

Contents

Introduction	xi
1 Gaussian measures in Hilbert spaces	1
1.1 General concepts of probability measures in Hilbert spaces	1
1.2 Gaussian probability measures	3
1.2.1 Gaussian measures in \mathbb{R}	3
1.2.2 Gaussian measures in Hilbert spaces	4
1.3 Gaussian random variables	5
1.3.1 Random variables	5
1.4 Gaussian random variables	6
1.4.1 Product measures	7
1.4.2 Independent real variables	8
1.5 Linear functional in $L^2(H, \mu)$	8
1.5.1 Linear bounded functionals	9
1.5.2 Linear unbounded functional	10
2 L^2 and Sobolev spaces with respect to a Gaussian measure	13
2.1 Some useful approximation results	15
2.2 Sobolev spaces	16
2.2.1 Generalities on closable operators	16
2.2.2 Weak derivatives in $L^2(H, \mu)$	17
2.3 The Malliavin derivative	19
2.4 Calculus rules for Malliavin derivatives	20
2.5 The basic integration by parts formula	22
2.6 The adjoint of the Malliavin derivative	23
2.6.1 Generalities on adjoint operators	23
2.6.2 The adjoint operator of M	24

3	Brownian motion	27
3.1	Generalities on stochastic Processes	27
3.1.1	The spaces $\mathcal{C}([0, T]; L^p(\Omega))$ and $C([0, T]; L^p(\Omega))$	28
3.1.2	The spaces $\mathcal{L}^p(\Omega; C([0, T]))$ and $L^p(\Omega; C([0, T]))$	28
3.2	Construction of a Brownian motion	29
3.3	The standard Brownian motion	33
3.4	Quadratic variation of the Brownian motion	34
3.5	Wiener integral	36
3.6	Multidimensional Brownian motions	40
4	Markov property of the Brownian motion	43
4.1	Cylindrical sets	44
4.2	Filtration, zero-one law	45
4.3	Stopping times	47
4.4	The Brownian motion $B(t + \tau) - B(\tau)$	50
4.5	Transition semigroup	51
4.6	Markov property	51
4.6.1	Strong Markov property	52
4.7	Reflection Principle	53
4.8	Application to partial differential equations	56
4.8.1	The Dirichlet problem	56
4.8.2	The Neumann problem	57
4.8.3	The Ventzell problem	58
5	The Itô integral	59
5.1	Itô's integral for elementary processes	60
5.2	General definition of Itô's integrals	61
5.3	Some basic spaces of processes	63
5.3.1	The space $L^2([0, T] \times \Omega, \mathcal{P}, \lambda \times \mathbb{P})$	63
5.3.2	The space $C_B([0, T]; L^p(\Omega))$	64
5.3.3	The space $L^p_B(\Omega; C([0, T]))$	64
5.4	Itô's integral for mean square continuous processes	66
5.5	The Itô integral as a stochastic process	68
5.6	Itô's integral with stopping times	70
5.7	Multidimensional Itô's integrals	71
6	The Itô formula	73
6.1	Itô's formula for real processes	74
6.1.1	Itô's formula for functions of a real Brownian motion	74
6.1.2	Itô's formula for a function of an Itô's integral	77
6.1.3	Itô's formula for unbounded functions	82

6.1.4	The Itô formula for functions of general processes	84
6.2	The Itô formula for functions of multi-dimensional processes	86
6.2.1	The Itô formula for Brownian motions	86
6.2.2	The Itô formula for functions of multi-dimensional processes	87
6.3	The Bismut-Elworthy formula	89
7	Stochastic differential equations	91
7.1	Existence and uniqueness	92
7.1.1	Differential stochastic equations with random coefficients	98
7.2	Continuous dependence on data	99
7.3	Differentiability of $X(t, s, x)$ with respect to x	101
7.3.1	Differentiability of $X(t, s, x)$ with respect to x	102
7.3.2	Existence of $X_{xx}(t, s, x)$	103
7.4	Itô Differentiability of $X(t, s, x)$ with respect to s	106
7.4.1	The deterministic case	106
7.4.2	The stochastic case	107
7.4.3	Backward Itô's formula	108
8	Transition evolution operators	111
8.1	The deterministic case	111
8.1.1	The autonomous case	113
8.2	Stochastic case	114
8.3	Markov property of $X(t, s, x)$	115
8.3.1	The Chapman-Kolmogorov equation	117
8.4	Basic properties of transition operators	117
8.5	Parabolic equations	118
8.6	Examples	121
8.7	The Bismut-Elworthy formula	122
9	Formulae of Feynman-Kac and Girsanov	125
9.1	The Feynman-Kac formula	126
9.2	The Girsanov formula	129
9.2.1	The autonomous case	133
9.3	The Girsanov transform	135
9.3.1	Change of probability	135
9.3.2	An application	139

10	One dimensional Malliavin calculus	141
10.1	Some further properties of the white noise function . . .	142
10.2	The Malliavin derivative	143
10.2.1	Malliavin derivative of Itô's integrals	144
10.2.2	The Skorohod integral	145
10.3	Malliavin derivative of a stochastic flows	147
10.4	Regularity of the transition semigroup	148
10.5	Existence of the density of the law of $X(t, x)$	150
11	Malliavin calculus in several dimensions	153
11.1	Some properties of the white noise function	153
11.2	The Malliavin derivative	154
11.3	The Clark-Ocone formula	155
11.4	The Skorohod integral	156
11.5	Malliavin derivative of a stochastic flow	157
11.6	Regularity of the transition semigroup	158
11.6.1	The case when $\det G \neq 0$	159
11.6.2	The case when $\det G = 0$	159
11.7	Existence of a density	161
12	Asymptotic behaviour of the transition semigroup	163
12.1	Preliminary properties of P_t	163
12.2	Invariant measures	165
12.2.1	The Prokhorov theorem	166
12.2.2	The Krylov-Bogoliubov theorem	166
12.3	Ergodicity	168
12.3.1	Ergodic averages	169
12.3.2	The Von Neumann theorem	170
12.3.3	Necessary and sufficient conditions for ergodicity	172
A	Conditional expectation	175
A.1	Definition	175
A.2	Basic properties of the conditional expectation	175
B	λ-systems and π-systems	179
C	Martingales	181
C.1	Definitions	181
C.2	The basic inequality for martingales	182
C.3	Square integrable martingales	183

D	Fixed points depending on parameters	185
D.1	Introduction	185
D.2	The main result	186