

# Contents

Preface . . . . .	ix
<b>1 Introduction . . . . .</b>	<b>1</b>
1.1 A priori and a posteriori methods of error estimation . . . . .	1
1.2 Book structure . . . . .	2
1.3 The error control problem . . . . .	5
1.4 Mathematical background and notation . . . . .	8
1.4.1 Vectors and tensors . . . . .	8
1.4.2 Spaces of functions . . . . .	11
1.4.3 Inequalities . . . . .	15
1.4.4 Convex functionals . . . . .	17
<b>2 Overview . . . . .</b>	<b>22</b>
2.1 Error indicator by Runge . . . . .	22
2.2 Prager–Synge estimate . . . . .	23
2.3 Mikhlin estimate . . . . .	25
2.4 Ostrowski estimates for contractive mappings . . . . .	26
2.5 Error estimates based on monotonicity . . . . .	30
2.6 A posteriori error indicators for finite element approximations . . . . .	31
2.6.1 Explicit residual methods . . . . .	32
2.6.2 Implicit residual methods . . . . .	35
2.6.3 A posteriori estimates based on post-processing of approximate solutions . . . . .	37
2.6.4 A posteriori methods using adjoint problems . . . . .	42
<b>3 Poisson’s equation . . . . .</b>	<b>45</b>
3.1 The variational method . . . . .	45
3.2 The method of integral identities . . . . .	50
3.3 Properties of a posteriori estimates . . . . .	52
3.4 Two-sided bounds in combined norms . . . . .	57
3.5 Modifications of estimates . . . . .	59
3.5.1 Galerkin approximations . . . . .	59
3.5.2 Advanced forms of error bounds . . . . .	60
3.5.3 Decomposition of the domain . . . . .	62
3.5.4 Estimates with partially equilibrated fluxes . . . . .	64
3.6 How can one use functional a posteriori estimates in practical computations? . . . . .	65
3.6.1 Post-processing of fluxes . . . . .	65

3.6.2	Runge type estimate . . . . .	66
3.6.3	Minimization of the majorant . . . . .	66
3.6.4	Error indicators generated by error majorants . . . . .	70
<b>4</b>	<b>Linear elliptic problems . . . . .</b>	<b>75</b>
4.1	Two-sided estimates for stationary diffusion problem . . . . .	75
4.1.1	Estimates for problems with mixed boundary conditions . . . . .	75
4.1.2	Modifications of estimates . . . . .	78
4.1.3	Estimates for problems with Neumann boundary condition . . . . .	80
4.2	The stationary reaction-diffusion problem . . . . .	81
4.3	Diffusion problems with convective term . . . . .	87
4.3.1	The stationary convection-diffusion problem . . . . .	88
4.3.2	The reaction-convection-diffusion problem . . . . .	92
4.3.3	Special cases and modifications . . . . .	95
4.3.4	Estimates for fluxes . . . . .	99
4.4	Notes for the chapter . . . . .	103
<b>5</b>	<b>Elasticity . . . . .</b>	<b>104</b>
5.1	The linear elasticity problem . . . . .	104
5.2	Estimates for displacements . . . . .	107
5.3	Estimates for stresses . . . . .	109
5.4	Isotropic linear elasticity . . . . .	110
5.4.1	3D problems . . . . .	110
5.4.2	The plane stress problem . . . . .	111
5.4.3	The plane strain problem . . . . .	113
5.4.4	Error of the plane stress model . . . . .	114
5.5	Notes for the chapter . . . . .	116
<b>6</b>	<b>Incompressible viscous fluids . . . . .</b>	<b>117</b>
6.1	The Stokes problem . . . . .	117
6.2	A posteriori estimates for the stationary Stokes problem . . . . .	123
6.2.1	Estimates for the velocity field . . . . .	123
6.2.2	Estimates for pressure . . . . .	127
6.2.3	Estimates for stresses . . . . .	128
6.2.4	Estimates in combined norms . . . . .	128
6.2.5	Lower bounds of errors . . . . .	130
6.2.6	Mixed boundary conditions . . . . .	131
6.2.7	Problems for almost incompressible fluids . . . . .	137
6.2.8	Problems with the condition $\operatorname{div} u = \phi$ . . . . .	139
6.3	Generalized Stokes problem . . . . .	140
6.3.1	Estimates for solenoidal approximations . . . . .	141
6.3.2	Estimates for nonsolenoidal fields . . . . .	145
6.3.3	Estimates for the pressure field . . . . .	146

