14

Pseudo-Anosov diffeomorphisms in pseudo-surfaces

There are diffeomorphisms on a compact surface S with uniformly hyperbolic 1 dimensional stable and unstable foliations if and only if S is a torus: the Anosov diffeomorphisms. What is happening on the other surfaces? This question leads to the study of pseudo-Anosov maps. Both Anosov and pseudo-Anosov maps appear as periodic points of the geodesic Teichmüller flow T_t on the unitary tangent bundle of the moduli space over S. We observe that the points of pseudo-Anosov maps are regular (the foliations are like the ones for the Anosov automorphisms) except for a finite set of points, called singularities, which are characterized by their number of prongs k. The stable and unstable foliations near the singularities are determined by the real and the imaginary parts of the quadratic differential $\sqrt{z^{k-2}(dz)^2}$. By a coordinate change $u(z) = z^{k/2}$ the quadratic differential $z^{k-2}(dz)^2$ gives rise to the quadratic differential $(du)^2$ and, in this new coordinates, the pseudo-Anosov maps are uniform contractions and expansions of the stable and unstable foliations. This fact inspired the construction of Pinto-Rand's pseudo-smooth structures, near the singularities, such that the pseudo-Anosov maps are smooth for this pseudo-smooth structures, and have the property that the stable and unstable foliations are uniformly contracted and expanded by the pseudo-Anosov dynamics. We define a pseudo-linear algebra, the first step in constructing the notion of the derivative of a map at a singularity. In this way, we obtain a pseudo-smooth structure at the singularity, leading to Pinto-Rand's pseudo-smooth manifolds, pseudo-smooth submanifolds, pseudo-smooth splittings and pseudo-smooth diffeomorphisms. The Stable Manifold Theorem, for pseudo-smooth manifolds, is presented giving the associated pseudo-Anosov diffeomorphisms.

14.1 Affine pseudo-Anosov maps

Let A_c be a conformal structure on a compact surface S. Two conformal structures A_c and B_c are equivalent if, and only if, there is a conformal map

h such that $A_c = h^*(B_c)$. The moduli space $M_S = \{[A_c]\}$ has a natural metric given by the minimal quasi-conformal distortion of the maps from the elements of a class $[A_c]$ to the elements of the other class $[B_c]$.

The geodesic (Teichmüller) flow T_t on the unitary tangent bundle of the moduli space has a dense set of periodic orbits. If the surface S is a torus, then the periodic points correspond to Anosov automorphisms. If the surface S is not a torus, then the periodic points correspond to pseudo-Anosov maps.

All the points of an Anosov automorphism are regular. The points of a pseudo-Anosov maps are regular, except for a finite set of points called singularities. A regular point is locally characterized by a quadratic differential $(dz)^2$. The stable and unstable foliations are determined by the real and the imaginary parts of $\sqrt{(dz)^2} = \pm dz$.

The singularities of pseudo-Anosov maps are characterized by their number of prongs k. A k-prong singularity is locally characterized by a quadratic differential $z^{k-2}(dz)^2$. The stable and unstable foliations are determined by the real and the imaginary parts of $\sqrt{z^{k-2}(dz)^2}$. If the pseudo-Anosov map has a singularity with an odd number of prongs, then the stable and unstable foliations are non-orientable.

By a coordinate change $u(z) = z^{k/2}$, the quadratic differential $z^{k-2}(dz)^2$ gives rise to the quadratic differential $(du)^2$. In this new coordinates, the pseudo-Anosov maps are locally affine contractions and expansions of the stable and unstable foliations by λ^{-1} and λ, respectively.

How can we regard the image of $u(z) = z^{k/2}$? The answer to this question leads us to the construction of Pinto-Rand's paper models, where the pseudo-Anosov maps constructed above are affine.

14.2 Paper models Σ_k

Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ denote the upper half plane with the Euclidean metric d_E. Consider the space $\sqcup_{j\in\mathbb{Z}_k}\mathbb{H}_{j\pi}$ which is the disjoint union of k copies of $\mathbb{H}$, with $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. Let the *paper models* Σ_k be the space obtained from $\sqcup_{j\in\mathbb{Z}_k}\mathbb{H}_{j\pi}$ by identifying $(x, 0) \in \mathbb{H}_{(j+1)\pi}$ with $(-x, 0) \in \mathbb{H}_{j\pi}$, for all $x \geq 0$. Let $s \in \Sigma_k$ be the point determined by $(0, 0) \in \mathbb{H}_{j\pi}$ for every $j \in \mathbb{Z}_k$. The Euclidean metric d_E on the upper half planes $\mathbb{H}_{j\pi}$ naturally define a flat metric on $\Sigma_k \setminus \{s\}$ which extends to a *continuous* metric d_k on Σ_k (see Figure 14.1).

The map $i : \mathbb{R} \to \Sigma_k$ is an *isometry* if, and only if, there is an isometry $i_{\mathbb{H}} : \mathbb{H} \to \Sigma_k$ such that $i_{\mathbb{H}}(x, 0) = i(x)$, for all $x \in \mathbb{R}$ (see Figure 14.2).

We say that:

- $l \subset \Sigma_k$ is a *straight line* in Σ_k if, and only if, there is an isometry $i : \mathbb{R} \to \Sigma_k$ such that $l = i(\mathbb{R})$;
- $l_{a\to b} \subset \Sigma_k$ is a *semi-straight line* in Σ_k, with origin at a and passing through b, if, and only if, there is an isometry $i : \mathbb{R} \to \Sigma_k$ such that $l_{a\to b} = i([a', +\infty))$ with $i(a') = a$ and $i(b') = b$, for some points $a' < b'$;

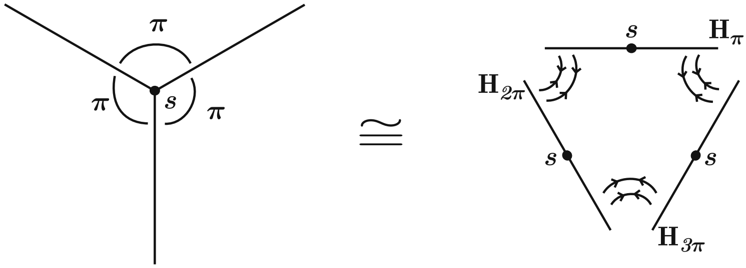

Fig. 14.1. $k = 3$.

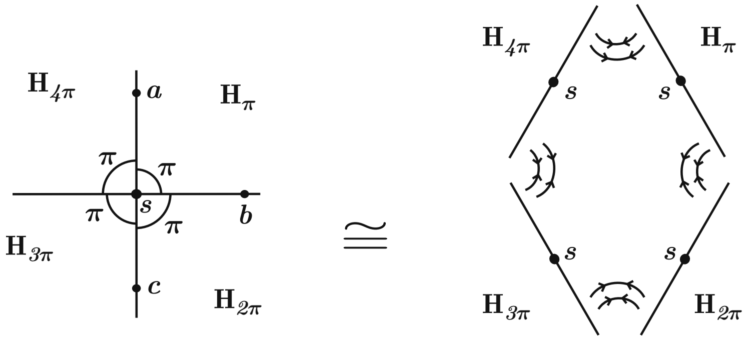

Fig. 14.2. There is a straight line passing through a and b. There is no straight line passing through a and c.

