CHAPTER II

The Extension Theories Based on Regularity

The theme of the present chapter is the construction of contents and measures from more primitive set functions. The construction is based on interrelated regularity and continuity conditions. These conditions are either both of outer or both of inner type. We want to demonstrate that the outer and inner theories are identical. To achieve this we have to work with the unconventional notions introduced in the first chapter, with set systems which avoid the empty set like the entire set, and with isotone set functions which take values in $\mathbb{R}$ or $\overline{\mathbb{R}}$. We start with the complete development of the outer extension theory. Then the upside-down transform method initiated in the first chapter will transform the outer into the inner extension theory. The chapter concludes with a detailed bibliographical annex.

4. The Outer Extension Theory: Concepts and Instruments

The Basic Definition

Let $\mathfrak{S}$ be a lattice in a nonvoid set X. We start with the basic definition which describes the final aim of the outer enterprise.

DEFINITION. Let $\varphi : \mathfrak{S} \to]-\infty, \infty]$ be an isotone set function $\not\equiv \infty$. For $\bullet = \star\sigma\tau$ we define an **outer $\bullet$ extension** of φ to be an extension of φ which is a $\dot{+}$ content $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ on an oval $\mathfrak{A}$, such that also $\mathfrak{S}^\bullet \subset \mathfrak{A}$ and that

α is outer regular $\mathfrak{S}^\bullet$, and

$\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous; in this connection note that $\alpha|\mathfrak{S}^\bullet > -\infty$.

We define φ to be an **outer $\bullet$ premeasure** iff it admits outer $\bullet$ extensions. Thus an outer $\bullet$ premeasure is modular and upward $\bullet$ continuous.

The principal aim is to characterize those φ which are outer $\bullet$ premeasures, and then to describe all outer $\bullet$ extensions of φ. We shall obtain a beautiful answer in natural terms. Our approach will be based on two formations due to Carathéodory: On the one hand the so-called outer measure, and on the other hand the so-called measurable sets. Both of them need substantial reformulation. These two tasks will be attacked in the present section.

The restriction $\varphi > -\infty$ imposed in the definition will be justified by success. Without it the presentation would be burdened, at least in the cases $\bullet = \sigma\tau$, with useless and unpleasant complications. We shall not pursue this point.

The Outer Envelopes

Let $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ be an isotone set function on a lattice $\mathfrak{S}$ in X. It is natural to form its **crude outer envelope** $\varphi^\star : \mathfrak{P}(X) \to \overline{\mathbb{R}}$, defined to be

$$\varphi^\star(A) = \inf\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \supset A\} \quad \text{for } A \subset X.$$

However, this set function does not allow an adequate treatment of our outer $\bullet$ extension problem for $\bullet = \sigma\tau$. The decisive idea is to form for $\bullet = \sigma$ the set function $\varphi^\sigma : \mathfrak{P}(X) \to \overline{\mathbb{R}}$, defined to be

$$\varphi^\sigma(A) = \inf\{\lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } S_l \uparrow\supset A\} \quad \text{for } A \subset X.$$

It is a variant of the traditional Carathéodory outer measure which itself will not be used below. One of the benefits of φ^σ is that it has an immediate nonsequential counterpart. This is the set function $\varphi^\tau : \mathfrak{P}(X) \to \overline{\mathbb{R}}$, defined to be

$$\varphi^\tau(A) = \inf\{\sup_{S\in\mathfrak{M}} \varphi(S) : \mathfrak{M} \text{ paving } \subset \mathfrak{S} \text{ with } \mathfrak{M} \uparrow\supset A\} \quad \text{for } A \subset X.$$

These are the three **outer envelopes** $\varphi^\bullet : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ of φ for $\bullet = \star\sigma\tau$ which will dominate the outer extension theory. From 1.3 we obtain the common formula

$$\varphi^\bullet(A) = \inf\{\sup_{S\in\mathfrak{M}} \varphi(S) : \mathfrak{M} \text{ paving } \subset \mathfrak{S} \text{ of type } \bullet \text{ with } \mathfrak{M} \uparrow\supset A\}.$$

We turn to the basic properties of these formations.

4.1. PROPERTIES. 1) $\varphi^\star|\mathfrak{S} = \varphi$. 2) $\varphi^\star \geqq \varphi^\sigma \geqq \varphi^\tau$. 3) *$\varphi^\bullet$ is isotone.* 4) *$\varphi^\bullet$ is outer regular* $[\varphi^\bullet|\mathfrak{S}^\bullet < \infty] \subset \mathfrak{S}^\bullet$. 5) *Assume that φ is submodular $\dot{+}$. Then $\varphi^\star$ is submodular $\dot{+}$, and $\varphi^\bullet$ for $\bullet = \sigma\tau$ is submodular $\dot{+}$ when either $\varphi > -\infty$ or $\varphi^\bullet < \infty$.*

Proof. 1)2)3) are obvious. 4) Fix $A \subset X$ with $\varphi^\bullet(A) < \infty$. For fixed real $c > \varphi^\bullet(A)$ there exists a paving $\mathfrak{M} \subset \mathfrak{S}$ of type $\bullet$ such that $\mathfrak{M} \uparrow$ some $M \supset A$ and $\sup_{S\in\mathfrak{M}} \varphi(S) \leqq c$. Then $M \in \mathfrak{S}^\bullet$, and by definition $\varphi^\bullet(M) \leqq c$. The assertion follows. 5) Fix $A, B \subset X$. We can assume that $\varphi^\bullet(A), \varphi^\bullet(B) < \infty$. For fixed real $a > \varphi^\bullet(A)$ and $b > \varphi^\bullet(B)$ there exist pavings $\mathfrak{M}, \mathfrak{N} \subset \mathfrak{S}$ of type $\bullet$ such that

$$\mathfrak{M} \uparrow \text{ some } M \supset A \text{ and } \sup_{S\in\mathfrak{M}} \varphi(S) \leqq a\,, \; \mathfrak{N} \uparrow \text{ some } N \supset B \text{ and } \sup_{S\in\mathfrak{N}} (T) \leqq b.$$

From them we have the pavings

$$\{S\cup T : S\in\mathfrak{M} \text{ and } T\in\mathfrak{N}\} \quad\uparrow\quad M\cup N\supset A\cup B,$$
$$\{S\cap T : S\in\mathfrak{M} \text{ and } T\in\mathfrak{N}\} \quad\uparrow\quad M\cap N\supset A\cap B.$$

Now we start with $\bullet = \star$. Here $M\in\mathfrak{M}$ and $N\in\mathfrak{N}$. Thus we have $\varphi^\star(A\cup B)\leqq\varphi(M\cup N)$ and $\varphi^\star(A\cap B)\leqq\varphi(M\cap N)$. It follows that

$$\varphi^\star(A\cup B)\dot{+}\varphi^\star(A\cap B)\leqq\varphi(M\cup N)\dot{+}\varphi(M\cap N)\leqq\varphi(M)\dot{+}\varphi(N)\leqq a+b,$$

and hence the assertion. We turn to the cases $\bullet=\sigma\tau$. We fix

$P,Q\in\mathfrak{M}$ and then $S\in\mathfrak{M}$ with $P,Q\subset S$,
$U,V\in\mathfrak{N}$ and then $T\in\mathfrak{N}$ with $U,V\subset T$,

and obtain

$$\varphi(P\cup U)\dot{+}\varphi(Q\cap V)\leqq\varphi(S\cup T)\dot{+}\varphi(S\cap T)\leqq\varphi(S)\dot{+}\varphi(T)\leqq a+b.$$

Therefore $\varphi(P\cup U),\varphi(Q\cap V)<\infty$. If some $\varphi(Q\cap V)$ is $\in\mathbb{R}$ then

$$\varphi^\bullet(A\cup B) \quad\leqq\quad \sup\{\varphi(P\cup U) : P\in\mathfrak{M} \text{ and } U\in\mathfrak{N}\}\in\mathbb{R},$$
$$\varphi^\bullet(A\cap B) \quad\leqq\quad \sup\{\varphi(Q\cap V) : Q\in\mathfrak{M} \text{ and } V\in\mathfrak{N}\}\in\mathbb{R},$$

and hence $\varphi^\bullet(A\cup B)\dot{+}\varphi^\bullet(A\cap B)\leqq a+b$. If not then $\varphi^\bullet(A\cap B)=-\infty$, and by assumption $\varphi^\bullet(A\cup B)<\infty$. Both times the assertion follows.

We shall later need a counterpart of the first assertion in 4.1.5) for supermodular $\dot{+}$.

4.2. Remark. *Let φ be supermodular $\dot{+}$. Assume that $A,B\subset X$ are* **separated $\mathfrak{S}$** *in the sense that*

for each $M\in\mathfrak{S}$ with $A\cap B\subset M$
there exist $S,T\in\mathfrak{S}$ with $A\subset S$ and $B\subset T$ such that $S\cap T\subset M$.

Then $\varphi^\star(A\cup B)\dot{+}\varphi^\star(A\cap B)\geqq\varphi^\star(A)\dot{+}\varphi^\star(B)$.

Proof. We can assume that $\varphi^\star(A\cup B)<\infty$ and hence all other values $\varphi^\star(\cdot)<\infty$ as well. Fix $M,N\in\mathfrak{S}$ with

$$M\supset A\cap B \text{ and } \varphi(M)<\infty,\quad N\supset A\cup B \text{ and } \varphi(N)<\infty.$$

Then choose $S,T\in\mathfrak{S}$ as assumed. It follows that

$$\begin{aligned}\varphi(N)+\varphi(M) &\geqq \varphi\big(N\cap(S\cup T)\big)+\varphi\big(N\cap(S\cap T)\big)\\ &= \varphi\big((N\cap S)\cup(N\cap T)\big)+\varphi\big((N\cap S)\cap(N\cap T)\big)\\ &\geqq \varphi(N\cap S)+\varphi(N\cap T)\geqq\varphi^\star(A)+\varphi^\star(B).\end{aligned}$$

This implies the assertion.

4.3. Exercise. *Let $\varnothing\in\mathfrak{S}$ and $\varphi(\varnothing)=0$ and φ be superadditive. Assume that $A,B\subset X$ with $A\cap B=\varnothing$ are separated $\mathfrak{S}$, that is*

there exist $S,T\in\mathfrak{S}$ with $A\subset S$ and $B\subset T$ such that $S\cap T=\varnothing$.

Then $\varphi^\star(A\cup B)\geqq\varphi^\star(A)+\varphi^\star(B)$.

4.4. Exercise. The second assertion in 4.1.5) becomes false without additional assumptions. Hint for an example: Let X be the disjoint union of two infinite countable subsets U and V, and let $\mathfrak{S}$ consist of its finite subsets. Define $\varphi : \mathfrak{S} \to [-\infty, \infty[$ to be $\varphi(S) = \#(S)$ if S meets both U and V, and $\varphi(S) = -\infty$ otherwise.

In contrast to $\varphi^\star|\mathfrak{S} = \varphi$ the relation $\varphi^\bullet|\mathfrak{S} = \varphi$ need not be true for $\bullet = \sigma\tau$. It can be characterized as follows.