6.3.4	Error minorant . . . . .	148
6.3.5	Models with polymerization . . . . .	148
6.3.6	Models with rotation . . . . .	149
6.4	The Oseen problem . . . . .	151
6.5	Stationary Navier–Stokes problem for $d = 2$ . . . . .	153
6.6	Notes for the chapter . . . . .	156
<b>7</b>	<b>Generalizations</b> . . . . .	<b>158</b>
7.1	Linear elliptic problem . . . . .	158
7.1.1	The variational method . . . . .	159
7.1.2	The method of integral identities . . . . .	164
7.1.3	Error estimates for the dual variable . . . . .	168
7.1.4	Two-sided estimates for combined norms . . . . .	168
7.2	Elliptic problems with lower terms . . . . .	171
7.3	Problems with solutions defined in subspaces . . . . .	173
7.3.1	Abstract problem . . . . .	173
7.3.2	Estimate for approximations lying in the subspace . . . . .	173
7.3.3	Estimate for approximations lying in the energy space . . . . .	174
7.4	Derivation of a posteriori estimates from saddle point relations . . . . .	176
<b>8</b>	<b>Nonlinear problems</b> . . . . .	<b>178</b>
8.1	Variational inequalities . . . . .	178
8.1.1	Variational inequalities of the first kind . . . . .	179
8.1.2	Variational inequalities of the second kind . . . . .	185
8.2	General elliptic problem. Variational method. . . . .	186
8.3	General elliptic problem. Nonvariational method . . . . .	191
8.4	A posteriori estimates for special classes of nonlinear elliptic problems . . . . .	196
8.4.1	$\alpha$ -Laplacian . . . . .	196
8.4.2	Problems with nonlinear boundary conditions . . . . .	201
8.4.3	Generalized Newtonian fluids . . . . .	211
8.5	Notes for the chapter . . . . .	214
<b>9</b>	<b>Other problems</b> . . . . .	<b>218</b>
9.1	Differential equations of higher order . . . . .	218
9.2	Equations with the operator curl . . . . .	224
9.3	Evolutionary problems . . . . .	229
9.3.1	The linear evolutionary problem . . . . .	229
9.3.2	First form of the error majorant . . . . .	231
9.3.3	Second form of the error majorant . . . . .	235
9.3.4	Equivalence of the deviation and majorant . . . . .	238
9.3.5	Comments . . . . .	240
9.4	A posteriori estimates for optimal control problems . . . . .	242
9.4.1	Two-sided bounds for cost functionals . . . . .	243

9.4.2	Estimates for state and control functions . . . . .	248
9.4.3	Estimate in a combined norm . . . . .	251
9.4.4	Generalizations . . . . .	252
9.4.5	Comments . . . . .	254
9.5	Estimates for nonconforming approximations . . . . .	254
9.5.1	Estimates based on projecting to the energy space . . . . .	255
9.5.2	Estimates based on the Helmholtz decomposition . . . . .	257
9.5.3	Accuracy of approximations obtained by the Trefftz method . . . . .	263
9.5.4	Comments . . . . .	264
9.6	Uncertain data . . . . .	265
9.6.1	Introduction . . . . .	265
9.6.2	Errors caused by indeterminacy in coefficients . . . . .	270
9.6.3	Errors owing to uncertain $\Omega$ . . . . .	276
9.6.4	Comments . . . . .	278
9.7	Error estimates in terms of functionals and nonenergy norms . . . . .	279
9.7.1	General framework . . . . .	279
9.7.2	Estimates in local norms . . . . .	280
9.7.3	Estimates in terms of linear functionals . . . . .	281
9.7.4	Estimates based on the Poincaré inequality . . . . .	284
9.7.5	Estimates based on multiplicative inequalities . . . . .	285
9.7.6	Estimates based on the maximum principle . . . . .	285
9.7.7	Estimates in weighted norms . . . . .	287
	Bibliography . . . . .	291
	Index . . . . .	314