- $l_{a,b} \subset \Sigma_k$ is a *segment straight line* in Σ_k, with endpoints a and b, if, and only if, there is an isometry $i : \mathbb{R} \to \Sigma_k$ such that $l_{a,b} = i([a', b'])$ with $i(a') = a$ and $i(b') = b$, for some points $a' < b'$. The interior $\text{int}l_{a,b}$ of $l_{a,b}$ is equal to $l_{a,b} \setminus \{a, b\}$.

Let $l_{s\to a}$ and $l_{s\to b}$ be two semi-straight lines in Σ_k. To fix ideas, let us suppose that $l_{s\to a} \subset \mathbb{H}_{j\pi}$ and $l_{s\to b} \subset \mathbb{H}_{(j+n)\pi}$, with $j, j+n \in \mathbb{Z}_k$. Let $l_{s\to c}$ be the semi-straight line formed by the points of $\mathbb{H}_{j\pi}$ and $\mathbb{H}_{(j+1)\pi}$ that were identified at the construction of Σ_k. Analogously, let $l_{s\to d}$ be the semi-straight line formed by the points of $\mathbb{H}_{(j+n-1)\pi}$ and $\mathbb{H}_{(j+n)\pi}$ that were identified at the construction of Σ_k. Let $\alpha \in [0, \pi]$ be the angle $\sphericalangle(l_{s\to a}, l_{s\to c})$ between the semi-straight lines $l_{s\to a}$ and $l_{s\to c}$, and let $\beta \in [0, \pi]$ be the angle $\sphericalangle(l_{s\to d}, l_{s\to b})$ between the semi-straight lines $l_{s\to d}$ and $l_{s\to b}$. We say that the *angle* $\sphericalangle(l_{s\to a}, l_{s\to b})$ *between the semi-straight lines* $l_{s\to a}$ *and* $l_{s\to b}$ is given by

$$\sphericalangle(l_{s\to a}, l_{s\to b}) = \alpha + (n-1)\pi + \beta.$$

Given $\alpha \in \mathbb{R}/k\pi\mathbb{R}$ and two points $x, y \in \Sigma_k$, we say that they are in an *α-angular region*, if $\sphericalangle(l_{s\to x}, l_{s\to y}) \le \alpha$ (see Figure 14.3).

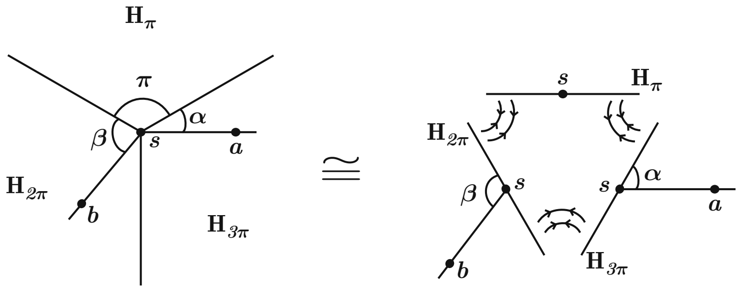

Fig. 14.3. The angle $\sphericalangle(l_{s\to a}, l_{s\to b}) = \alpha + \pi + \beta$.

14.3 Pseudo-linear algebra

Given two points $x, y \in \Sigma_k$, we say that $\mathbf{y} = y - x$ is a *vector* if, and only if, there is a segment straight line $l_{x,y} \subset \Sigma_k$ with endpoints x and y; we call x the *origin* and y the *endpoint* of the vector $y - x$. The *norm* $\|y - x\|$ of the vector $y - x$ is given by $d_k(x, y)$.

Given a vector $\mathbf{y} = y - x$ and a constant $\lambda \in \mathbb{R}$, the vector $w - x = \lambda(y - x)$ is well-defined if, and only if, there is an isometry $i_{\mathbb{H}} : \mathbb{H} \to \Sigma_k$ with the following property: there are points $x_{\mathbb{H}}, y_{\mathbb{H}}, w_{\mathbb{H}} \in \mathbb{H}$ such that

(i) $x = i_{\mathbb{H}}(x_{\mathbb{H}})$, $y = i_{\mathbb{H}}(y_{\mathbb{H}})$ and $w = i_{\mathbb{H}}(w_{\mathbb{H}})$;
(ii) $w_{\mathbb{H}} - x_{\mathbb{H}} = \lambda(y_{\mathbb{H}} - x_{\mathbb{H}})$;
(iii) if $s \in \mathrm{int} l_{x,w}$, then $s \in \mathrm{int} l_{x,y}$;
(iv) if $s = x$, then $\lambda \geq 0$.

We note that the vector $\lambda(y - x)$ is well-defined, for all $0 \leq \lambda \leq 1$. The above conditions (iii) and (iv) imply that the vector $w - x$ does not depend upon the isometry considered, and so $w - x$ is uniquely determined.

Given two vectors $\mathbf{y} = y - x$ and $\mathbf{z} = z - x$ with the same origin, the vector $\mathbf{w} = w - x$, with $\mathbf{w} = \mathbf{y} + \mathbf{z}$, is equal to the *sum* of the vectors $y - x$ with $z - x$ if, and only if, there is an isometry $i_{\mathbb{H}} : \mathbb{H} \to \Sigma_k$ with the following property: there exists a constant $\lambda > 0$ and there are points $x_{\mathbb{H}}, y_{\mathbb{H}}, z_{\mathbb{H}}, w_{\mathbb{H}} \in \mathbb{H}$ such that (see Figure 14.4)

(i) the vectors $y' - x = \lambda(y - x)$, $z' - x = \lambda(z - x)$ and $w' - x = \lambda(w - x)$ are well-defined;
(ii) $x = i_{\mathbb{H}}(x_{\mathbb{H}})$, $y' = i_{\mathbb{H}}(y_{\mathbb{H}})$, $z' = i_{\mathbb{H}}(z_{\mathbb{H}})$ and $w' = i_{\mathbb{H}}(w_{\mathbb{H}})$;
(iii) $w_{\mathbb{H}} = y_{\mathbb{H}} + z_{\mathbb{H}} - x_{\mathbb{H}}$;
(iv) if $s \in \mathrm{int} l_{x,w}$, then $s \in \mathrm{int} l_{x,y} \cup \mathrm{int} l_{x,z}$.

The above condition (iv) implies that the vector $\mathbf{w} = w - x$ does not depend upon the isometry considered. If s is a singularity, with order k, then there are k distinct vectors $x_1 - s, \ldots, x_k - s$, all with norm equal to one, such that $x_i - s + x_{i+1} - s = s - s$, for all $i \in \mathbb{Z}_k$ (see Figure 14.5).

