Contents

Preface ix

Introduction: Problems of Graph Theory 1
- Path Problems . . . 1
- Coloring Problems . . . 2
- Isomorphic Graphs . . . 4
- Planar Graphs . . . 4
- Disjoint Paths . . . 5
- Shortest Paths . . . 6
- ... and More . . . 7

A Basic Concepts 9
- Equivalent Graphs . . . 9
- Multigraphs . . . 10
- Directed Graphs and Mixed Graphs . . . 11
- Complete Graphs . . . 11
- Cycle Graphs . . . 11
- Paths in a Graph . . . 11
- Open and Closed Paths; Cycles . . . 12
- Subgraphs . . . 13
- The Complement of a Graph . . . 13
- Degrees of Vertices . . . 13
- The Degree Sequence of a Graph . . . 14
- Regular Graphs . . . 15
- Connected and Disconnected Graphs . . . 15
- Components of a Graph . . . 15
- More Problems . . . 16

B Isomorphic Graphs 21
- More Problems . . . 23

C Bipartite Graphs 25
Complete Bipartite Graphs . . . 26
Bipartite Graphs and Matrices . . . 26
Cycles in a Bipartite Graph . . . 27
Cycle Theorem for Bipartite Graphs . . . 27
Proof of the Cycle Theorem . . . 28
More Problems . . . 29

D Trees and Forests 31
Pruning a Tree . . . 32
Directed Trees . . . 33
Spanning Trees . . . 34
Counting Spanning Trees . . . 34
Codewords for Trees: Prufer's Method . . . 35
More Problems . . . 36

E Spanning Tree Algorithms 41
Constructing Spanning Trees . . . 41
Weighted Graphs . . . 41
Minimal Spanning Trees . . . 42
Prim's Algorithm . . . 42
Tables for Prim's Algorithm . . . 43
The Reduction Algorithm . . . 43
Spanning Trees and Shortest Paths . . . 44
Minimal Paths in a Weighted Graph . . . 45
Minimal Path Algorithm, first attempt . . . 45
Minimal Path Algorithm, revised . . . 46
Tables for Dijkstra's Algorithm . . . 46
Minimal Paths in a Directed Graph . . . 48
Negative Weights . . . 48
More Problems . . . 51

F Euler Paths 57
The Königsberg Bridge Problem . . . 57
Euler Paths in Directed Graphs and Directed Multigraphs . . . 59
Application of Euler Paths: State diagrams,
DeBruijn sequences, and rotating wheels . . . 60
More Problems . . . 61

G Hamilton Paths and Cycles 65
Some Negative Tests . . . 65
Positive Tests for Hamilton Cycles . . . 67
Some Proofs . . . 70
More Problems . . . 73

H Planar Graphs 77
Regions Formed by a Plane Diagram . . . 78
Proof that K_5 is Non-Planar, Using Euler's Formula . . . 80

Non-Planar Graphs and Kuratowski's Theorem 81
More Problems 83

I Independence and Covering **85**
The Independence Numbers of a Graph 85
A Graph Game 88
Covering Sets and Covering Numbers 89
More Problems 91

J Connections and Obstructions **95**
Internally Disjoint Paths 95
Edge-Disjoint Paths 95
Path Connection Numbers 96
Blocking Sets 96
k-Connected Graphs 98
Vertex Cut Sets and Vertex Cut Numbers 99
More Problems 100

K Vertex Coloring **103**
The Vertex Coloring Number of a Graph 103
Vertex Coloring Theorems 104
Map Coloring 110
More Problems 111

L Edge Coloring **119**
The Edge Coloring Number of a Graph 119
Edge Coloring of Complete Graphs 120
Edge Coloring of Bipartite Graphs 122
Edge Color Switch 122
Proof of Edge Coloring Theorem #3 123
Application of Edge Coloring: the Scheduling Problem 124
More Problems 124

M Matching Theory for Bipartite Graphs **131**
The Max/Min Principle 132
Proof of the König–Egervary Theorem 133
The Colored Digraph Construction 133
Matching Extension Algorithm 135
Proof of the Colored Digraph Theorem 135
Matrix Interpretation of the König–Egervary Theorem 136
Hall's Theorem and Its Consequences 138
More Problems 139

N Applications of Matching Theory **143**
Sets and Representatives 143
Latin Squares 144
Permutation Matrices 145

The Optimal Assignment Problem 146
More Problems 149

O Cycle-Free Digraphs **153**
Chains and Antichains 153
Chain Decompositions 154
Proof of Dilworth's Theorem 156
More Problems 158

P Network Flow Theory **161**
Flows in a Network 161
Cuts and Capacities 169
More Problems 172

Q Flow Problems with Lower Bounds **175**
The Supply and Demand Problem 175
More Problems 184

Answers to Selected Problems **187**

Index **201**

About the Author **205**