4.5. Proposition. *For an isotone set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ and $\bullet = \sigma\tau$ the following are equivalent.*

i) $\varphi^\bullet|\mathfrak{S} = \varphi$;
ii) *φ is upward $\bullet$ continuous.*

In this case we have furthermore

iii) *$\varphi^\bullet|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous;*
iv) *if $\{S \in \mathfrak{S}^\bullet : \varphi^\bullet(S) < \infty\} \subset \mathfrak{S}$ then $\varphi^\bullet = \varphi^\star$.*

Proof. i) $\Rightarrow$ ii) Let $A \in \mathfrak{S}$ and $\mathfrak{M} \subset \mathfrak{S}$ be a paving of type $\bullet$ with $\mathfrak{M} \uparrow A$. By i) and the definition of $\varphi^\bullet$ then

$$\varphi(A) = \varphi^\bullet(A) \leqq \sup_{S \in \mathfrak{M}} \varphi(S) \text{ and hence } = \sup_{S \in \mathfrak{M}} \varphi(S).$$

ii) $\Rightarrow$ i) Let $A \in \mathfrak{S}$ and $\mathfrak{M} \subset \mathfrak{S}$ be a paving of type $\bullet$ with $\mathfrak{M} \uparrow\supset A$. Then $\{S \cap A : S \in \mathfrak{M}\}$ is a paving $\subset \mathfrak{S}$ of type $\bullet$ with $\uparrow A$. By ii) therefore

$$\varphi(A) = \sup_{S \in \mathfrak{M}} \varphi(S \cap A) \leqq \sup_{S \in \mathfrak{M}} \varphi(S).$$

It follows that $\varphi(A) \leqq \varphi^\bullet(A)$, and hence from $\varphi^\bullet(A) \leqq \varphi^\star(A) \leqq \varphi(A)$ the assertion. i) $\Rightarrow$ iv) Assume that this is false. Fix $A \subset X$ with $\varphi^\bullet(A) < \varphi^\star(A)$. By 4.1.4) there exists $S \in \mathfrak{S}^\bullet$ with $S \supset A$ and $\varphi^\bullet(S) < \varphi^\star(A)$. By assumption then $S \in \mathfrak{S}$ and $\varphi(S) = \varphi^\bullet(S) < \varphi^\star(A)$. This is a contradiction.

The most involved part of the proof is for the implication i) $\Rightarrow$ iii). We first prove a lemma.

4.6. Lemma. *Let $\mathfrak{M} \subset \mathfrak{S}^\bullet$ be a paving of type $\bullet$ with $\mathfrak{M} \uparrow A$. Then of course $A \in \mathfrak{S}^\bullet$. Furthermore there exists a paving $\mathfrak{N} \subset \mathfrak{S}$ of type $\bullet$ with $\mathfrak{N} \uparrow A$ such that $\mathfrak{N} \subset (\sqsubset \mathfrak{M})$.*

Proof. Nontrivial are the cases $\bullet = \sigma\tau$. The case $\bullet = \sigma$: Choose a sequence $(M_n)_n$ in $\mathfrak{M}$ with $M_n \uparrow$ such that each member of $\mathfrak{M}$ is contained in some M_n. Then $M_n \uparrow A$. Now for each $n \in \mathbb{N}$ there exists a sequence $(S_n^l)_l$ in $\mathfrak{S}$ with $S_n^l \uparrow M_n$. We put $S_l := S_1^l \cup \cdots \cup S_l^l \in \mathfrak{S}$. Then $S_l \uparrow$some $S \subset X$. We have on the one hand $S_l \subset M_1 \cup \cdots \cup M_l = M_l$, and on the other hand $S_l \supset S_n^l$ for $1 \leqq n \leqq l$. It follows that $S \subset A$ and $S \supset M_n$ for all $n \in \mathbb{N}$, so that $S = A$. Thus the paving $\mathfrak{N} := \{S_l : l \in \mathbb{N}\} \subset \mathfrak{S}$ is as required. The case $\bullet = \tau$: Define $\mathfrak{N} := \{S \in \mathfrak{S} : S \subset \text{some } M \in \mathfrak{M}\} \subset \mathfrak{S}$. Then $\mathfrak{N}$ is nonvoid and has $\cup$ and is therefore upward directed. Furthermore

$$\bigcup_{S\in\mathfrak{N}} S = \bigcup_{M\in\mathfrak{M}} \bigcup_{S\in\mathfrak{S}, S\subset M} S = \bigcup_{M\in\mathfrak{M}} M = A.$$

Thus $\mathfrak{N}$ is as required.

Proof of 4.5.i) $\Rightarrow$ iii). Consider a paving $\mathfrak{M} \subset \mathfrak{S}^\bullet$ of type $\bullet$ with $\mathfrak{M} \uparrow A \in \mathfrak{S}^\bullet$, and take $\mathfrak{N} \subset \mathfrak{S}$ as obtained in 4.6. By i) then

$$\varphi(S) = \varphi^\bullet(S) \leqq \sup_{M\in\mathfrak{M}} \varphi^\bullet(M) \text{ for each } S \in \mathfrak{N},$$

and therefore by definition

$$\varphi^\bullet(A) \leqq \sup_{S\in\mathfrak{N}} \varphi(S) \leqq \sup_{M\in\mathfrak{M}} \varphi^\bullet(M).$$

The assertion follows.

The most remarkable fact about the outer envelopes is the sequential continuity theorem which follows. Note that there is no continuity assumption on the set function φ itself.

4.7. THEOREM. *Assume that $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ is isotone and submodular $\dot{+}$. Then φ^σ and φ^τ are almost upward σ continuous.*

4.8. LEMMA. *Assume that $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ is isotone and submodular $\dot{+}$. For $P_1, \cdots, P_n, Q \in \mathfrak{S}$ with $\varphi(P_1), \cdots, \varphi(P_n), \varphi(Q) < \infty$ then $\varphi(P_1 \cup \cdots \cup P_n \cup Q) < \infty$ and*

$$\varphi(P_1 \cup \cdots \cup P_n \cup Q) + \sum_{l=1}^{n} \varphi(P_l \cap Q) \leqq \sum_{l=1}^{n} \varphi(P_l) + \varphi(Q).$$

Proof of 4.8. The case $n = 1$ is obvious. The induction step $1 \leqq n \Rightarrow n+1$: Let $P_0, P_1, \cdots, P_n, Q \in \mathfrak{S}$ with $\varphi(P_0), \varphi(P_1), \cdots, \varphi(P_n), \varphi(Q) < \infty$. We know from 2.4 that $[\varphi < \infty]$ is a lattice. Thus from the induction hypothesis we obtain

$$\begin{aligned}
&\varphi(P_0 \cup P_1 \cup \cdots \cup P_n \cup Q) + \sum_{l=0}^{n} \varphi(P_l \cap Q) \\
&= \varphi(P_1 \cup \cdots \cup P_n \cup (P_0 \cup Q)) + \sum_{l=1}^{n} \varphi(P_l \cap Q) + \varphi(P_0 \cap Q) \\
&\leqq \sum_{l=1}^{n} \varphi(P_l) + \varphi(P_0 \cup Q) + \varphi(P_0 \cap Q) \leqq \sum_{l=0}^{n} \varphi(P_l) + \varphi(Q).
\end{aligned}$$

Proof of 4.7. We fix a sequence $(A_n)_n$ of subsets of X with $A_n \uparrow A$ and $\varphi^\bullet(A_n) > -\infty\ \forall n \in \mathbb{N}$. Then $\varphi^\bullet(A_n) \uparrow R \leqq \varphi^\bullet(A)$. To be shown is $\varphi^\bullet(A) \leqq R$. We can assume that $R < \infty$, so that the $\varphi^\bullet(A_n)$ and R are finite. We fix $\varepsilon > 0$, and then for each $n \in \mathbb{N}$ a paving $\mathfrak{M}(n) \subset \mathfrak{S}$ of type $\bullet$ such that $\mathfrak{M}(n) \uparrow$ some $M_n \supset A_n$ and

$$\sup_{S\in\mathfrak{M}(n)} \varphi(S) \leqq \varphi^\bullet(A_n) + \varepsilon 2^{-n-1}.$$

1) We claim that

$$\varphi(S_1 \cup \cdots \cup S_n) < R + \varepsilon \text{ for } S_l \in \mathfrak{M}(l)\ (l = 1, \cdots, n) \text{ and } n \in \mathbb{N}.$$

To see this fix $l \in \{1, \cdots, n\}$. Then $\{P \cap Q : P \in \mathfrak{M}(l) \text{ and } Q \in \mathfrak{M}(n+1)\}$ is a paving $\subset \mathfrak{S}$ of type $\bullet$ which $\uparrow M_l \cap M_{n+1} \supset A_l \cap A_{n+1} = A_l$. Hence there exist $P_l \in \mathfrak{M}(l)$ and $Q_l \in \mathfrak{M}(n+1)$ such that $\varphi(P_l \cap Q_l) \geqq \varphi^\bullet(A_l) - \varepsilon 2^{-l-1}$. We can assume that $P_l \supset S_l$. Also there exists $Q \in \mathfrak{M}(n+1)$ with $Q \supset Q_1 \cup \cdots \cup Q_n$. It follows that

$$\varphi(P_l \cap Q) \geqq \varphi^\bullet(A_l) - \varepsilon 2^{-l-1} \; (l = 1, \cdots, n).$$

Now from 4.8 we have $\varphi(P_1 \cup \cdots \cup P_n \cup Q) < \infty$ and

$$\varphi(P_1 \cup \cdots \cup P_n \cup Q) + \sum_{l=1}^{n} \varphi(P_l \cap Q) \leqq \sum_{l=1}^{n} \varphi(P_l) + \varphi(Q).$$

From the above we see that in this formula

$$\begin{aligned} \text{the left side is} \quad &\geqq \quad \varphi(S_1 \cup \cdots \cup S_n) + \sum_{l=1}^{n} \big(\varphi^\bullet(A_l) - \varepsilon 2^{-l-1}\big), \\ \text{the right side is} \quad &\leqq \quad \sum_{l=1}^{n} \big(\varphi^\bullet(A_l) + \varepsilon 2^{-l-1}\big) + \big(\varphi^\bullet(A_{n+1}) + \varepsilon 2^{-n-2}\big). \end{aligned}$$

It follows that

$$\varphi(S_1 \cup \cdots \cup S_n) < \varphi^\bullet(A_{n+1}) + \sum_{l=1}^{n+1} \varepsilon 2^{-l} < R + \varepsilon.$$

2) Let $\mathfrak{M}$ consist of all unions $S_1 \cup \cdots \cup S_n$ with $S_l \in \mathfrak{M}(l)$ $(l = 1, \cdots, n)$ and $n \in \mathbb{N}$. Then $\mathfrak{M}$ is a paving $\subset \mathfrak{S}$ of type $\bullet$. It is clear that $\mathfrak{M}$ is upward directed and $\mathfrak{M} \uparrow \bigcup_{n=1}^{\infty} M_n \supset \bigcup_{n=1}^{\infty} A_n = A$. Thus we have $\varphi^\bullet(A) \leqq \sup_{S \in \mathfrak{M}} \varphi(S)$. Combined with 1) we obtain $\varphi^\bullet(A) \leqq R + \varepsilon$ for all $\varepsilon > 0$ and hence the assertion.

Complements for the Nonsequential Situation

The sequential continuity theorem 4.7 has no nonsequential counterpart. The present subsection is a short discussion of the complications which arise from this fact.