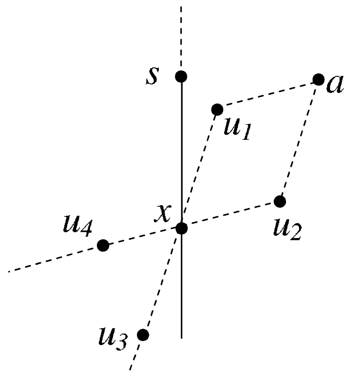

Fig. 14.4. $u_1 + u_2 = a$ and $\langle (u_1, u_3), (u_2, u_4) \rangle$ is a basis of $\mathbb{V}_x$.

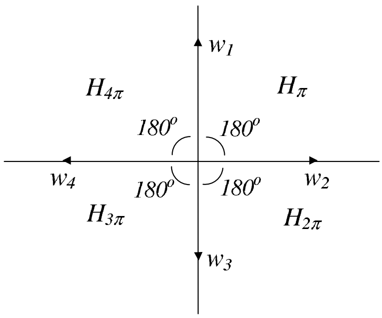

Fig. 14.5. + is not associative: $(\mathbf{w_1} + \mathbf{w_2}) + \mathbf{w_3} = \mathbf{w_3}$; $\mathbf{w_1} + (\mathbf{w_2} + \mathbf{w_3}) = \mathbf{w_1}$. There is not a unique "inverse": $\mathbf{w_1} + \mathbf{w_2} = \mathbf{0}$; $\mathbf{w_1} + \mathbf{w_4} = \mathbf{0}$, where $\mathbf{0} = s - s$. $\mathbf{w_2} + \mathbf{w_4}$ is not well-defined.

The *pseudo-linear space* $\mathbb{V}_x$ at x is the set of all vectors with origin at x, together with the operations of addition of vectors and of multiplication of a vector by a constant, as constructed above. Let l_x be either (i) the empty set or (ii) a semi-straight line contained in a semi-straight line with origin at x. The *branched linear space* $\mathbb{V}_{l_x}$ is given by $\mathbb{V}_x \setminus \mathrm{int} l_x$ (see Figure 14.6).

A *pseudo-linear subspace* $\mathbb{S}_x$ of a pseudo-linear space $\mathbb{V}_x$ (see Figure 14.7) is a subset of $\mathbb{V}_x$ with the following properties:

(i) For all $\mathbf{u}, \mathbf{v} \in \mathbb{S}_x$ such that $\mathbf{u} + \mathbf{v}$ is well-defined, we have that $\mathbf{u} + \mathbf{v} \in \mathbb{S}_x$;
(ii) For all $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{S}_x$ such that $\lambda \mathbf{u}$ is well-defined, we have that $\lambda \mathbf{u} \in \mathbb{S}_x$.

A *full pseudo-linear space* $\mathbb{S}_x$ is a pseudo-linear subspace $\mathbb{S}_x$ with the following property: If $\mathbf{u} \in \mathbb{S}_x$ and $\mathbf{v} \in \mathbb{V}_x$ are such that $\mathbf{u} + \mathbf{v} = \mathbf{0}$, then $\mathbf{v} \in \mathbb{S}_x$. Hence, a full pseudo-linear subspace $\mathbb{S}_s$, $\mathbb{S}_s \neq \mathbb{V}_s$, at the singularity s, with order k, is the image of an isometry $i : \Sigma_k^1 \to \mathbb{V}_s$.

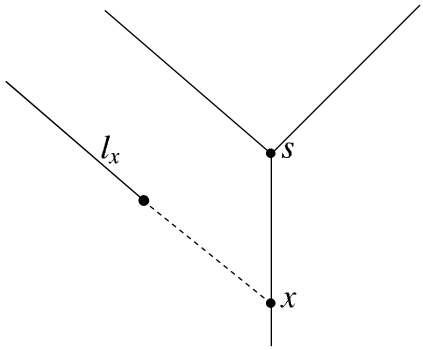

Fig. 14.6. The branched linear space $\mathbb{V}_{l_x}$.

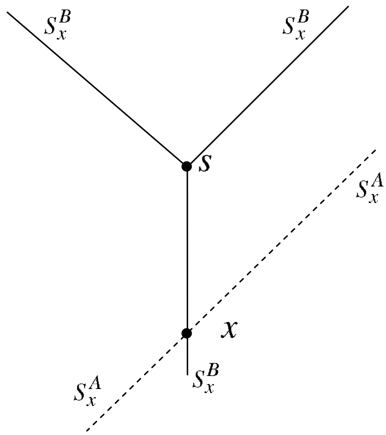

Fig. 14.7. Pseudo-linear subspaces $\mathbb{S}_x^A$ and $\mathbb{S}_x^B$ at x.

A pseudo-affine subspace $\mathbb{S}$ at a point $x \in \Sigma_k \setminus \{s\}$, with $\mathbb{S}_x \neq \mathbb{V}_x$, is the image of an isometry $i : A \to \mathbb{V}_x$ with A equal either $\mathbb{R}$ or Σ_k^1.

A map $L : \mathbb{V}_{l_x} \to \mathbb{V}_y$ is *linear* (see Figures 14.8 and 14.9), if the set $\mathbb{V}_{l_x}$ is a branched linear space in Σ_k, $\mathbb{V}_y$ is a pseudo-linear space in $\Sigma_{k'}$ and L satisfies the following properties:

(i) For every $\mathbf{v}, \mathbf{w} \in \mathbb{V}_{x,l}$ such that the vectors $\mathbf{v} + \mathbf{w}$ and $L(\mathbf{v}) + L(\mathbf{w})$ are well-defined, we have $L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$;
(ii) For every $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_{x,l}$ such that the vectors $\lambda\mathbf{v}$ and $L(\lambda\mathbf{v})$ are well-defined, we have $L(\lambda\mathbf{v}) = \lambda L(\mathbf{v})$;
(iii) $L(a - x) = s - y$, where a is the origin of l_x, $a - x \in \mathbb{V}_x$ is the vector with origin at x and $s - y \in \mathbb{V}_y$ is the vector with origin at y.

Given two linear maps $L_1 : \mathbb{V}_{l_x} \to \mathbb{V}_y$ and $L_2 : \mathbb{V}_{l_y} \to \mathbb{V}_z$, there is a unique linear map $L_3 : \mathbb{V}_{l'_x} \to \mathbb{V}_z$ such that $L_3 | \mathbb{V}_{l_x} \cap \mathbb{V}_{l'_x} = L_2 \circ L_1$, where l'_x might be

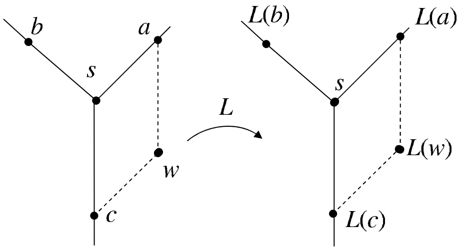

Fig. 14.8. Linear map at the singularity s.

