

# Contents

<b>Preface</b> .....	vii
<b>1 Introduction</b> .....	1
<hr/>	
<b>Part I General Mechanisms of Synchronization</b>	
<hr/>	
<b>2 General Remarks</b> .....	9
2.1 What Are We Going to Talk About? .....	9
2.2 Topics to Consider .....	10
2.3 Self-Sustained Oscillations: A Key Concept in Synchronization Theory .....	12
2.3.1 Features of Self-Oscillations .....	12
2.3.2 Features of Self-Oscillating Systems .....	13
2.3.3 Modern Revisions of the Definition of a Self-Sustained System .....	16
2.3.4 Self-Sustained Oscillations and Attractors .....	17
2.3.5 Synchronization as a Control Tool .....	17
2.4 Duality of the Description of Synchronization .....	17
2.5 Oscillations Helping Each Other Out .....	18
2.6 Terms of Bifurcations Theory .....	19
<b>3 1 : 1 Forced Synchronization of Periodic Oscillations</b> .....	21
3.1 Phase of Quasiharmonic Oscillations .....	24
3.2 Derivation of Truncated Equations for Phase Difference and Amplitude .....	26
3.3 Amplitude of Unperturbed Oscillations at Small Non-linearity .....	31
3.4 Analysis of Truncated Equations for Weak Forcing .....	32
3.5 Derivation of Truncated Equations in Descartes Coordinates .....	34
3.6 Analysis of Truncated Equations in Descartes Coordinates .....	37
3.7 Synchronization Region from the Truncated Equations: Non-bifurcational Approach .....	45
3.8 Fourier Power Spectra at Strong Forcing .....	50
3.9 Phase Locking and Suppression: Numerical Simulation .....	56

3.9.1	Phase Locking . . . . .	56
3.9.2	Suppression of Natural Dynamics . . . . .	60
3.10	Phase Locking and Suppression: Experiment . . . . .	62
3.10.1	Amplitudes from Oscilloscopes . . . . .	64
3.11	Beat Frequency: Theory, Simulations and Experiment . . . . .	67
3.11.1	Theory . . . . .	67
3.11.2	Numerical Simulation . . . . .	71
3.11.3	Experiment . . . . .	72
<b>4</b>	<b>1 : 1 Mutual Synchronization of Periodic Oscillations . . . . .</b>	<b>75</b>
4.1	Truncated Equations for Weakly Non-linear Oscillators . . . . .	77
4.2	Periodic Oscillators with Dissipative Coupling . . . . .	80
4.2.1	Symmetric Solutions . . . . .	81
4.2.2	Asymmetric Solutions . . . . .	83
4.2.3	Oscillation Death . . . . .	84
4.3	Dissipative Coupling: Numerical Simulation . . . . .	84
4.3.1	Locking . . . . .	85
4.3.2	Bifurcations . . . . .	86
4.3.3	Suppression . . . . .	87
4.4	Reactive Coupling . . . . .	89
4.4.1	Locking . . . . .	90
4.4.2	Suppression . . . . .	92
4.4.3	Bifurcations . . . . .	92
4.4.4	Phase Multistability . . . . .	94
4.5	Reactive Coupling and the Saddle Torus . . . . .	95
4.5.1	Hypothesized Structure of the Phase Space . . . . .	96
4.6	Generality of Bifurcational Transitions at Reactive Coupling . . . . .	97
4.7	Experiment . . . . .	99
4.7.1	Phase Locking . . . . .	100
4.7.2	Suppression . . . . .	101
4.8	Comparison of Synchronization Transitions in Forced and in Mutually Coupled Oscillators . . . . .	103
<b>5</b>	<b>Homoclinic Mechanism of Synchronization of Periodic Oscillations . .</b>	<b>105</b>
5.1	Global Bifurcation . . . . .	108
5.1.1	Features of a Homoclinic Bifurcation of a Cycle . . . . .	110
5.2	Homoclinics Inside Synchronization Tongue? . . . . .	111
5.3	How Homoclinics Leads to Synchronization . . . . .	114
5.4	Synchronization in a Bacteria–Viruses Model . . . . .	117
5.5	Summary . . . . .	120
<b>6</b>	<b><math>n : m</math> Synchronization of Periodic Oscillations . . . . .</b>	<b>121</b>
6.1	Important Definitions Relevant to $n : m$ Synchronization . . . . .	121
6.1.1	Poincaré Return Time . . . . .	121
6.1.2	Phase of Oscillations . . . . .	122

6.1.3 Phase of Oscillations via Poincaré Section . . . . . 122

6.1.4 Poincaré Winding (Rotation) Number . . . . . 123

6.1.5 Synchronization Order  $n : m$  . . . . . 123

6.2 1 : 1 Forced Synchronization in Weakly Non-linear Oscillators . . . . 123

6.2.1 3 : 1 Phase (Frequency) Locking . . . . . 128

6.2.2 3 : 1 Suppression of Natural Dynamics . . . . . 131

6.3  $n : m$  Synchronization in Strongly Non-linear Oscillators with Spiky Forcing . . . . . 133

