Contents

Introduction 1

Chapter 1. Elliptic Riemann-Roch-Grothendieck and flat vector bundles 11

1.1 The Clifford algebra 11
1.2 The standard Hodge theory 12
1.3 The Levi-Civita superconnection 14
1.4 Superconnections and Poincaré duality 15
1.5 A group action 16
1.6 The Lefschetz formula 16
1.7 The Riemann-Roch-Grothendieck theorem 17
1.8 The elliptic analytic torsion forms 19
1.9 The Chern analytic torsion forms 21
1.10 Analytic torsion forms and Poincaré duality 22
1.11 The secondary classes for two metrics 22
1.12 Determinant bundle and Ray-Singer metric 23

Chapter 2. The hypoelliptic Laplacian on the cotangent bundle 25

2.1 A deformation of Hodge theory 25
2.2 The hypoelliptic Weitzenböck formulas 29
2.3 Hypoelliptic Laplacian and standard Laplacian 30
2.4 A deformation of Hodge theory in families 33
2.5 Weitzenböck formulas for the curvature 35
2.6 $\mathfrak{F}^{\mathcal{M}}_{\phi_b,\pm\mathcal{H}-b\omega^H}$, $\mathfrak{C}^{\mathcal{M},2}_{\phi_b,\pm\mathcal{H}-b\omega^H}$ and Levi-Civita superconnection 40
2.7 The superconnection $A^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ and Poincaré duality 40
2.8 A 2-parameter rescaling 41
2.9 A group action 43

Chapter 3. Hodge theory, the hypoelliptic Laplacian and its heat kernel 44

3.1 The cohomology of $\mathbf{T}^*\mathbf{X}$ and the Thom isomorphism 44
3.2 The Hodge theory of the hypoelliptic Laplacian 45
3.3 The heat kernel for $\mathfrak{A}^2_{\phi,\mathcal{H}^c}$ 50
3.4 Uniform convergence of the heat kernel as $\mathbf{b} \to \mathbf{0}$ 53
3.5 The spectrum of $\mathfrak{A}'^2_{\phi_\mathbf{b},\pm\mathcal{H}}$ as $\mathbf{b} \to \mathbf{0}$ 55
3.6 The Hodge condition 58
3.7 The hypoelliptic curvature 60

Chapter 4. Hypoelliptic Laplacians and odd Chern forms 62

4.1 The Berezin integral 63

4.2 The even Chern forms 64
4.3 The odd Chern forms and a 1-form on $\mathbf{R}^{*2}$ 65
4.4 The limit as $\mathbf{t} \to \mathbf{0}$ of the forms $\mathbf{u}_{\mathbf{b},\mathbf{t}}, \mathbf{v}_{\mathbf{b},\mathbf{t}}, \mathbf{w}_{\mathbf{b},\mathbf{t}}$ 68
4.5 A fundamental identity 68
4.6 A rescaling along the fibers of $\mathbf{T}^*\mathbf{X}$ 69
4.7 Localization of the problem 70
4.8 Replacing $\mathbf{T}^*\mathbf{X}$ by $\mathbf{T}_{\mathbf{x}}\mathbf{X} \oplus \mathbf{T}^*_{\mathbf{x}}\mathbf{X}$ and the rescaling of Clifford variables on $\mathbf{T}^*\mathbf{X}$ 76
4.9 The limit as $t \to 0$ of the rescaled operator 80
4.10 The limit of the rescaled heat kernel 82
4.11 Evaluation of the heat kernel for $\frac{\Delta^{\mathbf{V}}}{4} + \mathbf{a}\nabla_{\mathbf{p}}$ 87
4.12 An evaluation of certain supertraces 91
4.13 A proof of Theorems 4.2.1 and 4.4.1 92

Chapter 5. The limit as $t \to +\infty$ and $b \to 0$ of the superconnection forms 98
5.1 The definition of the limit forms 98
5.2 The convergence results 101
5.3 A contour integral 102
5.4 A proof of Theorem 5.3.1 104
5.5 A proof of Theorem 5.3.2 104
5.6 A proof of the first equations in (5.2.1) and (5.2.2) 109

Chapter 6. Hypoelliptic torsion and the hypoelliptic Ray-Singer metrics 113
6.1 The hypoelliptic torsion forms 113
6.2 Hypoelliptic torsion forms and Poincaré duality 115
6.3 A generalized Ray-Singer metric on the determinant of the cohomology 116
6.4 Truncation of the spectrum and Ray-Singer metrics 120
6.5 A smooth generalized metric on the determinant bundle 122
6.6 The equivariant determinant 123
6.7 A variation formula 125
6.8 A simple identity 126
6.9 The projected connections 126
6.10 A proof of Theorem 6.7.2 127

Chapter 7. The hypoelliptic torsion forms of a vector bundle 131
7.1 The function $\tau(\mathbf{c}, \eta, \mathbf{x})$ 131
7.2 Hypoelliptic curvature for a vector bundle 133
7.3 Translation invariance of the curvature 134
7.4 An automorphism of E 135
7.5 The von Neumann supertrace of $\exp\left(-\mathfrak{L}^{\mathbf{E}}_{\mathbf{c}}\right)$ 136
7.6 A probabilistic expression for $\mathbf{Q}'_{\mathbf{c}}$ 138
7.7 Finite dimensional supertraces and infinite determinants 139
7.8 The evaluation of the form $\mathrm{Tr}_{\mathbf{s}}\left[\mathbf{g}\exp\left(-\mathfrak{L}^{\mathbf{E}}_{\mathbf{c}}\right)\right]$ 148
7.9 Some extra computations 152
7.10 The Mellin transform of certain Fourier series 155
7.11 The hypoelliptic torsion forms for vector bundles 160