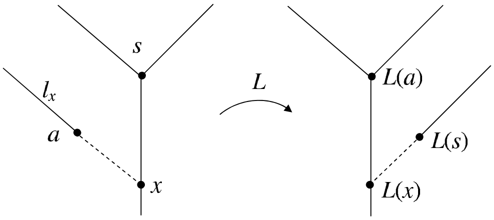

Fig. 14.9. Linear map at the point x.

distinct of l_x (see Figure 14.10). Hence, the *composition* $L_2 \circ L_1$ of two linear maps is well-defined by $L_3 = L_2 \circ L_1$, and so it is a linear map.

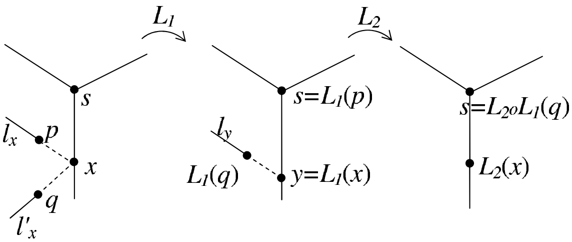

Fig. 14.10. The composition $L_3 = L_2 \circ L_1$ is well-defined.

A map $L_1 : \mathbb{V}_{l_x} \to \mathbb{V}_y$ is an *isomorphism* if, and only if, there is a linear map $L_2 : \mathbb{V}_{l_y} \to \mathbb{V}_x$ such that $L_2 \circ L_1 | \mathbb{V}_{l_x} \cap L_1^{-1}(\mathbb{V}_{l_y})$ and $L_1 \circ L_2 | \mathbb{V}_{l_y} \cap L_2^{-1}(\mathbb{V}_{l_x})$ are the identity maps. We note that if the linear map L_2 exists, then it is unique. Hence, the *inverse map* L_1^{-1} of L_1 is well-defined by $L_1^{-1} = L_2$. The kernel of a linear map $L : \mathbb{V}_{l_x} \to \mathbb{V}_y$ is equal to the intersection $\mathbb{V}_{l_x} \cap \mathbb{S}_x$ of a pseudo-linear subspace $\mathbb{S}_x$ with $\mathbb{V}_{l_x}$.

We say that a vector $y - x$ has a *parallel transport* from x to z (see Figure 14.11), if there are a vector $w - z$, a constant λ, with $|\lambda| \leq 1$, and an isometry $i_{\mathbb{H}} : \mathbb{H} \to \Sigma_k$ with the following property: there are points $x_{\mathbb{H}}, y_{\mathbb{H}}, z_{\mathbb{H}}, w_{\mathbb{H}} \in \mathbb{H}$ such that

(i) $w' - z = \lambda(w - z)$ and $y' - x = \lambda(y - x)$ are well-defined;

(ii) $x = i_{\mathbb{H}}(x_{\mathbb{H}})$, $z = i_{\mathbb{H}}(z_{\mathbb{H}})$, $y' = i_{\mathbb{H}}(y_{\mathbb{H}})$ and $w' = i_{\mathbb{H}}(w_{\mathbb{H}})$;
(iii) $w_{\mathbb{H}} - z_{\mathbb{H}} = y_{\mathbb{H}} - x_{\mathbb{H}}$;
(iv) if $s \in l_{z,w} \setminus \{w\}$, then $s \in \mathrm{int} l_{x,y}$.

The parallel transport is uniquely determined, if $s \notin l_{z,w} \setminus \{w\}$ or if $s \in l_{z,w} \setminus \{w\} \cap \mathrm{int} l_{x,y}$. Let $\mathbb{V}_{x \to z}$ be the set of all vectors that have a parallel transport from x to z. The *parallel transport map* $\mathbb{P}_{x \to z} : \mathbb{V}_{x \to z} \to \mathbb{V}_z$ is well-defined by $\mathbb{P}_{x \to z}(u) = v$, where the vector v is the parallel transport of the vector u from x to z, when $\mathbb{V}_{x \to z}$ is non-empty.

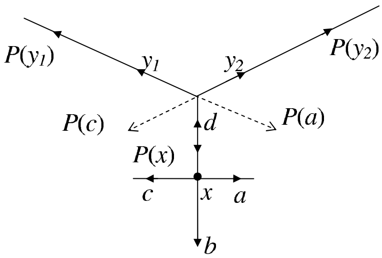

Fig. 14.11. Parallel transport from x to s.

We note that the parallel transport map $\mathbb{P}_{x \to z}$ is a linear map, except in the case where $x = s$ and $z \neq s$, because $\mathbb{P}_{s \to z}$ is just defined in an open 2π-angular region. However, $\mathbb{P}_{z \to s} : \mathbb{V}_{l_z} \to \mathbb{V}_s$ is a linear map and $\mathbb{P}_{s \to z} \circ \mathbb{P}_{z \to z} | \mathbb{V}_{l_z}$ is the identity.

We say that a map $G : \mathbb{V}_{x_1}^m \to \mathbb{V}_y$ is an *m-multilinear map*, if, for every $(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{0}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_m)$, there is $\mathbb{V}_{l_i}$, where l_i depends upon $(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{0}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_m)$, such that the map $g : \mathbb{V}_{l_i} \to \mathbb{V}_y$ defined by $g(\mathbf{a}_i) = G(\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_m)$ is a linear map.

Lemma 14.1. *Let $L : \mathbb{V}_{x_1}^m \to \mathbb{V}_{y_1}$ be an m-multilinear map. Let x_2 and y_2 be such that the parallel transport maps $\mathbb{P}_{x_1 \to x_2}$ and $\mathbb{P}_{y_1 \to y_2}$ are well-defined. Suppose that if x_1 is a singularity with order k, then x_2 is a singularity with order $2nk$, for some $n \geq 1$. Then, there is an m-multilinear map $L_P : \mathbb{V}_{x_2}^m \to \mathbb{V}_{y_2}$ such that*

$$L_P \left(P_{x_1 \to x_2}(\mathbf{v}_1), \ldots, P_{x_1 \to x_2}(\mathbf{v}_m) \right) = P_{y_1 \to y_2} \left(L(\mathbf{v}_1, \ldots, \mathbf{v}_m) \right),$$

whenever both sides are well-defined.

We call the above linear map L_P the *parallel transport of L from (x_1, y_1) to (x_2, y_2)*. We note that the parallel transport L_P of L is an isomorphism.

