.6.1 General remarks 179
7.6.2 The set of rational fractions 180
7.6.3 Equivalence 181
7.6.4 "Is greater than" or "is less than" 181
7.6.5 Addition 181
7.6.6 Multiplication 182
7.6.7 Subtraction 182
7.6.8 Division 182
7.6.9 Properties of operations with rational numbers that

parallel properties of operations with integers 182
7.6.10 New properties 183
7.6.11 Other useful properties 183
7.7 Pedagogical Remarks 184

part b algorithms and numeration 187

unit 1: the algorithm for addition of whole numbers 189

1.1 Introduction 189
1.1.1 Algorithms 189
1.1.2 Brief historical background of algorithms 191
1.1.3 Uses of addition 193
1.1.4 This text's approach to algorithms for whole number operations 195

1.2 Activity: The Addition Algorithm for Base-Four Whole Numbers 197
1.2.1 Base-four words 197
1.2.2 The trade rules for base-four blocks 199
1.2.3 Addition with base-four blocks 201
1.2.4 Trade rules for the base-four chip computer 203
1.2.5 Addition with the base-four chip computer 204
1.2.6 Trading and adding with the base-four rod computer 205

1.3 Activity: The Addition Algorithm for Base-Ten Whole Numbers 209
1.3.1 Addition with base-ten blocks 209
1.3.2 Addition with the base-ten chip computer 210
1.3.3 Addition with the base-ten rod computer 211

1.4 Problem Set: Analysis of the Addition Alogrithm 213
1.4.1 An intermediate form of the addition algorithm 213

1.5 Pedagogical Remarks 223

unit 2: the algorithm for subtraction of whole numbers 229

2.1 Introduction 229
2.1.1 Uses of subtraction 229
2.2 Activity: The Subtraction Algorithm for Base-Four Numbers 231
2.2.1 Subtraction with base-four blocks 231
2.2.2 Subtraction with the base-four chip computer 233
2.2.3 Subtraction with the base-four rod computer 233
2.3 Activity: The Subtraction Algorithm for Base-Ten Whole Numbers 235
2.3.1 Subtraction with base-ten blocks 235
2.3.2 Subtraction with the base-ten chip computer 235
2.3.3 Subtraction with the base-ten rod computer 236
2.3.4 Take-away and comparison methods of using embodiments for the subtraction algorithm 236
2.4 Problem Set: Analysis of the Subtraction Algorithm 238
2.5 Pedagogical Remarks 248

unit 3: the algorithm for multiplication of whole numbers 251

3.1 Introduction 251
3.1.1 Uses of multiplication 252
3.2 Activity: Development of the Shift Rules with Base-Eight Multiplication Pieces 253
3.2.1 How the multiplication pieces work 253
3.2.2 Shift rules: total approach with the multiplication pieces 256
3.2.3 Shift rules: piece-by-piece approach with the multiplication pieces 258
3.3 Activity: Development of Shift Rules with the Base-Eight Chip Computer 259
3.3.1 How the base-eight chip computer works 259
3.3.2 Shift rules: total approach

on the base-eight chip computer 261
3.3.3 Shift rules: place-by-place approach on the base-eight chip computer 263
3.4 Activity: The Copy Multiplication Algorithm for Base Eight 264
3.4.1 The copy multiplication algorithm with base-eight multiplication pieces 264
3.4.2 The copy multiplication algorithm on the base-eight chip computer 266
3.4.3 The copy multiplication algorithm symbolically 269
3.5 Activity: The Place-Value Multiplication Algorithm for Base Eight 270
3.5.1 The intermediate place-value algorithm on the base-eight chip computer 271
3.5.2 The intermediate place-value algorithm using base-eight tables 273
3.5.3 The compact place-value algorithm 274
3.6 Problem Set: Analysis of the Multiplication Algorithm for Whole Numbers 275
3.7 Pedagogical Remarks 289

unit 4: algorithms for the division of whole numbers 292

4.1 Introduction 292
4.1.1 Uses of division 292
4.2 Activity: Division Algorithms for Base-Eight Whole Numbers 294
4.2.1 The copy division algorithm with base-eight multiplication pieces 294
4.2.2 The copy algorithm on the base-eight chip computer 296
4.2.3 The division place-value algorithm on the base-eight chip computer 297

4.3 Activity: Division Algorithms for Base-Ten Whole Numbers 299
4.3.1 The copy division algorithm with base-ten multiplication pieces 299
4.3.2 The copy division algorithm on the base-ten chip computer 300
4.3.3 The place-value division algorithm on the base ten chip computer 301
4.4 Problem Set: Analysis of the Division Algorithms for Whole Numbers 302
4.5 Pedagogical Remarks 312

unit 5: decimals: basic operations, algorithms, and the metric system 316

5.1 Introduction 316
5.1.1 Historical background of decimals 317
5.1.2 Uses of decimal fractions 319
5.2 Activity and Problem Set: The Metric System 323
5.2.1 Language for the metric system 324
5.2.2 The metric measures 325
5.2.3 Summary of the metric system 332
5.2.4 English Metric-conversions 334
5.2.5 Developing continuing awareness of the metric system 337
5.3 Activity: The Meaning of Addition, Subtraction, Multiplication, and Division of Decimals 337
5.3.1 How the embodiments for decimals work 338
5.3.2 Relations for decimals 340
5.3.3 Addition of decimals 340
5.3.4 Subtraction of decimals 341
5.3.5 Multiplication of decimals 343
5.3.6 Division of decimals 348
5.4 Activity: Algorithms for Addition, Subtraction, Multiplication, and Division of Decimals 353
5.4.1 The chip computer for decimals 353