4.9. Remark. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ and $\varphi : \mathfrak{S} \to [0, \infty]$ be an isotone and modular set function with $\varphi(\varnothing) = 0$ which attains at least one finite positive value. Assume that φ is upward τ continuous and that $\mathfrak{S}^\tau = \mathfrak{S}$. Then 4.5 implies that $\varphi^\tau = \varphi^\sigma = \varphi^\star \geqq 0$. In this situation it can happen that $\varphi^\tau = \varphi^\sigma = \varphi^\star$ is not upward τ continuous. For example this is obvious when $\varphi^\tau(F) = \varphi^\sigma(F) = \varphi^\star(F) = 0$ for all finite $F \subset X$. As the simplest example we anticipate from 5.14 the Lebesgue measure on $\mathbb{R}^n$ restricted to $\mathrm{Op}(\mathbb{R}^n)$.

However, the outer $\bullet$ main theorem requires a certain touch of upward $\bullet$ continuity which is trivial for $\bullet = \star$ and a consequence of 4.7 for $\bullet = \sigma$.

Let us define an isotone set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ to be **upward • essential** iff

$$\begin{aligned} \varphi^\bullet(A) &= \sup\{\varphi^\bullet(A \cap S) : S \in [\varphi < \infty]\} \quad \text{for all } A \subset X \text{ with} \\ \infty > \varphi^\bullet(A) &\geqq \sup\{\varphi^\bullet(A \cap S) : S \in [\varphi < \infty]\} > -\infty. \end{aligned}$$

Then we obtain what follows.

4.10. PROPOSITION. *Let $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ be isotone. $\star$) φ is upward $\star$ essential. σ) If φ is submodular $\dot{+}$ then it is upward σ essential. τ) Assume that φ is submodular $\dot{+}$ and such that each $A \subset X$ with $\varphi^\tau(A) < \infty$ is upward enclosable $[\varphi < \infty]^\sigma$. Then φ is upward τ essential.*

Proof. $\star$) is obvious since $\varphi^\star(A) < \infty$ implies the existence of some $S \in [\varphi < \infty]$ with $S \supset A$ and hence $A \cap S = A$. $\sigma)\tau$) We prove for $\bullet = \sigma\tau$ and φ submodular $\dot{+}$ an intermediate assertion which implies both results: If $A \subset X$ is upward enclosable $[\varphi < \infty]^\sigma$ and fulfils

$$\begin{aligned} &\sup\{\varphi^\bullet(A \cap S) : S \in [\varphi < \infty]\} > -\infty \text{ then} \\ &\varphi^\bullet(A) = \sup\{\varphi^\bullet(A \cap S) : S \in [\varphi < \infty]\}. \end{aligned}$$

In fact, let $(S_l)_l$ be a sequence in $[\varphi < \infty]$ with $S_l \uparrow\supset A$, and let $T \in [\varphi < \infty]$ with $\varphi^\bullet(A \cap T) > -\infty$. Then $S_l \cup T \in [\varphi < \infty]$ since φ is submodular $\dot{+}$. Furthermore $A \cap (S_l \cup T) \uparrow A$ and $\varphi^\bullet(A \cap (S_l \cup T)) > -\infty$. Thus we obtain $\varphi^\bullet(A \cap (S_l \cup T)) \uparrow \varphi^\bullet(A)$ from 4.7.

In view of these results an isotone and submodular $\dot{+}$ set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ will be called **upward essential** instead of upward τ essential.

We conclude with an example which will illuminate the outer τ main theorems in the next section.

4.11. EXAMPLE. We fix $\mathfrak{S}$ in X and $\varphi : \mathfrak{S} \to [0, \infty]$ as described in 4.9 above. Define Y to consist of two disjoint copies of X, that is $Y := X \times \{0, 1\}$. We write the subsets $A \subset Y$ in the form $A = A^0 \sqcup A^1$ with $A^0, A^1 \subset X$. Define $\mathfrak{T}$ to consist of the subsets $A = A^0 \sqcup A^1 \subset Y$ with $A^0 \in \mathfrak{S}$ and A^1 finite $\subset A^0$. Thus $\mathfrak{T}$ is a lattice in Y with $\varnothing \in \mathfrak{T}$. Furthermore $\mathfrak{T}^\tau$ consists of the subsets $A = A^0 \sqcup A^1 \subset Y$ with $A^0 \in \mathfrak{S}$ and $A^1 \subset A^0$. Then define $\psi : \mathfrak{T} \to [0, \infty]$ to be $\psi(S) = \varphi(S^0)$ for $S = S^0 \sqcup S^1 \in \mathfrak{T}$. Thus ψ is isotone and modular with $\psi(\varnothing) = 0$. Also ψ is upward τ continuous. We prove three assertions.

1) ψ has no outer τ extension.
2) $\psi^\tau(A) = \varphi^\tau(A^0 \cup A^1)$ for $A = A^0 \sqcup A^1 \subset Y$.
3) ψ is not upward τ essential.

For 3) we use that $\varphi^\tau(F) = 0$ for all finite $F \subset X$.

Proof of 1). Let $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ be an outer τ extension of ψ. Then $\varnothing \in \mathfrak{A}$ and $\alpha(\varnothing) = 0$, so that α is a ccontent on a ring $\mathfrak{A}$. Furthermore $\alpha(A) = \psi(A) = \varphi(A^0)$ for $A = A^0 \sqcup A^1 \in \mathfrak{T}$, and hence $\alpha(A) = \varphi(A^0)$ for $A = A^0 \sqcup A^1 \in \mathfrak{T}^\tau$ since $\alpha|\mathfrak{T}^\tau$ is upward τ continuous. Now fix $E \in \mathfrak{S}$ with $c := \varphi(E) \in]0, \infty[$. Then on the one hand

$$\begin{array}{ll} E \sqcup \varnothing \in \mathfrak{T} & \text{with } \alpha(E \sqcup \varnothing) = \varphi(E) = c, \\ E \sqcup E \in \mathfrak{T}^\tau & \text{with } \alpha(E \sqcup E) = \varphi(E) = c. \end{array}$$

On the other hand $\varnothing \sqcup E \in \mathfrak{A}$ since $\mathfrak{A}$ is a ring, and

$$\begin{aligned} \alpha(\varnothing \sqcup E) &= \inf\{\alpha(A) : A \in \mathfrak{T}^\tau \text{ with } A \supset \varnothing \sqcup E\} \\ &= \inf\{\varphi(A^0) : A^0 \in \mathfrak{S} \text{ and } A^1 \subset A^0 \text{ with } A^1 \supset E\} \\ &= \varphi(E) = c, \end{aligned}$$

since α is outer regular $\mathfrak{T}^\tau$. These values combine to contradict the fact that α is modular.

Proof of 2). Both directions $\leqq$ and $\geqq$ are routine verifications. Proof of 3). Fix as above $E \in \mathfrak{S}$ with $c := \varphi(E) \in]0, \infty[$. For $A := \varnothing \sqcup E \subset Y$ then $\psi^\tau(A) = \varphi^\tau(E) = \varphi(E) = c$. On the other hand we obtain for $S = S^0 \sqcup S^1 \in \mathfrak{T}$ that $\psi^\tau(A \cap S) = \psi^\tau(\varnothing \sqcup (E \cap S^1)) = \varphi^\tau(E \cap S^1) = 0$ since S^1 is finite. The assertion follows.

The Extended Carathéodory Construction

We turn to the second task of the present section. We consider a set function $\phi : \mathfrak{P}(X) \to H$, defined on the full power set $\mathfrak{P}(X)$ of a nonvoid set X, and with values in a nonvoid set H which carries an associative and commutative addition $+$. We shall define and explore the so-called Carathéodory class $\mathfrak{C}(\phi)$ of ϕ, a paving in X. The definition is classical in case that H has the neutral element 0 and $\phi(\varnothing) = 0$. But in the present context there is no restriction for $\phi(\varnothing)$, it can in particular be a non-cancellable element of H. Recall that $a \in H$ is named **cancellable** iff for each pair $u, v \in H$ the implication $u + a = v + a \Rightarrow u = v$ holds true. Thus $H = \overline{\mathbb{R}}$ with $\dot{+}$ or $\underset{\cdot}{+}$ has the non-cancellable elements $\pm\infty$.

The new situation requires a drastic modification of the classical definition. We define the **Carathéodory class $\mathfrak{C}(\phi)$** of ϕ to consist of those subsets $A \subset X$ which fulfil

$$\phi(U) + \phi(V) = \phi(U|A|V) + \phi(U|A'|V) \quad \text{for all } U, V \subset X.$$

We proceed to list its basic properties.

4.12. Properties. 1) $\varnothing, X \in \mathfrak{C}(\phi)$. *Also* $\mathfrak{C}(\phi)$ *has* $\perp$. 2) *Assume that* $E \subset X$ *has cancellable value* $\phi(E) \in H$. *Then* $\mathfrak{C}(\phi)$ *consists of the subsets* $A \subset X$ *which fulfil*

$$\phi(P) + \phi(E) = \phi(P|A|E) + \phi(P|A'|E) \quad \textit{for all } P \subset X.$$

3) *In particular assume that* $\phi(\varnothing) = 0$ *is neutral in* H. *Then* $\mathfrak{C}(\phi)$ *consists of the subsets* $A \subset X$ *which fulfil*

$$\phi(P) = \phi(P \cap A') + \phi(P \cap A) \quad \textit{for all } P \subset X.$$

Thus we come back to the traditional definition of the class $\mathfrak{C}(\phi)$. 4) *If* $A \in \mathfrak{C}(\phi)$ *then*

$$(+) \qquad \phi(P) + \phi(A) = \phi(P \cup A) + \phi(P \cap A) \quad \textit{for all } P \subset X.$$

On the other hand a subset $A \subset X$ which satisfies (+) *need not be in $\mathfrak{C}(\phi)$; by* 2) *it is in $\mathfrak{C}(\phi)$ when $\phi(A) \in H$ is cancellable.* 5) *Assume that there exists an $E \subset X$ such that $\phi(E) \in H$ is cancellable. Then $\mathfrak{C}(\phi)$ is an algebra.*

Proof. 1) is obvious. 2) Let $A \subset X$ be as described above. In order to see that $A \in \mathfrak{C}(\phi)$ we fix $P, Q \subset X$ and form $U := P|A|Q$ and $V := P|A'|Q$. By assumption

$$\begin{aligned}
&\big(\phi(U) + \phi(E)\big) + \big(\phi(V) + \phi(E)\big)\\
&= \big(\phi(U|A|E) + \phi(U|A'|E)\big) + \big(\phi(V|A|E) + \phi(V|A'|E)\big)\\
&= \big(\phi(P|A|E) + \phi(Q|A'|E)\big) + \big(\phi(Q|A|E) + \phi(P|A'|E)\big)\\
&= \big(\phi(P|A|E) + \phi(P|A'|E)\big) + \big(\phi(Q|A|E) + \phi(Q|A'|E)\big)\\
&= \big(\phi(P) + \phi(E)\big) + \big(\phi(Q) + \phi(E)\big),
\end{aligned}$$

and hence $\phi(U) + \phi(V) = \phi(P) + \phi(Q)$ since $\phi(E)$ is cancellable. This is the assertion. 3) is an obvious special case of 2). 4) For $A \in \mathfrak{C}(\phi)$ the equation (+) is the definition with $V := A$. For the converse a counterexample will be in exercise 4.13 below. The last assertion is obvious. 5) We have to prove that $\mathfrak{C}(\phi)$ has $\cup$. Fix $A, B \in \mathfrak{C}(\phi)$. For $P \subset X$ we form $U := P|A \cup B|E$ and $V := P|(A \cup B)'|E$. With the notations $M := P|A|E$ and $N := P|A'|E$ one computes that

$$\begin{aligned}
M|B|E &= (M \cap B') \cup (E \cap B)\\
&= (P \cap A' \cap B') \cup (E \cap A \cap B') \cup (E \cap B)\\
&= \big(P \cap (A \cup B)'\big) \cup \big(E \cap (A \cup B)\big) = P|A \cup B|E = U,\\
M|B'|E &= (M \cap B) \cup (E \cap B')\\
&= (P \cap A' \cap B) \cup (E \cap A \cap B) \cup (E \cap B')\\
&= (P \cap A' \cap B) \cup (E \cap A' \cap B') \cup (E \cap A)\\
&= (V \cap A' \cap B) \cup (V \cap A' \cap B') \cup (E \cap A)\\
&= (V \cap A') \cup (E \cap A) = V|A|E,\\
N &= P|A'|E = V|A'|E.
\end{aligned}$$