Proof. The map $P_{y_1 \to y_2} \circ L_1 \circ P^{-1}_{x_1 \to x_2}$ has a unique extension to a linear map. □

Let $L_1 : \mathbb{V}_{x_1}^m \to \mathbb{V}_{y_1}$ and $L_2 : \mathbb{V}_{x_1}^m \to \mathbb{V}_{y_2}$ be two m-multilinear maps. Let $0 \le h \le 1$ be such that $L_1(\mathbf{v})$ and $L_2(\mathbf{v})$ are well-defined, for all $\mathbf{v}$ with $\|\mathbf{v}\| = h$, and such that there is $\mathbf{w}(\mathbf{v})$ with the property that $\mathbf{w}(\mathbf{v}) + L_1(\mathbf{v}) = L_2(\mathbf{v})$. We define the *distance $d(L_1, L_2)$ between the m-multilinear maps L_1 and L_2* as follows:

$$d(L_1, L_2) = \begin{cases} +\infty, & \text{if } h = 0 \\ \max_{\mathbf{v}} \frac{\|\mathbf{w}(\mathbf{v})\|}{h}, & \text{otherwise} \end{cases}$$

Let $L_1 : \mathbb{V}_{x_1}^m \to \mathbb{V}_{y_1}$ and $L_2 : \mathbb{V}_{x_2}^m \to \mathbb{V}_{y_2}$ be two m-multilinear maps. Let $\mathbb{L}$ be the set of all parallel transport L_P of L_2 from (x_2, y_2) to (x_1, y_1). We define the *distance $d(L_1, L_2)$ between the m-multilinear maps L_1 and L_2* as follows:

$$d(L_1, L_2) = \begin{cases} +\infty, & \text{if } \mathbb{L} = \emptyset \\ \min_{L_P \in \mathbb{L}} d(L_1, L_P), & \text{otherwise} \end{cases}$$

We note that $d(L_1, L_2) = d(L_2, L_1)$.

14.4 Pseudo-differentiable maps

Let $f : A \subset \Sigma_k \to \Sigma_{k'}$ be a map defined on an open neighbourhood A of x in Σ_k. We say that the map f is *pseudo-differentiable at x*, if there is a linear map $D_x f : \mathbb{V}_{l_x} \to \mathbb{V}_{f(x)}$ with the following property: For all $\mathbf{v} \in \mathbb{V}_{l_x}$, there exists a constant $h_0 > 0$ such that there is a unique vector $\mathbf{w}(h, \mathbf{v})$ satisfying

$$\mathbf{w}(h, \mathbf{v}) + f(x) = f(x + h\mathbf{v}),$$

for all $0 < h < h_0$, and

$$D_x f(\mathbf{v}) = \lim_{h \to 0} \frac{\mathbf{w}(h, \mathbf{v})}{h}.$$

By induction, let us suppose that the $(m-1)^{\text{th}}$-derivative $D_x^{m-1} f : \mathbb{V}_x^m \to \mathbb{V}_{f(x)}$ of f is well-defined in an open set A containing x. We say that f is m *pseudo-differentiable at x*, if there is an m-multilinear map

$$D_x^m f : \mathbb{V}_x^m \to \mathbb{V}_{f(x)}$$

with the following property: For all $\mathbf{v} \in \mathbb{V}_x^m$, there exists a constant $h_0(\mathbf{v}) > 0$ such that there is a unique vector $\mathbf{w}(h, \mathbf{v})$ satisfying

$$\mathbf{w}(h, \mathbf{v}_1, \ldots, \mathbf{v}_m) + D_x^{m-1} f(\mathbf{v}_1, \ldots, \mathbf{v}_m) = D_{x+h\mathbf{v}_1}^{m-1} f(\mathbf{v}_2, \ldots, \mathbf{v}_m),$$

for all $0 < h < h_0(\mathbf{v})$, and

$$D_x^m f(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \lim_{h \to 0} \frac{1}{h} \mathbf{w}(h, \mathbf{v}_1, \ldots, \mathbf{v}_m).$$

A map $f : A \to \Sigma_{k'}$ is C^m, with $m \in \mathbb{N}$, in the open set $A \subset \Sigma_k$, if f is m-differentiable for all $x \in A$, and the m-derivative $D_x f$ varies continuously with x. We say that f is a $C^{m+\alpha}$, with $m \in \mathbb{N}$ and $0 < \alpha \leq 1$, if f is C^m and there exists $c > 0$ such that

$$\|D_x f - D_y f\| \leq c\|x - y\|^{\alpha},$$

for all $x, y \in A$ with the property that there is a parallel transport L_p from x to y.

We say that $B_\varepsilon = B_\varepsilon(x, s) \subset A$ is an *avoid singularity cone*, if $d(x, y) = \varepsilon d(x, s)$ and $\alpha = d(x, s)/\varepsilon$ (see Figure 14.12).

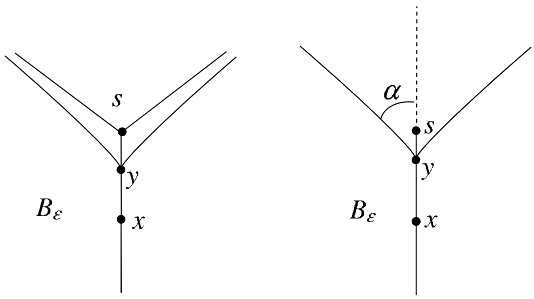

Fig. 14.12. Avoid singularity cone.

Theorem 14.2. *(Taylor's Theorem) Let $f : A \subset \Sigma_k \to \Sigma_k$ be a C^m pseudo-map defined on an open set A. Let $B_\varepsilon \subset A$ be an avoid singularity cone and $0 < \varepsilon < 1$ small. Then, for all $x, y \in B_\varepsilon$ with $\|y - x\| \leq \varepsilon$, the vectors $\mathbf{z}_m(x, y)$ and $\mathbf{w}_m(x, y)$ are well-defined by*

$$\begin{aligned} \mathbf{z}_m(x, y) &= \big(\ldots \big(D_x f(y - x) + D_x^2 f(y - x, y - x)\big) + \ldots\big) + \\ &\quad + \frac{1}{m!} D_x^m f(y - x, \ldots, y - x) \\ f(y) - f(x) &= \mathbf{z}_m(x, y) + \mathbf{w}_m(x, y). \end{aligned}$$

Furthermore,

$$\|\mathbf{w}_m(x, y)\| \leq \chi(\|y - x\|)\|y - x\|^m,$$

where $\chi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous map with $\chi(0) = 0$.

Let $l_1, \ldots, l_{2k}$ be semi-straight lines with origin at s such that $0 < \sphericalangle(l_i, l_{i+1}) < \pi$ and $\sphericalangle(l_i, l_{i+2}) = \pi$ for every $i \in \mathbb{Z}_{2k}$. Then, $S_s^1 = \cup_{i \in \mathbb{Z}_{2k}} l_{2i}$ and $S_s^2 = \cup_{i \in \mathbb{Z}_{2k}} l_{2i+1}$ are pseudo-linear subspaces at the singularity s. We call the *direct sum $S_s^1 \bigoplus S_s^2$ of S_s^1 and S_s^2* to the set of all pairs $(\mathbf{u}, \mathbf{v})$ of vectors with the property that if $\mathbf{u}_i \in l_i$, then $\mathbf{u}_{i+1} \in l_{i+1}$, for all $i \in \mathbb{Z}_{2k}$. By construction, there are one-to-one maps

$$\Theta_1 : \mathbb{V}_s \to S_s^1 \oplus S_s^2$$
$$\Theta_2 : \Sigma_k \to S_s^1 \oplus S_s^2$$

given by $\Theta_1^{-1}(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$ and $\Theta_2^{-1}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} + \mathbf{v}) + s$. We say that $\langle(\mathbf{u}_1, \ldots, \mathbf{u}_{2k-1}), (\mathbf{u}_2, \ldots, \mathbf{u}_{2k})\rangle$ is a *basis* of $\mathbb{V}_s$, if $\mathbf{u}_i \in l_i$ and $\mathbf{u}_i + \mathbf{u}_{i+2} = \mathbf{0}$, for every $i \in \mathbb{Z}_{2k}$ (see Figures 14.13 and 14.14).