5.4.2 Addition and subtraction algorithms for decimals 355
5.4.3 The multiplication algorithm for decimals 356
5.4.4 A division algorithm for decimals 359
5.5 Problem Set: Analysis of Decimals 361
5.5.2 Exponential notation 365
5.5.3 Scientific notation 367
5.5.4 Approximations using powers of ten 368
5.5.5 Orders of magnitude 370
5.5.6 A brief look at logarithms 371
5.6 Pedagogical Remarks 373

unit 6: the extension of the ional numbers to the real numbers 377

6.1 Introduction—Some Reminders and Some Questions 377
6.1.1 The basic definition of rational numbers 378
6.1.2 The number line 378
6.1.3 Terminating and repeating decimals 378
6.1.4 Closure of operations 379
6.1.5 Applications of mathematics to the common world 379
6.1.6 Summary 379
6.2 Changing Rational Fraction to Decimals and Vice Versa 379
6.2.1 Changing a rational fraction to a repeating decimal 379
6.2.2 Why *must* the decimal form of a rational fraction be a repeating decimal? 381
6.2.3 Changing repeating decimals to fractions 382
6.2.4 Two equivalent ways to define "rational number" 383
6.3 Problem Set: Do Non-Rational Numbers Exist? 384
6.3.1 The square root operation 384
6.3.2 $\sqrt{2}$ as the length of a segment 386
6.3.3 $\sqrt{2}$ on the number line 387

6.3.4 At least one nonrational number exists 387
6.3.5 Some other nonrational numbers 390

6.4 Properties of the Real Numbers 393

6.5 Ratios and Proportions Again 394
6.5.1 Examples of ratios, proportions, and percents from common life situations 395
6.5.2 Some examples typical of those found on school mathematics achievement tests 396

6.6 Pedagogical Remarks 397

unit 7: analysis of different systems of numeration 399

7.1 Introduction 399

7.2 Reading and Activity: A Brief History of Number Words and Number Symbols, of Means of Recordkeeping and of Calculation with Numbers 400
7.2.1 The needs for numbers 400
7.2.2 Number words 403
7.2.3 Number symbols 408
7.2.4 Recordkeeping 411
7.2.5 Calculation 413

7.3 Activity: Some Systems of Numeration 415
7.3.1 Babylonian numeration systems 416
7.3.2 Greek numeration systems 419
7.3.3 Chinese numeration systems 423

7.4 Problem Set: Analysis of Characteristics of Numeration Systems 426
7.4.1 Six characteristics of numeration systems 427
7.4.2 Characteristics of various numeration systems 427
7.4.3 Disparities between our number words and our number symbols 429

7.5 Problem Set: Advantages of Various Numeration Systems 431
7.5.1 A comparison of the systems from Section 7.3 431
7.5.2 Advantages and disadvantages of the decimal system of numeration 433
7.6 Non-ten Number Bases 436
7.6.1 Base four 437
7.6.2 Base eight 438
7.6.3 Base five 439
7.6.4 Base twelve 440
7.6.5 Base *n* 440
7.6.6 Advantages of different bases 441
7.7 Pedagogical Remarks 442

part c underlying mathematical concepts and structures 449

unit 1: sets and logic 451

1.1 Introduction 451
1.2 Number Concepts and Basic Ideas About Sets 451
1.2.1 Piaget's number conservation tasks 452
1.2.2 One-to-one correspondence 452
1.2.3 An area problem 456
1.2.4 A "number-area" problem 457
1.2.5 A problem about subsets 458
1.2.6 Nonanalytic thinking 459
1.2.7 Centering 459
1.3 Attribute Games 460
1.3.1 One-difference sequence 460
1.3.2 Two-difference sequences 461
1.3.3 One-difference matrices 462
1.3.4 Two-difference matrices 462
1.3.5 Graeco-Latin squares 463
1.3.6 Guess the block 464
1.4 The Words *Same* and *Different* 464
1.4.1 Some issues to consider 465
1.5 SETS: A Discussion Session 468
1.6 Problems 472
1.7 Operations on Sets and Relations Between Sets 473
1.7.1 Operations on sets 475

1.7.2 Intersections 476
1.7.3 Unions 477
1.7.4 Complements 477
1.7.5 Relations between sets 478
1.7.6 Some laws about sets 479