Since $A, B \in \mathfrak{C}(\phi)$ it follows that

$$\begin{aligned}
&\phi(U) + \big(\phi(V) + \phi(E)\big) = \phi(U) + \big(\phi(V|A|E) + \phi(V|A'|E)\big)\\
&= \big(\phi(M|B|E) + \phi(M|B'|E)\big) + \phi(N) = \big(\phi(M) + \phi(E)\big) + \phi(N)\\
&= \big(\phi(P|A|E) + \phi(P|A'|E)\big) + \phi(E) = \big(\phi(P) + \phi(E)\big) + \phi(E),
\end{aligned}$$

and hence that $\phi(U) + \phi(V) = \phi(P) + \phi(E)$. By 2) this is the assertion.

4.13. EXERCISE. Construct an example $\phi : \mathfrak{P}(X) \to H$ such that there exists a subset $A \subset X$ which fulfils (+) but is not in $\mathfrak{C}(\phi)$, and that furthermore there exists a subset $E \subset X$ with cancellable value $\phi(E) \in H$. Hint: Let $H := [0, \infty]$ with the usual addition, and let $X = Y \cup Z$ with nonvoid disjoint Y and Z. Define $\phi : \mathfrak{P}(X) \to H$ to be $\phi(Y) = \phi(Z) = 0$ and $\phi(A) = \infty$ for all other $A \subset X$. Then proceed as follows. 0) For $E \subset X : \phi(E) \in H$ is cancellable iff $E = Y$ or Z. 1) For $A \subset X : A$ fulfils (+) iff $A \neq Y, Z$. 2) For $A \subset X : A \in \mathfrak{C}(\phi)$ iff $A = \varnothing$ or X.

In the sequel we concentrate on the particular case that $H = \overline{\mathbb{R}}$ with one of the additions $\dot{+}$ and $\underset{\cdot}{+}$. For a set function $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ we then write $\mathfrak{C}(\phi, \dot{\underset{\cdot}{+}})$. No specification $\dot{\underset{\cdot}{+}}$ is needed when ϕ attains at most one of the values $\pm\infty$.

4.14. Remark. $\mathfrak{C}(\phi, \dot{\underset{\cdot}{+}})$ *is an algebra.*

Proof. By the above 4.12.5) it remains to consider the cases that the value set of ϕ is one of the singletons $\{\pm\infty\}$ or $\{-\infty, \infty\}$. Then it suffices to note that on $\{-\infty, \infty\}$ the element $-\infty$ is cancellable for $\dot{+}$ and the element ∞ is cancellable for $\underset{\cdot}{+}$.

4.15. Exercise. $\mathfrak{C}(\phi, \underset{\cdot}{+}) = \mathfrak{C}(\phi\bot, \dot{+})$.

4.16. Remark (Symmetrization). *Assume that $A \subset X$ satisfies*

$$\phi(P)\dot{\underset{\cdot}{+}}\phi(Q) \geqq \phi(P|A|Q)\dot{\underset{\cdot}{+}}\phi(P|A'|Q) \quad \textit{for all } P, Q \subset X.$$

Then $A \in \mathfrak{C}(\phi, \dot{\underset{\cdot}{+}})$.

Proof. We know from 1.1.5) that $U := P|A|Q$ and $V := P|A'|Q$ have $U|A|V = P$ and $U|A'|V = Q$. It follows that

$$\begin{aligned} \phi(P|A|Q)\dot{\underset{\cdot}{+}}\phi(P|A'|Q) &= \phi(U)\dot{\underset{\cdot}{+}}\phi(V) \\ &\geqq \phi(U|A|V)\dot{\underset{\cdot}{+}}\phi(U|A'|V) = \phi(P)\dot{\underset{\cdot}{+}}\phi(Q), \end{aligned}$$

and hence the assertion.

The Carathéodory Class in the Spirit of the Outer Theory

We proceed to consider the Carathéodory class $\mathfrak{C}(\phi, \dot{+})$ of an isotone set function $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ under assumptions in the spirit of the outer theory. We shall see that the definition of $\mathfrak{C}(\phi, \dot{+})$ then admits substantial simplifications. We start with a simple remark.

4.17. Remark. *Let $\mathfrak{T}$ be a paving in X and $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ be isotone and outer regular $\mathfrak{T}$. If $A \subset X$ satisfies*

$$\phi(P)\dot{+}\phi(Q) \geqq \phi(P|A|Q)\dot{+}\phi(P|A'|Q) \quad \textit{for all } P, Q \in \mathfrak{T},$$

then $A \in \mathfrak{C}(\phi, \dot{+})$.

Proof. By symmetrization it suffices to prove that

$$\phi(U)\dot{+}\phi(V) \geqq \phi(U|A|V)\dot{+}\phi(U|A'|V) \quad \text{for all } U, V \subset X.$$

We fix $U, V \subset X$ and can assume that $\phi(U), \phi(V) < \infty$. For fixed real $c > \phi(U) + \phi(V)$ there are $a, b \in \mathbb{R}$ with $c = a + b$ and with $\phi(U) < a$ and $\phi(V) < b$. By assumption there exist $P, Q \in \mathfrak{T}$ such that $P \supset U$ and $\phi(P) < a$, $Q \supset V$ and $\phi(Q) < b$. Hence

$$\phi(U|A|V)\dot{+}\phi(U|A'|V) \leqq \phi(P|A|Q)\dot{+}\phi(P|A'|Q) \leqq \phi(P)\dot{+}\phi(Q) < a + b = c.$$

The assertion follows.

4.18. Exercise. *Let $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ be a content $\dot{+}$ on an oval $\mathfrak{A}$. Then $\mathfrak{A} \subset \mathfrak{C}(\alpha^\star, \dot{+})$.*

The main point is that the verification of $A \in \mathfrak{C}(\phi, \dot{+})$ can be reduced to certain pairs of subsets $P \subset Q$ of X. The basis is the fundamental lemma which follows. We use the abbreviation $P \sqsubset Q := \{P\} \sqsubset \{Q\} = \{A \subset X : P \subset A \subset Q\}$.

4.19. LEMMA. *Let $P \subset Q \subset X$. Assume that $\varphi : P \sqsubset Q \to \mathbb{R}$ is submodular. If $A \subset X$ satisfies*

$$\varphi(P) + \varphi(Q) \geqq \varphi(P|A|Q) + \varphi(P|A'|Q),$$

then

$$\varphi(U) + \varphi(V) = \varphi(U|A|V) + \varphi(U|A'|V) \quad \textit{for all } U, V \in P \sqsubset Q.$$

Proof. By symmetrization it suffices to prove the assertion with $\geqq$. Fix $U, V \in P \sqsubset Q$. i) From the assumption we have

$$\varphi(P) + \varphi(Q) + 2\varphi(U) \geqq \big(\varphi(P|A|Q) + \varphi(U)\big) + \big(\varphi(P|A'|Q) + \varphi(U)\big).$$

Since φ is submodular and $P \subset U \subset Q$ this is

$$\geqq \big(\varphi(U|A|Q) + \varphi(P|A|U)\big) + \big(\varphi(U|A'|Q) + \varphi(P|A'|U)\big);$$

and when we use submodularity for the two first terms in the brackets and repeat the two second terms this is

$$\geqq \varphi(Q) + \varphi(U) + \varphi(P|A|U) + \varphi(P|A'|U).$$

Thus we have

$$\varphi(P) + \varphi(U) \geqq \varphi(P|A|U) + \varphi(P|A'|U).$$

Of course we have likewise

$$\varphi(P) + \varphi(V) \geqq \varphi(P|A|V) + \varphi(P|A'|V).$$

ii) We add the last two inequalities and use submodularity twice on the right for the two pairs of terms which were in crosswise position. Then we obtain

$$2\varphi(P) + \varphi(U) + \varphi(V) \geqq \big(\varphi(V|A|U) + \varphi(P)\big) + \big(\varphi(U|A|V) + \varphi(P)\big),$$

and hence the assertion.

From the lemma we deduce the next result which looks somewhat technical but will be a powerful tool.

4.20. PROPOSITION. *Assume that $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ is an isotone set function. Let $\mathfrak{P}\downarrow$ and $\mathfrak{Q}\uparrow$ be pavings in X with nonvoid $\mathfrak{P} \sqsubset \mathfrak{Q}$ such that*

$\phi|\mathfrak{P}$ and $\phi|\mathfrak{Q}$ are finite, and $\phi|\mathfrak{P} \sqsubset \mathfrak{Q}$ is submodular.

Furthermore let $\mathfrak{H}\uparrow$ be a paving in X with $\mathfrak{Q} \subset \mathfrak{H}$ such that

ϕ is outer regular $\mathfrak{P} \sqsubset \mathfrak{H}$,

$$\phi(T) = \sup_{Q \in \mathfrak{Q}} \phi(T \cap Q) \quad \textit{for all } T \in \mathfrak{P} \sqsubset \mathfrak{H}.$$

If $A \subset X$ satisfies

$$\phi(P) + \phi(Q) \geqq \phi(P|A|Q) + \phi(P|A'|Q) \quad \textit{for all } P \in \mathfrak{P} \textit{ and } Q \in \mathfrak{Q} \textit{ with } P \subset Q,$$

then $A \in \mathfrak{C}(\phi, \dot{+})$.