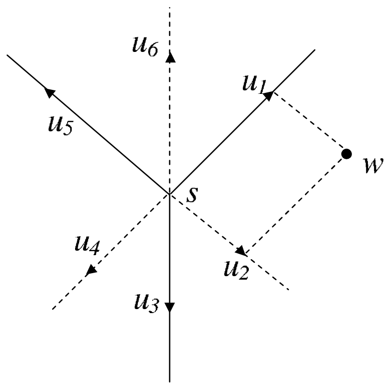

Fig. 14.13. $\mathbf{u_1} + \mathbf{u_2} = \mathbf{w}$ and $\langle(\mathbf{u_1}, \mathbf{u_3}, \mathbf{u_5}), (\mathbf{u_2}, \mathbf{u_4}, \mathbf{u_6})\rangle$ is a basis of $\mathbb{V}_s$.

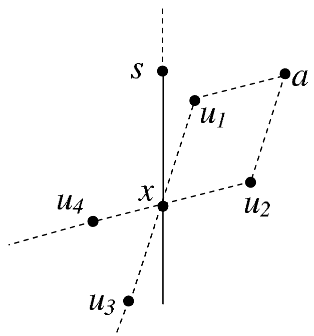

Fig. 14.14. $\mathbf{u_1} + \mathbf{u_2} = \mathbf{a}$ and $\langle(\mathbf{u_1}, \mathbf{u_3}), (\mathbf{u_2}, \mathbf{u_4})\rangle$ is a basis of $\mathbb{V}_x$.

For every $i \in \mathbb{Z}_{2k}$, let $\mathbf{u}_i \in l_i$ be such that $\|\mathbf{u_i}\| = 1$. Let $D_{K_i} = \mathbb{R}^2 \setminus ((-\infty, 0) \times \{0\})$. We define the map $K_i : D_{K_i} \to \mathbb{V}_s$ at the singularity by

$$K_i(a, b) = \begin{cases} a\mathbf{u}_i + b\mathbf{u}_{i+1}, & \text{if } a, b \geq 0 \\ a\mathbf{u}_i + b\mathbf{u}_{i-1}, & \text{if } a \geq 0, b \leq 0 \\ a\mathbf{u}_{i+2} + b\mathbf{u}_{i+1}, & \text{if } a \leq 0, b > 0 \\ a\mathbf{u}_{i-2} + b\mathbf{u}_{i-1}, & \text{if } a \leq 0, b < 0 \end{cases}$$

The set of maps $K_1, \ldots, K_{2k}$ is called a *coordinate system for* $\mathbb{V}_s$ ($\cong \Sigma_k$) given by $S_1 \bigoplus S_2$.

Lemma 14.3. *Let $K_1, \ldots, K_{2k}$ be a coordinate system for Σ_k.*

(i) Let $L : V_{l_s} \to V_{l'_s}$ *be a linear map at the singularity. Then, there is a unique linear map* $L' : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $L'(a,b) = K_j^{-1} \circ L \circ K_i = (a,b)$*, where* $j = j(i,a,b)$ *has the property that* $L \circ K_i(a,b) \in D_{K_j}$*.*
(ii) A map $f : A \to \Sigma_{k'}$ *is* C^r *on* $A \subset \Sigma_k$ *if, and only if,* $K_j^{-1} \circ f \circ K_i$ *is* C^r*, where* $j = j(i,a,b)$ *has the property that* $f \circ K_i(a,b) \in D_{K_j}$*.*

14.4.1 C^r pseudo-manifolds

Let M be a topological space. A *chart* $c : U \to \Sigma_k$ is a homeomorphism onto its image defined on an open set U of M (recall that $\Sigma_2 = \mathbb{R}^2$). If $k \neq 2$, then we call $c : U \to \Sigma_k$ a *singular chart*. A *topological atlas* $\mathcal{A}$ of M is a collection of charts

$$c_x : U_x \to \Sigma_{k_x}$$

such that the union $\cup_{x \in M} U_x$ of the open sets cover M. A C^r *pseudo-atlas* $\mathcal{A}$ of M is a topological atlas $\mathcal{A}$ of M with the following properties: (i) $\mathcal{A}$ has just a finite set of singular charts; (ii) the overlap maps

$$c_x \circ c_y^{-1} : c_y(U_x \cap U_y) \to c_x(U_x \cap U_y)$$

are C^r diffeomorphisms. A topological space M with a C^r pseudo-atlas $\mathcal{A}$ is called a C^r *pseudo-manifold*, that we will denote by the pair $(M, \mathcal{A})$. A topological space N contained in a C^r manifold $(M, \mathcal{A})$ is a *pseudo-submanifold* of M, if there is a collection $\mathcal{B}$ of charts

$$e_x : V_x \to \Sigma_{k_x}$$

with the following properties (see Figure 14.15):

(i) The set N is contained in the union $\cup_{x \in N} V_x$;
(ii) For all $x \in N$, $e_x(N \cap V_x)$ is the intersection of a pseudo-linear subspace $S_{e_x(x)}$ at $e_x(x)$ with an open set of M;
(iii) The dimension of $S_{e_x(x)}$ is 1;
(iv) The overlap maps

$$e_x \circ c_x^{-1} : c_x(U_x \cap V_x) \to e_x(U_x \cap V_x)$$

between the charts $c_x \in \mathcal{A}$ and $e_x \in \mathcal{B}$ are C^r diffeomorphisms.

Hence, the first derivative at every point is locally a bijection over a corresponding pseudo-linear subspace with dimension 1. We call the above charts e_x the *submanifold charts* of N.

Definition 14.4. *Let* $(M, \mathcal{A})$ *and* $(M', \mathcal{A}')$ *be* C^r *manifolds. The map* $f : M \to M'$ *is pseudo* C^r *if, and only if, the maps* $c_x \circ f \circ e_y^{-1}$ *are* C^r *with respect to charts* $c_x \in \mathcal{A}$ *and* $e_y \in \mathcal{A}'$*. The map* $f : M \to M'$ *is* C^r pseudo-diffeomorphism *if, and only if,* $f : M \to M'$ *is a homeomomorphism and the maps* $c_x \circ f \circ c_y^{-1}$ *are* C^r *with respect to charts* $c_x \in \mathcal{A}$ *and* $c_y \in \mathcal{A}'$*.*

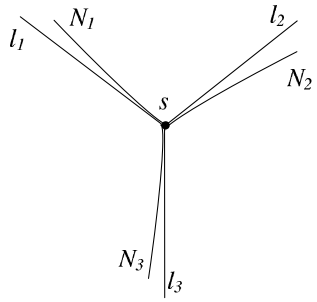

Fig. 14.15. The full subspace $\mathbb{S}_s = \cup_{i=1}^3 l_i$ at the singularity, and the pseudo-submanifold $N = \cup_{i=1}^3 N_i$.