Chapter 8. Hypoelliptic and elliptic torsions: a comparison formula 162
8.1 On some secondary Chern classes 162
8.2 The main result 163
8.3 A contour integral 164
8.4 Four intermediate results 165
8.5 The asymptotics of the $\mathbf{I}_{\mathbf{k}}^{\mathbf{0}}$ 166
8.6 Matching the divergences 169
8.7 A proof of Theorem 8.2.1 170

Chapter 9. A comparison formula for the Ray-Singer metrics 171

Chapter 10. The harmonic forms for $b \to 0$ and the formal Hodge theorem 173
10.1 A proof of Theorem 8.4.2 173
10.2 The kernel of $\mathbf{A}^{\mathbf{2}}_{\phi,\mathcal{H}^c}$ as a formal power series 175
10.3 A proof of the formal Hodge Theorem 178
10.4 Taylor expansion of harmonic forms near $\mathbf{b} = \mathbf{0}$ 180

Chapter 11. A proof of equation (8.4.6) 182
11.1 The limit of the rescaled operator as $\mathbf{t} \to \mathbf{0}$ 182
11.2 The limit of the supertrace as $\mathbf{t} \to \mathbf{0}$ 187
11.3 A proof of equation (8.4.6) 189

Chapter 12. A proof of equation (8.4.8) 190
12.1 Uniform rescalings and trivializations 190
12.2 A proof of (8.4.8) 192

Chapter 13. A proof of equation (8.4.7) 194
13.1 The estimate in the range $\mathbf{t} \geq \mathbf{b}^{\beta}$ 194
13.2 Localization of the estimate near $\pi^{-1}\mathbf{X}_{\mathbf{g}}$ 196
13.3 A uniform rescaling on the creation annihilation operators 198
13.4 The limit as $\mathbf{t} \to \mathbf{0}$ of the rescaled operator 200
13.5 Replacing $\mathbf{X}$ by $\mathbf{T}_{\mathbf{x}}\mathbf{X}$ 202
13.6 A proof of (13.2.11) 205
13.7 A proof of Theorem 13.6.2 206

Chapter 14. The integration by parts formula 214
14.1 The case of Brownian motion 215
14.2 The hypoelliptic diffusion 217
14.3 Estimates on the heat kernel 219
14.4 The gradient of the heat kernel 220

Chapter 15. The hypoelliptic estimates 224
15.1 The operator $\mathfrak{A}'^{2}_{\phi_b,\pm\mathcal{H}}$ 224
15.2 A Littlewood-Paley decomposition 226
15.3 Projectivization of $\mathbf{T}^*\mathbf{X}$ and Sobolev spaces 227
15.4 The hypoelliptic estimates 229
15.5 The resolvent on the real line 238
15.6 The resolvent on $\mathbf{C}$ 240

15.7 Trace class properties of the resolvent 243

Chapter 16. Harmonic oscillator and the J_0 function 247

16.1 Fock spaces and the Bargman transform 247
16.2 The operator $\mathbf{B}(\xi)$ 249
16.3 The spectrum of $\mathbf{B}(\mathbf{i}\xi)$ 251
16.4 The function $\mathbf{J_0}(\mathbf{y},\lambda)$ 253
16.5 The resolvent of $\mathbf{B}(\mathbf{i}\xi)+\mathbf{P}$ 261

Chapter 17. The limit of $\mathfrak{A}'^2_{\phi_b,\pm\mathcal{H}}$ as $b\to 0$ 264

17.1 Preliminaries in linear algebra 268
17.2 A matrix expression for the resolvent 268
17.3 The semiclassical Poisson bracket 270
17.4 The semiclassical Sobolev spaces 271
17.5 Uniform hypoelliptic estimates for $\mathbf{P_h}$ 272
17.6 The operator $\mathbf{P_h^0}$ and its resolvent $\mathbf{S_{h,\lambda}}$ for $\lambda\in\mathbf{R}$ 277
17.7 The resolvent $\mathbf{S_{h,\lambda}}$ for $\lambda\in\mathbf{C}$ 281
17.8 A trivialization over $\mathbf{X}$ and the symbols $\mathcal{S}^{\mathbf{d,k}}_{\rho,\delta,\mathbf{c}}$ 283
17.9 The symbol $\mathbf{Q_h^0}(\mathbf{x},\xi)-\lambda$ and its inverse $\mathbf{e_{0,h,\lambda}}(\mathbf{x},\xi)$ 289
17.10 The parametrix for $\mathbf{S_{h,\lambda}}$ 306
17.11 A localization property for $\mathbf{E_0},\mathbf{E_1}$ 307
17.12 The operator $\mathbf{P_\pm S_{h,\lambda}}$ 308
17.13 A proof of equation (17.12.9) 309
17.14 An extension of the parametrix to $\lambda\in\mathcal{V}$ 318
17.15 Pseudodifferential estimates for $\mathbf{P_\pm S_{h,\lambda}i_\pm}$ 319
17.16 The operator $\mathbf{\Theta_{h,\lambda}}$ 323
17.17 The operator $\mathbf{T_{h,\lambda}}$ 326
17.18 The operator $(\mathbf{J_1/J_0})\left(\mathbf{hD^X}/\sqrt{\mathbf{2}},\lambda\right)$ 329
17.19 The operator $\mathbf{U_{h,\lambda}}$ 331
17.20 Estimates on the resolvent of $\mathbf{T_{h,h^2\lambda}}$ 337
17.21 The asymptotics of $(\mathbf{L_c}-\lambda)^{-1}$ 340
17.22 A localization property 348

Bibliography 353

Subject Index 359

Index of Notation 361