Proof. i) We know from 1.8.1⋆) that $\mathfrak{P} \sqsubset \mathfrak{Q}$ and $\mathfrak{P} \sqsubset \mathfrak{H}$ are ovals. ii) For each pair $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ there exists a pair $A \in \mathfrak{P}$ and $B \in \mathfrak{Q}$ with $A \subset P$ and $Q \subset B$ such that $A \subset B$. In fact, by assumption there exist $U \in \mathfrak{P}$ and $V \in \mathfrak{Q}$ with $U \subset V$. Then by directedness there are $A \in \mathfrak{P}$ with $A \subset P, U$ and $B \in \mathfrak{Q}$ with $B \supset Q, V$. It is obvious that A and B are as required. iii) For each pair $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ with $P \subset Q$ we see from 4.19 that

$$\phi(U) + \phi(V) = \phi(U|A|V) + \phi(U|A'|V) \quad \text{for all } U, V \in P \sqsubset Q.$$

By directedness it follows that

$$\phi(U) + \phi(V) = \phi(U|A|V) + \phi(U|A'|V) \quad \text{for all } U, V \in \mathfrak{P} \sqsubset \mathfrak{Q}.$$

Note that $U|A|V$ and $U|A'|V$ are in $\mathfrak{P} \sqsubset \mathfrak{Q}$ as well. iv) In view of 4.17 applied to $\mathfrak{T} := \mathfrak{P} \sqsubset \mathfrak{H}$ it suffices to prove that

$$\phi(U) + \phi(V) \geqq \phi(U|A|V) + \phi(U|A'|V) \quad \text{for all } U, V \in \mathfrak{T}.$$

Note that $U|A|V$ and $U|A'|V$ are in $\mathfrak{T}$ as well, and that $\phi|\mathfrak{T} > -\infty$. v) Fix $U, V \in \mathfrak{T}$. Also fix $M, N \in \mathfrak{Q}$ and then $Q \in \mathfrak{Q}$ with $M, N \subset Q$. By ii) we can assume that Q is downward enclosable $\mathfrak{P}$. Thus $U \cap Q, V \cap Q \in \mathfrak{P} \sqsubset \mathfrak{Q}$. By iii) therefore

$$\begin{aligned} \phi(U) + \phi(V) &\geqq \phi(U \cap Q) + \phi(V \cap Q) \\ &= \phi(U \cap Q|A|V \cap Q) + \phi(U \cap Q|A'|V \cap Q) \\ &= \phi\big((U|A|V) \cap Q\big) + \phi\big((U|A'|V) \cap Q\big) \\ &\geqq \phi\big((U|A|V) \cap M\big) \dot{+} \phi\big((U|A'|V) \cap N\big). \end{aligned}$$

Now the supremum over $M, N \in \mathfrak{Q}$ of the right side is $= \phi(U|A|V) + \phi(U|A'|V)$ since the two partial suprema are both $> -\infty$. By iv) the proof is complete.

4.21. Addendum. *Assume in addition that $\phi|\mathfrak{P} \sqsubset \mathfrak{Q}$ is upward σ continuous. Then $\mathfrak{C}(\phi, \dot{+})$ is a σ algebra.*

Proof. Let $(A_l)_l$ be a sequence in $\mathfrak{C}(\phi, \dot{+})$ with $A_l \uparrow A$. To be shown is $A \in \mathfrak{C}(\phi, \dot{+})$. Fix $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ with $P \subset Q$. By assumption we have

$$\begin{aligned} \phi(P) + \phi(Q) &= \phi(P|A_l|Q) + \phi(P|A_l'|Q) \\ &= \phi\big(P \cup (Q \cap A_l)\big) + \phi\big(P \cup (Q \cap A_l')\big) \\ &\geqq \phi\big(P \cup (Q \cap A_l)\big) + \phi\big(P \cup (Q \cap A')\big), \end{aligned}$$

since $A_l \subset A$ and hence $A_l' \supset A'$. Here all arguments are in $P \sqsubset Q$ and hence all values are finite. By assumption it follows that

$$\begin{aligned} \phi(P) + \phi(Q) &\geqq \phi\big(P \cup (Q \cap A)\big) + \phi\big(P \cup (Q \cap A')\big) \\ &= \phi(P|A|Q) + \phi(P|A'|Q). \end{aligned}$$

Thus from 4.20 we obtain $A \in \mathfrak{C}(\phi, \dot{+})$.

We include another addendum to 4.20 for the sake of chapter VI.

4.22. Addendum. *Assume in addition that* $\theta : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ *is an isotone set function with* $\theta \geqq \phi$ *such that*

$$\theta|\mathfrak{P} = \phi|\mathfrak{P} \text{ and } \theta|\mathfrak{Q} = \phi|\mathfrak{Q},$$
$$\theta(T) = \sup_{Q\in\mathfrak{Q}} \theta(T \cap Q) \quad \text{for all } T \in \mathfrak{P} \sqsubset \mathfrak{H}.$$

Then $\phi|\mathfrak{C}(\phi, \dot{+})$ *is an extension of* $\theta|\mathfrak{C}(\theta, \dot{+})$.

Proof. Fix $A \in \mathfrak{C}(\theta, \dot{+})$. i) In order to prove that $A \in \mathfrak{C}(\phi, \dot{+})$ let $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ with $P \subset Q$. Then

$$\begin{aligned} \phi(P) + \phi(Q) = \theta(P) + \theta(Q) &= \theta(P|A|Q) + \theta(P|A'|Q) \\ &\geqq \phi(P|A|Q) + \phi(P|A'|Q). \end{aligned}$$

From 4.20 the assertion follows. ii) We claim that $\theta(P|A|Q) = \phi(P|A|Q) \in \mathbb{R}$ for all $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$. This follows at once from

$$\begin{aligned} \phi(P) + \phi(Q) = \theta(P) + \theta(Q) &= \theta(P|A|Q)\dot{+}\theta(P|A'|Q) \\ &\geqq \phi(P|A|Q)\dot{+}\phi(P|A'|Q) = \phi(P) + \phi(Q). \end{aligned}$$

iii) It remains to prove that $\theta(A) \leqq \phi(A)$ and hence $\theta(A) = \phi(A)$. We fix $V \in \mathfrak{P} \sqsubset \mathfrak{H}$ with $V \supset A$ and have to show that $\theta(A) \leqq \phi(V)$. Let $P \in \mathfrak{P}$ with $P \subset V$ and $Q \in \mathfrak{Q}$. Then on the one hand

$$\theta(P|A|Q) \geqq \theta(P \cap Q|A|V \cap Q) = \theta\big((P|A|V) \cap Q\big),$$

so that from the assumption and $P|A|V \in \mathfrak{P} \sqsubset \mathfrak{H}$ and from $P|A|V \supset V \cap A = A$ we obtain

$$\sup_{Q\in\mathfrak{Q}} \theta(P|A|Q) \geqq \theta(P|A|V) \geqq \theta(A).$$

On the other hand we have $P|A|Q \subset P \cup A \subset V$ and hence

$$\sup_{Q\in\mathfrak{Q}} \phi(P|A|Q) \leqq \phi(V).$$

From ii) the assertion follows.

5. The Outer Extension Theory: The Main Theorem

The Outer Main Theorem

In the last section we have developed the concepts and instruments which we need in order to reach our principal aim as formulated after the basic definition. The first theorem below is a clear hint that these devices are adequate.

The present subsection is under the assumption that $\varphi : \mathfrak{S} \to]-\infty, \infty]$ is an isotone set function $\not\equiv \infty$ on a lattice $\mathfrak{S}$ in X.

5.1. Theorem. *Assume that* $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ *is an outer* $\bullet$ *extension of* φ. *Then* α *is a restriction of* $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$.

Proof. i) $\varphi = \alpha|\mathfrak{S}$ is upward $\bullet$ continuous, and hence $\varphi = \varphi^\bullet|\mathfrak{S}$ by 4.5. Then $\alpha = \varphi = \varphi^\bullet$ on $\mathfrak{S}$ implies $\alpha = \varphi^\bullet$ on $\mathfrak{S}^\bullet$ by 4.3.iii), and hence $\alpha = \varphi^\bullet$ on $\mathfrak{A}$ by 4.1.4). ii) It remains to prove that $\mathfrak{A} \subset \mathfrak{C}(\varphi^\bullet, \dot{+})$. Fix $A \in \mathfrak{A}$. For $P, Q \in \mathfrak{S}^\bullet$ then

$$\alpha(P)\dot{+}\alpha(Q) = \alpha(P|A|Q)\dot{+}(P|A'|Q),$$

where all arguments are in $\mathfrak{A}$ since $\mathfrak{A}$ is an oval. In fact, since α is modular $\dot{+}$ both sides are $= \alpha(P \cup Q)\dot{+}\alpha(P \cap Q)$. Since now $\alpha = \varphi^\bullet$ on $\mathfrak{A}$ by i) we obtain $A \in \mathfrak{C}(\varphi^\bullet, \dot{+})$ from 4.17 applied to $\phi := \varphi^\bullet$ and $\mathfrak{T} := \mathfrak{S}^\bullet$.

We prepare the outer main theorem with the important next result.

5.2. Proposition. *Let φ be submodular with $\varphi^\bullet|\mathfrak{S} > -\infty$, and upward essential in case $\bullet = \tau$. Fix pavings*

$\mathfrak{P} \subset [\varphi < \infty]$ *downward cofinal, that is such that* $[\varphi < \infty] \subset (\sqsupset \mathfrak{P})$,
$\mathfrak{Q} \subset [\varphi < \infty]$ *upward cofinal, that is such that* $[\varphi < \infty] \subset (\sqsubset \mathfrak{Q})$.

If $A \subset X$ satisfies

$$\varphi^\bullet(P) + \varphi^\bullet(Q) \geqq \varphi^\bullet(P|A|Q) + \varphi^\bullet(P|A'|Q)$$
$$\text{for all } P \in \mathfrak{P} \text{ and } Q \in \mathfrak{Q} \text{ with } P \subset Q,$$

then $A \in \mathfrak{C}(\varphi^\bullet, \dot{+})$.

5.3. Addendum. *For $\bullet = \sigma\tau$ the class $\mathfrak{C}(\varphi^\bullet, \dot{+})$ is a σ algebra.*

Proof of 5.2 and 5.3. We deduce the assertions from 4.20 and 4.21. i) $[\varphi < \infty]$ is a lattice by 2.4 and nonvoid by assumption. Therefore $\mathfrak{P} \downarrow$ and $\mathfrak{Q} \uparrow$, and $\mathfrak{P} \sqsubset \mathfrak{Q} \supset [\varphi < \infty]$ is nonvoid. ii) $\phi := \varphi^\bullet$ is isotone and submodular $\dot{+}$ by 4.1.3)5). By assumption and 4.1.1)2) ϕ is finite on $[\varphi < \infty]$ and hence on $\mathfrak{P} \sqsubset \mathfrak{Q}$. iii) We define $\mathfrak{H} := [\varphi^\bullet|\mathfrak{S}^\bullet < \infty]$ and note that $\varphi^\bullet|\mathfrak{S}^\bullet > -\infty$. Then $[\varphi < \infty] \subset \mathfrak{H}$ and hence $\mathfrak{Q} \subset \mathfrak{H}$. By ii) and 2.4 $\mathfrak{H}$ is a lattice and hence $\mathfrak{H} \uparrow$. iv) $\sqsupset \mathfrak{P}$ contains $[\varphi < \infty]$ and hence $\mathfrak{S}$, therefore $\mathfrak{S}^\bullet$ and in particular $\mathfrak{H}$. Thus $\mathfrak{H} \subset \mathfrak{P} \sqsubset \mathfrak{H}$. By 4.1.4) therefore ϕ is outer regular $\mathfrak{P} \sqsubset \mathfrak{H}$. v) For $T \in \mathfrak{P} \sqsubset \mathfrak{H}$ we have by definition $\varphi^\bullet(T) < \infty$ and $\sup\{\varphi^\bullet(T \cap S) : S \in [\varphi < \infty]\} > -\infty$. Since φ is upward $\bullet$ essential by 4.10.$\star$)σ) and by assumption we have

$$\phi(T) = \sup\{\phi(T \cap S) : S \in [\varphi < \infty]\} = \sup\{\phi(T \cap S) : S \in \mathfrak{Q}\}.$$

vi) After this the assertions follow from 4.20 and 4.21.