14.4.2 Pseudo-tangent spaces

The *pseudo-tangent fiber bundle* $T\Sigma_k$ *of* Σ_k is the set $\cup_{x\in\Sigma_k}\mathbb{V}_x$, with the natural induced topology by Σ_k. We also call the pseudo-linear space $\mathbb{V}_x$ at x the *pseudo-tangent space* $T_x\Sigma_k$ *at* x ($T_x\Sigma_k \cong \mathbb{V}_x$).

The *pseudo-tangent space* T_xM *at* $x \in M$ of a C^r pseudo-manifold $(M, \mathcal{A})$ is a pseudo-linear space isomorphic to $T_{c_x(x)}\Sigma_{k_x}$, where $c_x : U_x \to \Sigma_{k_x}$ is a chart in $\mathcal{A}$ with $x \in U_x$. The *tangent fiber bundle* TM *of a* C^r *manifold* $(M, \mathcal{A})$ is the topological set $\cup_{x\in M}T_xM$, with the induced topology by the topological sets

$$\cup_{x\in U_x}T_{c_x(x)}\Sigma_{k_x}.$$

The *tangent space* T_xN *at* $x \in N$ *of a* C^r *submanifold* N of M is a pseudo-linear subspace $T_xN \subset T_xM$ isomorphic to the pseudo-linear subspace $S_{e_x(x)}$ at $e_x(x)$. The *tangent fiber subbundle* $TN \subset TM$ of a C^r submanifold N of M is the topological set $\cup_{x\in N}T_xN$.

14.4.3 Pseudo-inner product on Σ_k

Let $I_x \subset \mathbb{V}_x \times \mathbb{V}_x$ be the set of all pairs $(\mathbf{u}, \mathbf{v}) \in I_x$ such that $|\sphericalangle(\mathbf{u}, \mathbf{v})| \le \pi$. A *pseudo-inner product*

$$i : I_x \to \mathbb{R}$$

at a point $x \in \Sigma_k$ is a bi-linear map with the following properties:

- $i(\mathbf{u}, \mathbf{v}) = i(\mathbf{v}, \mathbf{u})$, for all $(\mathbf{u}, \mathbf{v}) \in I_x$;
- $i(\mathbf{u}, \mathbf{u}) \ge 0$, for all $(\mathbf{u}, \mathbf{u}) \in I_x$;
- $i(\mathbf{u}, \mathbf{u}) = 0$ if, and only if, $\mathbf{u} = 0$ $(= x - x)$.

A C^r *pseudo-Riemannian metric* in an open set $U \subset \Sigma_k$ is a map

$$<,>: \cup_{x\in U}I_x \to \mathbb{R}$$

with the following properties:

- $<,>_x = <,> |I_x$ is an inner product;
- For every isometry $i_{\mathbb{H}} : \mathbb{H} \to \Sigma_k$, the pullback by $i_{\mathbb{H}}$

$$< y-x, z-x >_{x,\mathbb{H}} = < i_{\mathbb{H}}(y) - i_{\mathbb{H}}(x), i_{\mathbb{H}}(z) - i_{\mathbb{H}}(x) >_{i_{\mathbb{H}}(x)}$$

of the inner products $<,>_{i_{\mathbb{H}}(x)}$ in U induces a C^r Riemannian metric in $i_{\mathbb{H}}^{-1}(U)$.

Let $(M, \mathcal{A})$ be a C^r manifold. Let $J_x \subset T_x M \times T_x M$ be the pull-back by the derivative of the chart $c_i : U_i \to \Sigma_{k_i}$ in $\mathcal{A}$ of $I_{c_i(x)}$. A C^r *pseudo-Riemannian metric in a* C^r *manifold* $(M, \mathcal{A})$ is a map

$$<,>: \cup_{x \in M} J_x \to \mathbb{R}$$

such that, for every chart $c_i : U_i \to \Sigma_{k_x}$ in $\mathcal{A}$, the push-forward of $<,>$ is a C^r Riemannian metric $<,>_{c_i(U_i)}$ in $c_i(U_i)$.

We say that $(x, \mathbf{u}_x)$ and $(x, \mathbf{v}_x)$ are *direction equivalent* $(x, \mathbf{u}_x) \sim (x, \mathbf{v}_x)$ if, and only if, $\mathbf{u}_x$ and $\mathbf{v}_x$ belong to a same dimension 1 full subspace $\mathbb{S}_x$. $T\Sigma_k / \sim$ is the /direction set. A C^r *direction field* is a continuous map

$$\phi : \Sigma_k \to T\Sigma_k / \sim$$

such that for every isometry $i_{\mathbb{H}} : \mathbb{H} \to \Sigma_k$, the map $\hat{\phi} : \mathbb{H} \to T\mathbb{H} / \sim$ given by $\hat{\phi} = di_{\mathbb{H}}^{-1} \circ \phi \circ i_{\mathbb{H}}$ is C^r.

A C^r *splitting* is a pair (ϕ_s, ϕ_u) of C^r direction fields such that, for every $x \in \Sigma_k$, we have

$$\mathbb{V}_x = \mathbb{S}_{\phi_s(x)} \oplus \mathbb{S}_{\phi_u(x)},$$

where $\mathbb{S}_{\phi_\iota(x)}$ is a dimension 1 full subspace containing $\phi_\iota(x)$.

Definition 14.5. *Let $(M, \mathcal{A})$ be a C^r pseudo-manifold with a pseudo-Riemannian metric. A C^r pseudo-diffeomorphism $f : M \to M$ is a C^r* pseudo-Anosov diffeomorphism, *if M has a 1 dimensional smooth splitting $E^s \oplus E^u$ of the tangent bundle, with the following properties: (i) the splitting is invariant under Tf, and (ii) Tf expands uniformly E^u and contracts uniformly E^s.*

The set of all C^r pseudo-Anosov diffeomorphisms on M is an open set.

Theorem 14.6. *(Stable Manifold Theorem) If $f : M \to M$ is a C^r pseudo-Anosov diffeomorphism, then the stable and unstable sets at the points of Λ are C^r pseudo-submanifolds with dimension 1.*

Proof. First, we prove that the stable and unstable sets through the singularities are C^r pseudo-submanifolds. Then, we prove that the stable and unstable sets through the other points are also C^r pseudo-submanifolds. The singularities are periodic points, because f is a pseudo-diffeomorphism and so the image of a singularity is a singularity with the same order. Let us construct the unstable manifold at the singularity s (for simplicity $f(s) = s$). Let

$E_{1,cut}, \ldots, E_{k,cut}$ at a singularity s be the *cut sets* represented in Figure 14.16. By the Whitney's extension theorem, there is a C^r diffeomorphism F_1 on the plane such that $F_1|_{E_{1,cut}} = f$. By the Hirsch and Pugh [48] Stable Manifold Theorem, the unstable set passing through $(0,0)$ of F_1 is a C^r submanifold $W^u = W_1^u \cup W_2^u$. Doing the same with respect to $E_{i,cut}$, we get that the unstable set

$$W^u(s) = \bigcup_{i=1}^{k} W_i^u$$

at the singularity is a C^r submanifold tangent to the unstable subspace (see Figure 14.17).