1.8 Logic and Electrical Circuits 480
1.8.1 Parallel and series circuits 480
1.8.2 More complicated circuits 482

1.9 Logic 485
1.9.1 A brain teaser 485
1.9.2 Compound sentences and truth tables 485
1.9.3 Conditional sentences 486
1.9.4 Logical precision 488
1.9.5 The quantifiers *all*, *some*, and *none* 490
1.9.6 Validity versus truth 491
1.9.7 Logic problem for fun 492

1.10 Pedagogical Remarks 494

unit 2: relations 496

2.1 Introduction 496
2.2 Family Relationships 497
2.3 Ordered Pairs 499
2.4 Structural Properties of Relations 502
2.5 Inverses 505
2.6 Reviewing Some Properties 506
2.6.1 "Is a divisor of" 506
2.6.2 "Is a subset of" 507
2.6.3 "Is relatively prime to" 507

2.7 Equivalence Relations 508
2.7.1 Equivalence classes 509
2.7.2 Activity 511

2.8 Functions 511
2.8.1 One-to-one, one-to-many, many-to-one, and many-to-many relations 512
2.8.2 Functions 514
2.8.3 Function machines 515
2.8.4 A function machine game 516
2.8.5 Problems 518

2.9 From Relations to Coordinate Graphs 518

2.10 Pedagogical Remarks 524

unit 3: displaying information with graphs 525

3.1 Introduction 525
3.2 Organizing and Simplifying Information 530
3.2.1 Interpreting achievement test scores 530
3.3 Some Common Graphs and Their Properties 533
3.4 Selecting and Drawing Appropriate Graphs 540
3.4.1 Selecting an appropriate graph 540
3.4.2 Choosing appropriate scales 542
3.4.3 Making comparisons between sets of data 544
3.4.4 Making estimations and predictions 545
3.4.5 Supporting a position 550
3.5 Picturing Locations 551
3.6 Coordinates 553
3.6.1 A geoboard tournament 553
3.6.2 Using numbers to describe positions 554
3.6.3 Rectangular coordinates 556
3.7 Some Introductory Ideas in Analytic Geometry 557
3.8 Slide Rules 565
3.8.1 The multiplicative slide rule 565
3.8.2 The additive slide rule 567
3.9 Pedagogical Remarks 568

unit 4: operations 569

4.1 Introduction 569
4.1.1 Refreshing your memory about some basic facts 570
4.1.2 "Finger multiplication" and "The basic facts" 571
4.2 Binary Operations 571
4.2.1 Operating on colored rods 572
4.2.2 Adding rows of numbers 573
4.2.3 Multiplying rows of numbers 573
4.2.4 Turning a square 574
4.2.5 Mod four addition 576
4.3 The Closure Property 577
4.3.1 Adding colored rods 578
4.3.2 Addition mod orange 578
4.3.3 The "mod ten" number line 579
4.3.4 The "different-from" operation 580
4.4 The Associative Property 580

4.5 The Commutative Property 581
4.5.1 Invariance moves on a triangle 583
4.5.2 Permutations on poker chips 586
4.5.3 Checking for properties 589
4.6 Identity Elements 590
4.6.1 The "turned-to" operation 591
4.7 Inverse Elements 591
4.7.1 Mod ten addition 592
4.7.2 Mod ten multiplication 593
4.8 Groups 593
4.9 Isomorphisms 595
4.9.1 Multiplication mod yellow 596
4.9.2 Permutations on poker chips 597
4.9.3 Turning a square 599
4.9.4 Comparing several systems 599
4.9.5 A mystery number system 601
4.9.6 Optional projects 602
4.10 The Distributive Property 604
4.10.1 Counting numbers 604
4.10.2 Colored rods 605
4.10.3 The mod ten number line 605
4.11 Exponentiation 606
4.11.1 Some properties of exponentiation 606
4.11.2 Summarizing the properties of exponentiation 609
4.12 Summary of Properties of Operations 610
4.13 Pedagogical Remarks 612

unit 5: problem solving in number theory 613

5.1 Introduction 613
5.2 Figurate Numbers 615
5.2.1 Pyramid numbers 615
5.2.2 Triangle numbers 616
5.2.3 Finding patterns 617
5.2.4 Investigating differences 618
5.2.5 Optional problems 620
5.2.6 Problems posed by students 620
5.3 Prime and Composite Numbers 621
5.3.1 Line and rectangle numbers 622
5.3.2 The sieve of Eratosthenes 622
5.3.3 Factorization 624
5.3.4 Least common multiple 625
5.3.5 Greatest common factor 626
5.3.6 Divisibility rules 627
5.3.7 Problems 628

5.4 Finding Number Patterns 629
5.4.1 Tow tables of numbers 631
5.5 Pascal's Triangle 632
5.5.1 Situations related to Pascal's triangle 633
5.5.2 Patterns in Pascal's triangle 636
5.6 The Fibonacci Sequence 638
5.6.1 The divine proportion 639
5.6.2 Human anatomy 640
5.6.3 Fibonacci numbers in biology 641
5.6.4 Fibonacci numbers in music 641
5.6.5 Patterns in the Fibonacci sequence 642
5.7 Problems 643
5.8 Pedagogical Remarks 646

appendix I: how to get the laboratory materials for this book 649

appendix II: bibliography and references 662

index 669