5.4. Consequence. *Let φ be submodular with $\varphi^\bullet|\mathfrak{S} > -\infty$, and upward essential in case $\bullet = \tau$. Assume that $\mathfrak{Q} \subset [\varphi < \infty]$ is upward cofinal. Then*

$$\mathfrak{Q}\top\mathfrak{C}(\varphi^\bullet, \dot{+}) \subset \mathfrak{C}(\varphi^\bullet, \dot{+}).$$

Proof. We put $\mathfrak{P} := [\varphi < \infty]$. Fix $A \in \mathfrak{Q}\top\mathfrak{C}(\varphi^\bullet, \dot{+})$. For $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ we have $A \cap Q \in \mathfrak{C}(\varphi^\bullet, \dot{+})$ and hence

$$\varphi^\bullet(P) + \varphi^\bullet(Q) = \varphi^\bullet(P|A \cap Q|Q) + \varphi^\bullet(P|(A \cap Q)'|Q).$$

In case $P \subset Q$ the right side is $= \varphi^\bullet(P|A|Q) + \varphi^\bullet(P|A'|Q)$. Thus from 5.2 we obtain $A \in \mathfrak{C}(\varphi^\bullet, \dot{+})$.

We come to the central result of the present chapter.

5.5. THEOREM (Outer Main Theorem). *Let* $\varphi : \mathfrak{S} \to]-\infty, \infty]$ *be an isotone and submodular set function* $\not\equiv \infty$ *on a lattice* $\mathfrak{S}$. *Fix pavings*

$\mathfrak{P} \subset [\varphi < \infty]$ *downward cofinal, and*
$\mathfrak{Q} \subset [\varphi < \infty]$ *upward cofinal.*

Then the following are equivalent.

1) *There exist outer* $\bullet$ *extensions of* φ, *that is* φ *is an outer* $\bullet$ *premeasure.*

2) $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$ *is an outer* $\bullet$ *extension of* φ. *Furthermore*

$$\begin{aligned} \text{if } \bullet = \star &: \quad \varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+}) \text{ is a content } \dot{+} \text{ on the algebra } \mathfrak{C}(\varphi^\bullet, \dot{+}), \\ \text{if } \bullet = \sigma\tau &: \quad \varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+}) \text{ is a measure } \dot{+} \text{ on the } \sigma \text{ algebra } \mathfrak{C}(\varphi^\bullet, \dot{+}). \end{aligned}$$

3) $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$ *is an extension of* φ *in the crude sense, that is* $\mathfrak{S} \subset \mathfrak{C}(\varphi^\bullet, \dot{+})$ *and* $\varphi = \varphi^\bullet|\mathfrak{S}$.

4) $\varphi(U) + \varphi(V) = \varphi(M)\dot{+}\varphi^\bullet(U|M'|V)$ *for all* $U \subset M \subset V$ *in* $\mathfrak{S}$; note that $M = U|M|V$. *In case* $\bullet = \tau$ *furthermore* φ *is upward essential.*

5) $\varphi = \varphi^\bullet|\mathfrak{S}$; *and* $\varphi(P) + \varphi(Q) \geqq \varphi(M) + \varphi^\bullet(P|M'|Q)$ *for all* $P \subset M \subset Q$ *with* $P \in \mathfrak{P}, Q \in \mathfrak{Q}$, *and* $M \in \mathfrak{S}$ *and hence* $\in [\varphi < \infty]$. *In case* $\bullet = \tau$ *furthermore* φ *is upward essential.*

Note that 5.4 then implies $\mathfrak{Q}\top\mathfrak{S}^\bullet \subset \mathfrak{C}(\varphi^\bullet, \dot{+})$.

A posteriori it turns out that condition 5) is independent of the pavings $\mathfrak{P}$ and $\mathfrak{Q}$. But the present formulation is important for later specializations.

Proof. We prove 2)⇒1)⇒3)⇒4)⇒5)⇒2). The implication 2)⇒1) is obvious, and 1)⇒3) follows from 5.1. 3)⇒4) The first assertion follows from the definition of $\mathfrak{C}(\varphi^\bullet, \dot{+})$. It remains to show in case $\bullet = \tau$ that φ is upward τ essential. If not, then there exists $A \subset X$ such that

$$\infty > \varphi^\tau(A) > \sup\{\varphi^\tau(A \cap S) : S \in [\varphi < \infty]\} > -\infty.$$

Let $\varepsilon := \varphi^\tau(A) - \sup\{\varphi^\tau(A \cap S) : S \in [\varphi < \infty]\} > 0$. For $S \in [\varphi < \infty]$ then $\varphi(S) = \varphi^\tau(S) \in \mathbb{R}$ and $S \in \mathfrak{C}(\varphi^\tau, \dot{+})$. Therefore

$$\varphi^\tau(S)\dot{+}\varphi^\tau(A) = \varphi^\tau(S|S|A)\dot{+}\varphi^\tau(S|S'|A) = \varphi^\tau(A \cap S)\dot{+}\varphi^\tau(A \cup S),$$

with all terms finite, and hence

$$\varphi(S) + \varphi^\tau(A) \leqq \varphi^\tau(A) - \varepsilon + \varphi^\tau(A \cup S), \text{ or } \varphi(S) + \varepsilon \leqq \varphi^\tau(A \cup S).$$

Fix a paving $\mathfrak{M} \subset [\varphi < \infty]$ with $\mathfrak{M} \uparrow M \supset A$ and $\sup\limits_{S \in \mathfrak{M}} \varphi(S) < \infty$. By 4.5.iii) then $M \in \mathfrak{S}^\tau$ and $\sup\limits_{S \in \mathfrak{M}} (S) = \varphi^\tau(M) \in \mathbb{R}$. It follows that $\varphi^\tau(M) + \varepsilon \leqq \varphi^\tau(M)$ and thus a contradiction. 4)⇒5) For $V \in \mathfrak{S}$ and $U = M \in [\varphi < \infty]$ contained in V we obtain from 4) that $\varphi(V) = \varphi^\bullet(V)$.

It remains to prove the implication 5)⇒2). For the remainder of the proof we assume 5). Then the assumptions of 5.2 and 5.3 are fulfilled. i) $\mathfrak{C}(\varphi^\bullet, \dot{+})$ is an algebra by 4.14, and $\alpha := \varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$ is isotone and modular $\dot{+}$ by 4.12.4). For $\bullet = \sigma\tau$ the class $\mathfrak{C}(\varphi^\bullet, \dot{+})$ is a σ algebra by 5.3.

ii) We conclude from 5.2 that $\mathfrak{S}^\bullet \subset \mathfrak{C}(\varphi^\bullet, \dot{+})$. Fix $A \in \mathfrak{S}^\bullet$. Then let $\mathfrak{M} \subset \mathfrak{S}$ be a paving of type $\bullet$ with $\mathfrak{M} \uparrow A$. Furthermore fix $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ with $P \subset Q$. For $S \in \mathfrak{M}$ we form

$$M := P|S|Q = P \cup (Q \cap S) \in \mathfrak{S} \text{ with } P \subset M \subset Q,$$

so that from 5) we obtain $\varphi(P) + \varphi(Q) \geqq \varphi(M) + \varphi^\bullet(P|M'|Q)$. Here

$$\begin{aligned} P|M'|Q &= P \cup (Q \cap M') = P \cup \big(Q \cap P' \cap (Q' \cup S')\big) \\ &= P \cup (Q \cap P' \cap S') = P \cup (Q \cap S') \supset P \cup (Q \cap A') = P|A'|Q, \end{aligned}$$

so that

$$\varphi(P) + \varphi(Q) \geqq \varphi\big(P \cup (Q \cap S)\big) + \varphi^\bullet(P|A'|Q).$$

Now $\{P \cup (Q \cap S) : S \in \mathfrak{M}\} \uparrow P \cup (Q \cap A) = P|A|Q$. From 4.5.iii) it follows that

$$\varphi(P) + \varphi(Q) \geqq \varphi^\bullet(P|A|Q) + \varphi^\bullet(P|A'|Q).$$

Thus 5.2 implies that $A \in \mathfrak{C}(\varphi^\bullet, \dot{+})$ as claimed.

iii) In particular $\mathfrak{S} \subset \mathfrak{C}(\varphi^\bullet, \dot{+})$, so that α is an extension of φ. Thus α attains at least one finite value and hence is a content $\dot{+}$. Furthermore α is outer regular $\mathfrak{S}^\bullet$ by 4.1.4), and $\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous by 4.5.iii). Therefore α is an outer $\bullet$ extension of φ. iv) It remains to prove that α is a measure $\dot{+}$ when $\bullet = \sigma\tau$. By 4.7 α is almost upward σ continuous. Thus by 2.11 α is almost downward σ continuous as well, provided that it is semifinite below, that is outer regular $[\alpha > -\infty]$. But we know from ii) that $\mathfrak{S}^\bullet$ is in $\mathfrak{C}(\varphi^\bullet, \dot{+})$ and hence in $[\alpha > -\infty]$. Thus α is outer regular $[\alpha > -\infty]$. The proof is complete.

The above outer main theorem fulfils the promise made after the basic definition: Conditions 4) and 5) characterize the outer $\bullet$ premeasures. Combined with 5.1 we see that for an outer $\bullet$ premeasure φ all outer $\bullet$ extensions are restrictions of a unique maximal one, which is $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$. Thus we arrive at a natural and simple situation, and our concepts and instruments prove to be adequate. We want to put particular emphasis on the role of the Carathéodory class, because its initial creation was not at all connected with regularity.

5.6. Remark. The outer main theorem would be false in case $\bullet = \tau$ if in 4)5) the condition that φ be upward essential would be omitted. To see this we return to example 4.11 and adopt the former notations. The set function $\psi : \mathfrak{T} \to [0, \infty]$ then violates 1) as shown in 4.11.1). On the other hand we anticipate from 5.14 that we could have started from a set function $\varphi : \mathfrak{S} \to [0, \infty]$ as described in 4.9 with $\varphi^\tau(F) = 0$ for all finite $F \subset X$ which has an outer τ extension and thus is an outer τ premeasure. Then the set function $\psi : \mathfrak{T} \to [0, \infty]$ fulfils the first part of 4). In fact, we see from 4.11.2) and since φ^τ is submodular that this condition reads

$$\varphi(U) + \varphi(V) = \varphi(M) + \varphi^\tau(U|M'|V) \quad \text{for all } U \subset M \subset V \text{ in } \mathfrak{S},$$

and therefore is satisfied.

5.7. SPECIAL CASE (Traditional Type). *Assume that* $\mathfrak{S}$ *is an oval. Then condition* 5) *simplifies to*

5∘) $\varphi = \varphi^{\bullet}|\mathfrak{S}$; *and* φ *is supermodular. In case* $\bullet = \tau$ *furthermore* φ *is upward essential.*

Proof of 5∘)⇒5). If $P \subset M \subset Q$ are as in 5) then $P|M'|Q = Q|M|P =: N \in \mathfrak{S}$ and hence $\in [\varphi < \infty]$. We have $M \cap N = P$ and $M \cup N = Q$, and hence $\varphi(M)+\varphi^{\bullet}(N) = \varphi(M)+\varphi(N) \leqq \varphi(P)+\varphi(Q)$. Proof of 5)⇒5∘). One notes that 5)⇒1) ⇒ φ is supermodular, or that 5) for $\mathfrak{P} = \mathfrak{Q} = [\varphi < \infty] \Rightarrow \varphi$ is supermodular.