6.3.1 2 : 3 Phase (Frequency) Locking . . . . . 136

6.3.2 The Route to 2 : 3 Suppression . . . . . 138

6.4 Circle Map: Derivation . . . . . 138

6.4.1 Amplitude and Phase of Oscillations . . . . . 139

6.4.2 From Differential to Discrete Equation for Phase . . . . . 141

6.5 Circle Map: Properties . . . . . 142

6.6 Arnold Tongues . . . . . 144

6.7  $n : m$  Synchronization: Experiment . . . . . 144

6.8 Summary . . . . . 147

**7 1 : 1 Forced Synchronization of Periodic Oscillations in the Presence of Noise . . . . . 149**

7.1 Introductory Comments on Random Processes . . . . . 150

7.1.1 One-Dimensional Probability Density, Mean and Variance . . 150

7.1.2 Two-Dimensional Probability Density, Correlation and Covariance . . . . . 152

7.1.3 Stationary Process . . . . . 154

7.1.4 Correlation Time . . . . . 154

7.1.5 Correlation Between Two Different Processes . . . . . 155

7.1.6 Spectrum of a Wide-Sense Stationary Process . . . . . 156

7.2 Truncated Equations . . . . . 158

7.3 Simplification of the Fluctuational Terms in Truncated Equations . . 158

7.4 Probability Density Distribution of the Phase Difference . . . . . 165

7.4.1 Case of  $Q > 0$  . . . . . 169

7.5 Bessel Functions . . . . . 170

7.6 Probability Density Distribution of the Phase Difference, Continued 172

7.7 Mean Frequency of Forced Oscillations with Noise . . . . . 174

7.8 Interpretation of Phase Dynamics . . . . . 177

7.9 Phase Diffusion . . . . . 180

7.10 Full-Scale Biological Experiment . . . . . 183

7.11 Effects of Noise on the Spectrum of a Synchronized System . . . . . 185

7.11.1 Effect of Noise on the Spectrum of Oscillations Synchronized by Suppression . . . . . 189

- 8 Chaos Synchronization** ..... 191
  - 8.1 What Is Chaos? ..... 192
    - 8.1.1 Exponential Divergence of Phase Trajectories ..... 192
    - 8.1.2 Chaos Properties in Terms of Phase Space ..... 193
    - 8.1.3 Chaos Properties in Terms of Spectra ..... 197
  - 8.2 What Does Synchronization of Chaos Encompass? ..... 197
    - 8.2.1 Chaos Synchronization: Different Manifestations ..... 197
    - 8.2.2 Chaos Synchronization in a Classical Sense ..... 198
  - 8.3 Phase and Basic Frequency of Chaotic Oscillations ..... 199
  - 8.4 Forcing Chaos Periodically: What to Expect? ..... 201
    - 8.4.1 Phase Locking of Chaos ..... 203
    - 8.4.2 Suppression of Chaos ..... 204
    - 8.4.3 Any Other Options? ..... 204
    - 8.4.4 Interacting Chaotic Systems ..... 205
  - 8.5 Synchronization of Chaos by Periodic Forcing ..... 205
    - 8.5.1 Experiment ..... 205
    - 8.5.2 Numerical Analysis ..... 211
  - 8.6 Synchronization of Periodic Oscillations by Chaos ..... 212
    - 8.6.1 Spectra ..... 213
    - 8.6.2 Poincaré Sections ..... 215
    - 8.6.3 Phase Difference ..... 216
    - 8.6.4 Lyapunov Exponents ..... 220
  - 8.7 Mutual Synchronization of Chaos ..... 222
    - 8.7.1 Phase/Frequency Locking ..... 222
    - 8.7.2 Suppression ..... 223
    - 8.7.3 Phase Behavior ..... 225
  - 8.8 Homoclinic Synchronization of Chaos ..... 227
  - 8.9 Effects of Noise on a Synchronized Chaos ..... 232
    - 8.9.1 Chaotic System Frequency-Locked by a Harmonic Signal .. 233
    - 8.9.2 Periodic System Suppressed by Chaotic Forcing ..... 237
  - 8.10 Summary ..... 237
  
- 9 Synchronization of Noise-Induced Oscillations** ..... 239
  - Stochastic Limit Cycle ..... 241
  - 9.1 Noise-Induced Oscillations ..... 242
  - 9.2 Models ..... 243
    - 9.2.1 Morris–Lecar Model ..... 243
    - 9.2.2 Monovibrator Circuit ..... 244
  - 9.3 Coherence Resonance Oscillator ..... 244
  - 9.4 Frequency and Phase Locking ..... 248
    - 9.4.1 Frequency Locking: Electronic Experiment ..... 249
    - 9.4.2 Phase Locking: Coupled Morris–Lecar Models ..... 251
    - 9.4.3 Phase Dynamics Inside the Synchronization Region:  
Electronic Experiment ..... 253
  - 9.5 Synchronization via Suppression ..... 255