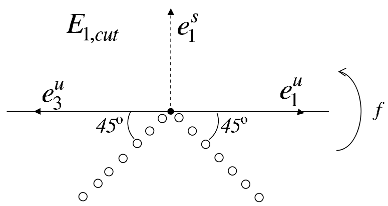

Fig. 14.16. A $E_{1,cut}$ cut set at a singularity.

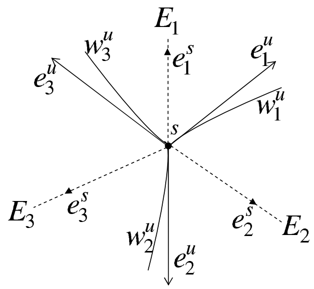

Fig. 14.17. The unstable set at a singularity $s \in \Sigma_3$.

Away from the singularities, let $(x_n)_{n\in\mathbb{Z}}$ be an orbit of f. If $x_n \in E_{i_n,cut}$, then we take the C^r diffeomorphism F_{i_n} such that $F_{i_n}|E_{i_n,cut} = f$ in a neighbourhood of x_n. Applying the Hirsch and Pugh [48] Stable Manifold Theorem to this orbit, we get that the unstable set at every point of the orbit is a C^r submanifold tangent to the unstable subspace. $\square$

14.5 C^r foliations

A C^{1+} pseudo-foliation satisfies the properties of a C^r foliation with the extra *turntable condition* that we now describe. If s is a singularity, with order $k = k(s)$, then a singular leaf W^ι on M, containing s, is such that $W^\iota \setminus \{s\}$ is the union of k disjoint leaves ℓ^ι_j, $j \in \mathbb{Z}_k$, whose closures intersect in s. The components $\ell^\iota_1, \dots, \ell^\iota_k$ of $W^\iota(s,\varepsilon) \setminus s$ are called *separatrices* of s. We call W^ι a *singular spinal set* and call the sets ℓ^ι_j emph the separatrices of s.

A C^{1+} foliation satisfies the *turntable condition*: if for all singular spinal sets W^ι with separatrices ℓ^ι_j, $j \in \mathbb{Z}_k$, there are leaf charts (i_j, ℓ^ι_j), such that the maps defined by $i_{j,l}|\ell^\iota_j = -i_j$ and $i_{j,l}|\ell^\iota_l = i_l$ are smooth. A C^{1+} foliation induced by a C^{1+} pseudo-Anosov diffeomorphism satisfies the turntable condition (see Pinto and Rand [160]).

The HR structures and the solenoid functions also apply to C^r pseudo-Anosov diffeomorphisms with the extra turntable condition that we now describe.

For any triple (v_1, v_2, v_3) of points v_1, v_2 and v_3 contained in same ι-leaf, we define the *solenoid limit* $s^z_\iota(v_1, v_2, v_3)$ as follows. For all $i \geq 0$, let

$$(z^i_1, z^i_2, z^i_3), (z^i_2, z^i_3, z^i_4), \dots, (z^i_{n_i-2}, z^i_{n_i-1}, z^i_{n_i}) \in \mathrm{sol}^\iota$$

be a sequence of triples such that for some $1 < j_i < n_i$

$$v_1 = \lim_{i\to\infty} f^i_\iota(z^i_1) \quad , \quad v_2 = \lim_{i\to\infty} f^i_\iota(z^i_{j_i}) \;\text{ and }\; v_3 = \lim_{i\to\infty} f^i_\iota(z^i_{n_i}).$$

The solenoid *limit* $s^z_\iota(v_1, v_2, v_3)$ is equal to

$$s^z_\iota(v_1, v_2, v_3) = \frac{\sum_{j=j_i-1}^{n_i-2} (s_\iota(z_1, z_2, z_3) \dots s_\iota(z_j, z_{j+1}, z_{j+2}))}{\sum_{j=1}^{j_i-2} (s_\iota(z_1, z_2, z_3) \dots s_\iota(z_j, z_{j+1}, z_{j+2}))}.$$

For all singularities s, with order $k = k(s)$, and for all $i \in \mathbb{Z}_k$, let $a_i = (v_i, s, v_{i+1})$ be a triple contained in a leaf ℓ^ι_i which intersects an ι' boundary of a Markov rectangle just in the points v_i and v_{i+1} or in the points v_i, s and v_{i+1}. The limit solenoids $s^{z_i}_\iota(a_i)$ satisfy the following *turntable condition*:

$$\prod_{i=1}^{k} s^{z_i}_\iota(a_i) = 1.$$

If $k(s) = 1$ and $v_1 = v_2$, then $s^z_\iota(v_1, s, v_2) = 1$.

The solenoid functions determined by C^r pseudo-Anosov diffeomorphisms satisfy the turntable condition (see Pinto and Rand [160]).

The train-tracks and the self-renormalizable structures also apply to C^r pseudo-Anosov diffeomorphisms with the extra turntable condition that we now describe.

A C^{1+} atlas $\mathcal{B}$ satisfies the *turntable condition* at a singularity s, with order $k = k(s)$: if for all singular spinal sets on the train-track with separatrices ℓ_j^ι, $j \in \mathbb{Z}_k$, there are leaf charts (i_j, ℓ_j^ι), such that the maps defined by $i_{j,l}|\ell_j^\iota = -i_j$ and $i_{j,l}|\ell_l^\iota = i_l$ are smooth.

A C^{1+} foliation induced by a C^{1+} pseudo-Anosov determines a C^{1+} train-track atlas satisfying the turntable condition that comes from the turntable condition of a C^{1+} foliation. For example, let s be a singularity with order 3, as in Figure 14.1. The Markov partition determines a singular spinal set S^ι with separatrices l_j^ι, $j \in \mathbb{Z}_k$, such that there are train-track charts (i_j, ℓ_j^ι), whose maps defined by $i_{j,l}|\ell_j^\iota = -i_j$ and $i_{j,l}|\ell_l^\iota = i_l$ are smooth.

14.6 Further literature

The theory developed in this book has a natural extension to C^r pseudo-Anosov diffeomorphisms using the turntable conditions (see Pinto and Rand [160]). Pinto and Pujals [155] relate the pseudo-Anosov diffeomorphisms with the Pujals and Sambarino [181, 184] non-uniformly hyperbolic diffeomorphisms. The sympletic forms are defined similarly to the Riemannian metric. Let $(M, \mathcal{A})$ be a C^r pseudo-manifold with a pseudo-volume form ω. Pinto and Viana [176] proved that there is a residual set $\mathcal{R}$ contained in the set of all C^1 pseudo-diffeomorphisms, preserving the volume form, such that if $f \in \mathcal{R}$, then either f is a C^1 pseudo-diffeomorphism or has almost everywhere both Lyapunov exponents zero. In that way we recover the duality given by Mañé-Bochi Theorem in the torus to the other surfaces. This chapter is based on Pinto [152] and Pinto and Rand [160].