

# Contents

## Chapter I Introduction to Analysis of the Infinite

<b>I.1 Cartesian Coordinates and Polynomial Functions</b> .....	2
Algebra .....	2
“Algebra Nova” .....	6
Descartes’s Geometry .....	8
Polynomial Functions .....	10
Exercises .....	14
<b>I.2 Exponentials and the Binomial Theorem</b> .....	17
Binomial Theorem .....	18
Exponential Function .....	25
Exercises .....	28
<b>I.3 Logarithms and Areas</b> .....	29
Computation of Logarithms .....	30
Computation of Areas .....	33
Area of the Hyperbola and Natural Logarithms .....	34
Exercises .....	39
<b>I.4 Trigonometric Functions</b> .....	40
Basic Relations and Consequences .....	43
Series Expansions .....	46
Inverse Trigonometric Functions .....	49
Computation of Pi .....	52
Exercises .....	55
<b>I.5 Complex Numbers and Functions</b> .....	57
Euler’s Formula and Its Consequences .....	58
A New View on Trigonometric Functions .....	61
Euler’s Product for the Sine Function .....	62
Exercises .....	66
<b>I.6 Continued Fractions</b> .....	68
Origins .....	68
Convergents .....	71
Irrationality .....	76
Exercises .....	78

## Chapter II Differential and Integral Calculus

<b>II.1 The Derivative</b> .....	81
The Derivative .....	81
Differentiation Rules .....	84
Parametric Representation and Implicit Equations .....	88
Exercises .....	89
<b>II.2 Higher Derivatives and Taylor Series</b> .....	91
The Second Derivative .....	91
De Conversione Functionum in Series .....	94
Exercises .....	97
<b>II.3 Envelopes and Curvature</b> .....	98
Envelope of a Family of Straight Lines .....	98
The Caustic of a Circle .....	99
Envelope of Ballistic Curves .....	101
Curvature .....	101
Exercises .....	105
<b>II.4 Integral Calculus</b> .....	107
Primitives .....	107

Applications	109
Integration Techniques	112
Taylor's Formula with Remainder	116
Exercises	117
<b>II.5 Functions with Elementary Integral</b>	118
Integration of Rational Functions	118
Useful Substitutions	123
Exercises	125
<b>II.6 Approximate Computation of Integrals</b>	126
Series Expansions	126
Numerical Methods	128
Asymptotic Expansions	131
Exercises	132
<b>II.7 Ordinary Differential Equations</b>	134
Some Types of Integrable Equations	139
Second Order Differential Equations	140
Exercises	143
<b>II.8 Linear Differential Equations</b>	144
Homogeneous Equation with Constant Coefficients	145
Inhomogeneous Linear Equations	148
Cauchy's Equation	152
Exercises	152
<b>II.9 Numerical Solution of Differential Equations</b>	154
Euler's Method	154
Taylor Series Method	156
Second Order Equations	158
Exercises	159
<b>II.10 The Euler-Maclaurin Summation Formula</b>	160
Euler's Derivation of the Formula	160
De Usu Legitimo Formulae Summatoriae Maclaurinianae	163
Stirling's Formula	165
The Harmonic Series and Euler's Constant	167
Exercises	169
<b>Chapter III Foundations of Classical Analysis</b>	
<b>III.1 Infinite Sequences and Real Numbers</b>	172
Convergence of a Sequence	172
Construction of Real Numbers	177
Monotone Sequences and Least Upper Bound	182
Accumulation Points	184
Exercises	185
<b>III.2 Infinite Series</b>	188
Criteria for Convergence	189
Absolute Convergence	192
Double Series	195
The Cauchy Product of Two Series	197
Exchange of Infinite Series and Limits	199
Exercises	200
<b>III.3 Real Functions and Continuity</b>	202
Continuous Functions	204
The Intermediate Value Theorem	206
The Maximum Theorem	206
Monotone and Inverse Functions	208
Limit of a Function	209
Exercises	210

**III.4 Uniform Convergence and Uniform Continuity** ..... 213  
 The Limit of a Sequence of Functions ..... 213  
 Weierstrass's Criterion for Uniform Convergence ..... 216  
 Uniform Continuity ..... 217  
 Exercises ..... 220

**III.5 The Riemann Integral** ..... 221  
 Definitions and Criteria of Integrability ..... 221  
 Integrable Functions ..... 226  
 Inequalities and the Mean Value Theorem ..... 228  
 Integration of Infinite Series ..... 230  
 Exercises ..... 232

**III.6 Differentiable Functions** ..... 235  
 The Fundamental Theorem of Differential Calculus ..... 239  
 The Rules of de L'Hospital ..... 242  
 Derivatives of Infinite Series ..... 245  
 Exercises ..... 246

**III.7 Power Series and Taylor Series** ..... 248  
 Determination of the Radius of Convergence ..... 249  
 Continuity ..... 250  
 Differentiation and Integration ..... 251  
 Taylor Series ..... 252  
 Exercises ..... 255

**III.8 Improper Integrals** ..... 257  
 Bounded Functions on Infinite Intervals ..... 257  
 Unbounded Functions on a Finite Interval ..... 260  
 Euler's Gamma Function ..... 261  
 Exercises ..... 262

**III.9 Two Theorems on Continuous Functions** ..... 263  
 Continuous, but Nowhere Differentiable Functions ..... 263  
 Weierstrass's Approximation Theorem ..... 265  
 Exercises ..... 269

**Chapter IV Calculus in Several Variables**

**IV.1 Topology of  $n$ -Dimensional Space** ..... 273  
 Distances and Norms ..... 273  
 Convergence of Vector Sequences ..... 275  
 Neighborhoods, Open and Closed Sets ..... 278  
 Compact Sets ..... 283  
 Exercises ..... 285

**IV.2 Continuous Functions** ..... 287  
 Continuous Functions and Compactness ..... 289  
 Uniform Continuity and Uniform Convergence ..... 290  
 Linear Mappings ..... 293  
 Hausdorff's Characterization of Continuous Functions ..... 294  
 Integrals with Parameters ..... 297  
 Exercises ..... 298

**IV.3 Differentiable Functions of Several Variables** ..... 300  
 Differentiability ..... 302  
 Counter-Examples ..... 304  
 A Geometrical Interpretation of the Gradient ..... 305  
 The Mean Value Theorem ..... 308  
 The Implicit Function Theorem ..... 309  
 Differentiation of Integrals with Respect to Parameters ..... 311  
 Exercises ..... 313

**IV.4 Higher Derivatives and Taylor Series** ..... 316  
 Taylor Series for Two Variables ..... 319

Taylor Series for $n$ Variables .....	320
Maximum and Minimum Problems .....	323
Conditional Minimum (Lagrange Multiplier) .....	325
Exercises .....	328
<b>IV.5 Multiple Integrals</b> .....	<b>330</b>
Double Integrals over a Rectangle .....	330
Null Sets and Discontinuous Functions .....	334
Arbitrary Bounded Domains .....	336
The Transformation Formula for Double Integrals .....	338
Integrals with Unbounded Domain .....	345
Exercises .....	347
<b>Appendix: Original Quotations</b> .....	<b>351</b>
<b>References</b> .....	<b>358</b>
<b>Symbol Index</b> .....	<b>369</b>
<b>Index</b> .....	<b>371</b>