We shall soon turn to the most important special cases. But first we want to terminate the present context with a short comparison of the three cases $\bullet = \star\sigma\tau$.

Comparison of the three Outer Theories

In the present subsection we assume that $\varphi : \mathfrak{S} \rightarrow]-\infty, \infty]$ is an isotone and submodular set function $\not\equiv \infty$ on a lattice $\mathfrak{S}$.

5.8. PROPOSITION. σ) *In case* $\varphi = \varphi^{\sigma}|\mathfrak{S}$ *we have* $\mathfrak{C}(\varphi^{\star}, \dot{+}) \subset \mathfrak{C}(\varphi^{\sigma}, \dot{+})$. τ) *In case* $\varphi = \varphi^{\tau}|\mathfrak{S}$ *and* φ *upward essential we have* $\mathfrak{C}(\varphi^{\sigma}, \dot{+}) \subset \mathfrak{C}(\varphi^{\tau}, \dot{+})$.

Proof. Combine $\varphi^{\star} \geqq \varphi^{\sigma} \geqq \varphi^{\tau}$ with 5.2 for $\mathfrak{P} = \mathfrak{Q} = [\varphi < \infty]$.

5.9. PROPOSITION. *Assume that* φ *is modular.* σ) *In case* $\varphi = \varphi^{\sigma}|\mathfrak{S}$ *we have* $\varphi^{\star}(A) = \varphi^{\sigma}(A)$ *for all* $A \in \mathfrak{C}(\varphi^{\star}, \dot{+})$ *with* $\varphi^{\star}(A) < \infty$. τ) *In case* $\varphi = \varphi^{\tau}|\mathfrak{S}$ *we have* $\varphi^{\sigma}(A) = \varphi^{\tau}(A)$ *for all* $A \in \mathfrak{C}(\varphi^{\sigma}, \dot{+})$ *with* $\varphi^{\sigma}(A) < \infty$.

Proof. σ) Fix $A \in \mathfrak{C}(\varphi^{\star}, \dot{+})$ with $\varphi^{\star}(A) < \infty$. For $P, Q \in [\varphi < \infty]$ we have by 4.1.5)

$$\begin{aligned}
\varphi(P) + \varphi(Q) &= \varphi^{\star}(P) + \varphi^{\star}(Q) = \varphi^{\star}(P|A|Q) + \varphi^{\star}(P|A'|Q) \\
&\geqq \varphi^{\sigma}(P|A|Q) + \varphi^{\sigma}(P|A'|Q) \geqq \varphi^{\sigma}(P \cup Q) + \varphi^{\sigma}(P \cap Q) \\
&= \varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q),
\end{aligned}$$

where all arguments are between $P \cap Q$ and $P \cup Q$ and hence all values are finite. It follows that

$$\varphi^{\star}(P|A|Q) = \varphi^{\sigma}(P|A|Q) \text{ for all } P, Q \in [\varphi < \infty].$$

Since $\varphi^{\star}(A) < \infty$ there exists $Q \in [\varphi < \infty]$ with $Q \supset A$. Then $\varphi^{\star}(P \cup A) = \varphi^{\sigma}(P \cup A)$ for all $P \in [\varphi < \infty]$. Now we have to prove that $\varphi^{\star}(A) \leqq \varphi^{\sigma}(A)$ and can thus assume that $\varphi^{\sigma}(A) < \infty$. Consider $U \in \mathfrak{S}^{\sigma}$ with $U \supset A$, and recall that φ^{σ} is outer regular $\mathfrak{S}^{\sigma}$. There are subsets $P \in [\varphi < \infty]$ such that $P \subset U$. It follows that

$$\varphi^{\star}(A) \leqq \varphi^{\star}(P \cup A) = \varphi^{\sigma}(P \cup A) \leqq \varphi^{\sigma}(U),$$

and hence $\varphi^{\star}(A) \leqq \varphi^{\sigma}(A)$.

τ) Fix $A \in \mathfrak{C}(\varphi^\sigma, \dot{+})$ with $\varphi^\sigma(A) < \infty$. For $P, Q \in [\varphi < \infty]$ we have as above

$$\begin{aligned} \varphi(P) + \varphi(Q) &= \varphi^\sigma(P) + \varphi^\sigma(Q) = \varphi^\sigma(P|A|Q) + \varphi^\sigma(P|A'|Q) \\ &\geqq \varphi^\tau(P|A|Q) + \varphi^\tau(P|A'|Q) \geqq \varphi^\tau(P \cup Q) + \varphi^\tau(P \cap Q) \\ &= \varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q), \end{aligned}$$

where all values are finite. It follows that

$$\varphi^\sigma(P|A|Q) = \varphi^\tau(P|A|Q) \text{ for all } P, Q \in [\varphi < \infty].$$

Since φ^σ and φ^τ are upward σ continuous by 4.7 this implies that

$$\varphi^\sigma(P|A|Q) = \varphi^\tau(P|A|Q) \text{ for all } P, Q \in [\varphi < \infty]^\sigma.$$

Since $\varphi^\sigma(A) < \infty$ there exists $Q \in [\varphi < \infty]^\sigma$ with $Q \supset A$. Then $\varphi^\sigma(P \cup A) = \varphi^\tau(P \cup A)$ for all $P \in [\varphi < \infty]^\sigma$. Now we have to prove that $\varphi^\sigma(A) \leqq \varphi^\tau(A)$ and can thus assume that $\varphi^\tau(A) < \infty$. Consider $U \in \mathfrak{S}^\tau$ with $U \supset A$, and recall that φ^τ is outer regular $\mathfrak{S}^\tau$. There are subsets $P \in [\varphi < \infty]$ such that $P \subset U$. It follows that

$$\varphi^\sigma(A) \leqq \varphi^\sigma(P \cup A) = \varphi^\tau(P \cup A) \leqq \varphi^\tau(U),$$

and hence $\varphi^\sigma(A) \leqq \varphi^\tau(A)$.

5.10. Exercise. σ) Construct an example which shows that 5.9.σ) becomes false without the condition $\varphi^\star(A) < \infty$. Hint: Let $\mathfrak{S}$ consist of the finite subsets of an infinite countable X, and let $\varphi = 0$. Determine $\mathfrak{C}(\varphi^\star, \dot{+})$ with the aid of 5.2. τ) Do the same for 5.9.τ).

However, we shall see that the three properties of φ to be an outer $\bullet$ premeasure for $\bullet = \star\sigma\tau$ are independent, except that as a consequence of 5.5.5) the combination $+ - +$ of these properties cannot occur. The independence is plausible after 5.5.5): This condition can be subdivided into two partial ones, such that the one increases and the other decreases with $\bullet = \star\sigma\tau$. We shall come back to this point in 5.15 in the frame of the conventional outer situation.

The Conventional Outer Situation

The above central theorem of the chapter will be most important in two particular cases. These are the specializations

$\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$, and

$X \in \mathfrak{S}$ and $\varphi(X) = 0$.

The first one is called the conventional outer situation. It will be the theme of the present subsection. This specialization contains, and unifies and clarifies those earlier extension procedures which were in visible or invisible manner based on outer regularity. The other one is what later will become the conventional inner situation. It will achieve the same for the earlier extension procedures based on inner regularity. It is obvious that this specialization should be treated via the upside-down transform method. However, it seems more natural to perform the upside-down procedure for the entire development, and then to specialize to the case $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ as before. This will be done in the next section.

For the present we consider a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and an isotone set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$. There are certain immediate simplifications: An outer $\bullet$ extension of φ is an extension of φ which is a ccontent $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A}$, with the further properties as above. Furthermore we have $\varphi^\bullet : \mathfrak{P}(X) \to [0, \infty]$ with $\varphi^\bullet(\varnothing) = 0$. Thus we can write $\mathfrak{C}(\varphi^\bullet)$ instead of $\mathfrak{C}(\varphi^\bullet, \dot{+})$. Also the definition of upward essential simplifies in an obvious manner. It is natural to specialize 5.2 and 5.5 to $\mathfrak{P} = \{\varnothing\}$, and for simplicity we take $\mathfrak{Q} = [\varphi < \infty]$. Let us then rewrite the outer main theorem with these simplifications.

5.11. Theorem (Conventional Outer Main Theorem). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$, and $\varphi : \mathfrak{S} \to [0, \infty]$ be an isotone and submodular set function with $\varphi(\varnothing) = 0$. Then the following are equivalent.*

1) *There exist outer $\bullet$ extensions of φ, that is φ is an outer $\bullet$ premeasure.*

2) *$\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is an outer $\bullet$ extension of φ. Furthermore*

$$\text{if } \bullet = \star \quad : \quad \varphi^\bullet|\mathfrak{C}(\varphi^\bullet) \text{ is a ccontent on the algebra } \mathfrak{C}(\varphi^\bullet),$$
$$\text{if } \bullet = \sigma\tau \quad : \quad \varphi^\bullet|\mathfrak{C}(\varphi^\bullet) \text{ is a cmeasure on the } \sigma \text{ algebra } \mathfrak{C}(\varphi^\bullet).$$

3) *$\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is an extension of φ in the crude sense, that is $\mathfrak{S} \subset \mathfrak{C}(\varphi^\bullet)$ and $\varphi = \varphi^\bullet|\mathfrak{S}$.*

4) *$\varphi(B) = \varphi(A) + \varphi^\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$. In case $\bullet = \tau$ furthermore φ is upward essential.*

5) *$\varphi = \varphi^\bullet|\mathfrak{S}$; and $\varphi(B) \geqq \varphi(A) + \varphi^\bullet(B \setminus A)$ for all $A \subset B$ in $[\varphi < \infty]$. In case $\bullet = \tau$ furthermore φ is upward essential.*

Note that 5.4 then implies $[\varphi < \infty]\top\mathfrak{S}^\bullet \subset \mathfrak{C}(\varphi^\bullet)$.

Assume that $\mathfrak{S}$ is a lattice with $\varnothing \in \mathfrak{S}$. We define an isotone set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ to be **outer $\bullet$ tight** iff it fulfils

$$\varphi(B) \geqq \varphi(A) + \varphi^\bullet(B \setminus A) \quad \text{for all } A \subset B \text{ in } \mathfrak{S},$$

as it appears in condition 5) above. It is obvious that

$$\text{outer } \star \text{ tight } \Rightarrow \text{ outer } \sigma \text{ tight } \Rightarrow \text{ outer } \tau \text{ tight }.$$

We show on the spot that both converses $\Leftarrow$ are false. The counterexamples will be isotone and modular set functions $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ which are upward τ continuous.