**10 Conclusions to Part I** ..... 259

**Part II Case Studies in Synchronization**

**11 Synchronization of Anisochronous Oscillators** ..... 265

- 11.1 Phase Velocity Field and Coupling Vector ..... 266
- 11.2 Effective Coupling Function ..... 268
  - 11.2.1 Asymptotic Phase ..... 268
  - 11.2.2 Effective Coupling Function ..... 269
- 11.3 Dephasing ..... 270
- 11.4 Examples of 2D Anisochronous Oscillators ..... 273
- 11.5 Synchronization near the Homoclinic Bifurcation ..... 279
  - 11.5.1 Weak Coupling Limit ..... 282
  - 11.5.2 Finite Coupling Strength ..... 285
  - 11.5.3 Strong Coupling with Moderate  $\mu$  ..... 288
  - 11.5.4 Summary on Synchronization near Homoclinic Bifurcation ..... 289
- 11.6 Phase Locking Patterns of Coupled Fast-and-Slow Oscillators ..... 290
  - 11.6.1 Antiphase Locking in Coupled FitzHugh–Nagumo Models ..... 290
  - 11.6.2 Out-of-phase Synchronization via Slow Channels ..... 293
- 11.7 Synchronous Patterns in Coupled Morris–Lecar Models ..... 296
  - 11.7.1 Model ..... 296
  - 11.7.2 Overview of the Dynamics ..... 298
  - 11.7.3 Structure of Arnold Tongue for Antiphase Solution ..... 300
    - Chaotic Bursting and Torus Breakdown ..... 306
  - 11.7.4 Crises at the Boundary of Quasiperiodic Regions ..... 308
  - 11.7.5 Transition to In-phase Synchronization ..... 311
  - 11.7.6 Mechanism of Torus Folding in the Vicinity of Unstable Orbit ..... 312
  - 11.7.7 Remarks on Synchronization in Morris–Lecar Systems ..... 314
- 11.8 Summary ..... 314

**12 Phase Multistability** ..... 317

- 12.1 Period-Doubling Oscillations ..... 318
  - 12.1.1 Dynamics of Coupled Rössler Systems ..... 320
  - 12.1.2 Mapping Approach to Multistability ..... 330
- 12.2 Self-Modulated Oscillations ..... 335
  - 12.2.1 Methods of Analysis ..... 335
  - 12.2.2 Phase Dynamics of Coupled Oscillators ..... 337
- 12.3 Bursting Dynamics ..... 339
  - 12.3.1 Simple Qualitative Approach to Phase Multistability ..... 342
  - 12.3.2 Dynamics of Coupled Bursters ..... 344
  - 12.3.3 Multistability Induced by Dephasing ..... 349
- 12.4 Summary ..... 352

- 13 Synchronization in Systems with Complex Multimode Dynamics . . . .** 353
  - 13.1 Synchronization of Chaotic Systems with Fast and Slow Time Scales . . . . . 355
    - 13.1.1 Single System with Two Time Scales . . . . . 355
    - 13.1.2 Coupled Systems with Two Mode Dynamics . . . . . 360
    - 13.1.3 Conclusions . . . . . 363
  - 13.2 Generation and Synchronization of Oscillations with Several Noise-Induced Modes . . . . . 363
    - 13.2.1 Description of Experiment . . . . . 364
    - 13.2.2 Characterizing Collective Response by Spectra . . . . . 364
    - 13.2.3 Mutually Coupled Excitable Units . . . . . 365
    - 13.2.4 Three Coupled Excitable Units . . . . . 369
    - 13.2.5 Two Mutually Coupled Excitable Units with Inhibitory Coupling . . . . . 369
  - 13.3 Synchronization of Chaotic Systems with Denumerable Set of Equilibrium States . . . . . 371
  - 13.4 Summary . . . . . 376
  
- 14 Synchronization of Systems with Resource Mediated Coupling . . . . .** 377
  - 14.1 Neural Synchronization via Potassium Signaling . . . . . 379
    - 14.1.1 Model . . . . . 380
    - 14.1.2 Identical Cells: Competing In-phase and Antiphase Synchronization . . . . . 383
    - 14.1.3 Heterogeneous Cells: Dynamical Patterns . . . . . 386
  - 14.2 Multimode Dynamics in Linear Array of Electronic Oscillators . . . . 388
    - 14.2.1 Model . . . . . 388
    - 14.2.2 Clustering . . . . . 390
    - 14.2.3 Intracluster Synchronization . . . . . 393
  - 14.3 Cascaded Microbiological Oscillators . . . . . 395
    - 14.3.1 Model . . . . . 396
    - 14.3.2 Spatial Dynamics . . . . . 397
  - 14.4 Synchronization Patterns in Kidney Autoregulation . . . . . 401
    - 14.4.1 Vascular-Nephron Model . . . . . 402
    - 14.4.2 Coupling-Induced Inhomogeneity . . . . . 405
  - 14.5 Summary . . . . . 408
  
- 15 Conclusions to Part II . . . . .** 411
  - And finally . . . . . 412
  
- References . . . . .** 413
  
- Index . . . . .** 423