5.12. Exercise. We recall from 2.3.1) for $a \in X$ the Dirac set functions $\delta_a : \mathfrak{P}(X) \to \{0, 1\}$. δ_a is a cmeasure and upward and downward τ continuous. Now assume that X is a Hausdorff topological space. We consider the set functions $\varphi := \delta_a|\mathrm{Op}(X)$ and $\psi := \delta_a|\mathrm{Cl}(X)$. 1) We have $\varphi^\bullet = \delta_a$ for all $\bullet = \star\sigma\tau$. Therefore φ is outer $\bullet$ tight and an outer $\bullet$ premeasure. 2) We have the equivalences

$$\psi \text{ outer } \bullet \text{ tight } \Longleftrightarrow \psi^\bullet(\{a\}') = 0 \Longleftrightarrow \{a\} \in (\mathrm{Op}(X))_\bullet.$$

The condition on the right side can be different for $\bullet = \star\sigma\tau$: For $\bullet = \star$ it means that a is an isolated point of X. For $\bullet = \sigma$ it means that in classical notation $\{a\}$ is a G_δ set. For $\bullet = \tau$ it is always fulfilled. Thus we obtain obvious counterexamples as announced above. Furthermore $\psi^\tau = \delta_a$; therefore ψ is upward essential and hence an outer τ premeasure.

5.13. Special Case (Traditional Type). *Assume that* $\mathfrak{S}$ *is a ring. Then condition* 5) *simplifies to*
5∘) $\varphi = \varphi^\bullet|\mathfrak{S}$; *and* φ *is supermodular. In case* $\bullet = \tau$ *furthermore* φ *is upward essential.*

The conventional outer main theorem will henceforth be one of our systematic tools. A fundamental achievement will be the extension 7.12.1) of the last special case. For the present it will be applied to the former main example $\lambda : \mathfrak{K} = \mathrm{Comp}(\mathbb{R}^n) \to [0,\infty[$ in order to obtain the Lebesgue measure on $\mathbb{R}^n$ and its basic properties in the spirit of the outer theory.

The decisive fact follows from a simple observation which will be systematized below: For each pair $A \subset B$ in $\mathfrak{K}$ we have $B \setminus A \in \mathfrak{K}^\sigma$, that is there exists a sequence $(K_l)_l$ in $\mathfrak{K}$ such that $K_l \uparrow B \setminus A$. Then $A \cap K_l = \varnothing$ and $A \cup K_l \uparrow B$. We conclude from 2.26 that $\lambda(A) + \lambda(K_l) = \lambda(A \cup K_l) \leqq \lambda(B)$ and hence

$$\lambda^\sigma(B \setminus A) \leqq \lim_{l\to\infty} \lambda(K_l) \leqq \lambda(B) - \lambda(A);$$

note that 2.27 even implies that $\lambda(A \cup K_l) \uparrow \lambda(B)$ and hence $\lambda(K_l) \uparrow \lambda(B) - \lambda(A)$. Thus λ is outer σ tight. From 2.27 it follows that λ is an outer σ premeasure. The achievement of the conventional outer main theorem is then the cmeasure

$$\Lambda := \lambda^\sigma|\mathfrak{C}(\lambda^\sigma) = \lambda^\sigma|\mathfrak{L} \quad \text{on} \quad \mathfrak{L} := \mathfrak{C}(\lambda^\sigma),$$

defined to be the **Lebesgue measure** on $\mathbb{R}^n$. The last assertion in 5.11 furnishes

$$\mathrm{Cl}(\mathbb{R}^n) \subset \mathfrak{K}\top\mathfrak{K} \subset \mathfrak{K}\top\mathfrak{K}^\sigma \subset \mathfrak{C}(\lambda^\sigma) = \mathfrak{L} \quad \text{and hence} \quad \mathrm{Bor}(\mathbb{R}^n) \subset \mathfrak{L}.$$

The restriction $\Lambda|\mathrm{Bor}(\mathbb{R}^n)$ is called the **Borel-Lebesgue measure** on $\mathbb{R}^n$. All this is the first statement in the comprehensive theorem which follows.

5.14. Theorem. 1) $\lambda : \mathfrak{K} = \mathrm{Comp}(\mathbb{R}^n) \to [0,\infty[$ *is an outer* σ *premeasure. The Lebesgue measure* $\Lambda := \lambda^\sigma|\mathfrak{C}(\lambda^\sigma)$ *has the domain* $\mathfrak{L} := \mathfrak{C}(\lambda^\sigma) \supset \mathrm{Bor}(\mathbb{R}^n)$. 2) λ *is not upward* τ *continuous and hence not an outer* τ *premeasure.* 3) λ *is not outer* $\star$ *tight and hence not an outer* $\star$ *premeasure.*

4) λ^σ *and hence* Λ *are outer regular* $\mathrm{Op}(\mathbb{R}^n)$. 5) Λ *is inner regular* $\mathfrak{K} = \mathrm{Comp}(\mathbb{R}^n)$. 6) $\Lambda|\mathrm{Op}(\mathbb{R}^n)$ *is upward* τ *continuous.* 7) $\Lambda|\mathrm{Op}(\mathbb{R}^n) =: \omega$ *is an outer* $\bullet$ *premeasure for all* $\bullet = \star\sigma\tau$. *It satisfies* $\omega^\bullet = \lambda^\sigma$ *and hence* $\omega^\bullet|\mathfrak{C}(\omega^\bullet) = \Lambda$.

Proof. 1) has been proved above. 2) is obvious since $\lambda(F) = 0$ for all finite $F \subset \mathbb{R}^n$. The next proofs require the preparations which follow. i) For $K \in \mathfrak{K}$ and $\varepsilon > 0$ there exists an open $U \supset K$ with $\Lambda(U) \leqq \lambda(K) + \varepsilon$. In fact, for $K \neq \varnothing$ and $K(\delta) := \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) \leqq \delta\}$ we see from 2.24 that $\Lambda(\mathrm{Int}K(\delta)) \downarrow \lambda(K)$ for $\delta \downarrow 0$. ii) For $A \in \mathfrak{K}^\sigma$ with $\Lambda(A) < \infty$

and $\varepsilon > 0$ there exists an open $U \supset A$ with $\Lambda(U) \leqq \Lambda(A) + \varepsilon$. To see this choose a sequence $(K_l)_l$ in $\mathfrak{K}$ with $K_l \uparrow A$ and open subsets $U_l \supset K_l$ with $\Lambda(U_l) \leqq \lambda(K_l) + \varepsilon 2^{-l}$. For the $V_l := U_1 \cup \cdots \cup U_l$ one obtains via induction $\Lambda(V_l) \leqq \lambda(K_l) + \varepsilon(1 - 2^{-l})$. Thus $V_l \uparrow V$ furnishes an open $V \supset A$ with $\Lambda(V) \leqq \Lambda(A) + \varepsilon$.

3) Let $B := \{x \in \mathbb{R}^n : 0 \leqq x_1, \cdots, x_n \leqq 1\}$ be the unit cube of $\mathbb{R}^n$ and $D \subset \mathrm{Int}B$ be a countable dense subset. From $\Lambda(D) = 0$ and ii) we obtain an open subset $U \supset D$ of $\mathrm{Int}B$ with $\Lambda(U) < 1$. Note that $\lambda^\star(U) = 1$. Thus for the compact $A := B \setminus U \subset B$ we have $U = B \setminus A$ and

$$\lambda(B) = \lambda(A) + \Lambda(U) < \lambda(A) + 1 = \lambda(A) + \lambda^\star(B \setminus A).$$

It follows that λ is not outer $\star$ tight. 4) follows from 4.1.4) and the above ii). 5) Since $\mathbb{R}^n$ is in $\mathfrak{K}^\sigma$ we can restrict ourselves to $A \in \mathfrak{L}$ such that $A \subset$ some $K \in \mathfrak{K}$. Fix $\varepsilon > 0$. By 4) there exists an open $U \supset K \cap A'$ with $\Lambda(U) \leqq \Lambda(K \cap A') + \varepsilon$. Then $K \cap U'$ is compact $\subset A$, and we have

$$\begin{aligned} \Lambda(A) + \Lambda(K \cap U) &\leqq \Lambda(A) + \Lambda(U) \leqq \Lambda(A) + \Lambda(K \cap A') + \varepsilon \\ &= \lambda(K) + \varepsilon = \lambda(K \cap U') + \Lambda(K \cap U) + \varepsilon, \end{aligned}$$

and hence $\Lambda(A) \leqq \lambda(K \cap U') + \varepsilon$. 6) Let $A \in \mathrm{Op}(\mathbb{R}^n)$, and $\mathfrak{M} \subset \mathrm{Op}(\mathbb{R}^n)$ be a paving with $\mathfrak{M} \uparrow A$. For real $c < \Lambda(A)$ we obtain from 5) a compact $K \subset A$ with $c < \lambda(K)$. Now $K \subset$ some $M \in \mathfrak{M}$, and hence $c < \sup\{\Lambda(M) : M \in \mathfrak{M}\}$. The assertion follows. 7) We see from 4)6) that Λ is an outer $\bullet$ extension of ω, so that ω is an outer $\bullet$ premeasure. Now $\omega^\bullet$ and λ^σ coincide on $\mathrm{Op}(\mathbb{R}^n)$ and are both outer regular $\mathrm{Op}(\mathbb{R}^n)$. Therefore $\omega^\bullet = \lambda^\sigma$. The proof is complete.

5.15. Exercise. We can now prove that for the isotone and modular set functions $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ the three properties to be an outer $\bullet$ premeasure for $\bullet = \star\sigma\tau$ are independent, as announced at the end of the last subsection. There are $2^3 = 8$ combinations of these properties. We know that the combination $+-+$ cannot occur. 1) Deduce examples for $--+$ and $-++$ from 5.12.2). 2) Deduce examples for $---$ and $+--$ from 3.11. 3) Deduce examples for $-+-$ and $+++$ and also for $++-$ from 5.14.

6. The Inner Extension Theory

The basic part of the present section obtains the inner extension theory as a mere transcription of the outer extension theory via the upside-down transform method. Thus the two extension theories are in fact identical. We shall add a subsection on further results in the τ case, which in practice is much more important in the inner than in the outer situation. Then we specialize to the conventional inner situation as announced.

The Basic Definition

Let as before $\mathfrak{S}$ be a lattice in a nonvoid set X.

DEFINITION. Let $\varphi : \mathfrak{S} \to [-\infty, \infty[$ be an isotone set function $\not\equiv -\infty$. For $\bullet = \star\sigma\tau$ we define an **inner $\bullet$ extension** of φ to be an extension of φ which is a $\dot{+}$ content $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ on an oval $\mathfrak{A}$, such that also $\mathfrak{S}_\bullet \subset \mathfrak{A}$ and that

α is inner regular $\mathfrak{S}_\bullet$, and

$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous; in this context note that $\alpha|\mathfrak{S}_\bullet < \infty$.

We define φ to be an **inner $\bullet$ premeasure** iff it admits inner $\bullet$ extensions. Thus an inner $\bullet$ premeasure is modular and downward $\bullet$ continuous.

As before the principal aim is to characterize those φ which are inner $\bullet$ premeasures, and then to describe all inner $\bullet$ extensions of φ.

6.1. EXERCISE. Let $\varphi : \mathfrak{S} \to [-\infty, \infty[$ be an isotone set function $\not\equiv -\infty$, and hence $\varphi\bot : \mathfrak{S}\bot \to]-\infty, \infty]$ an isotone set function $\not\equiv \infty$. Then a set function $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ is an inner $\bullet$ extension of φ iff the set function $\alpha\bot : \mathfrak{A}\bot \to \overline{\mathbb{R}}$ is an outer $\bullet$ extension of $\varphi\bot$.

We also refer to the instructive exercise 9.21 below. It is ab-ovo and could have been placed here, but will be postponed until it will be needed.

The Inner Envelopes

Let $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ be an isotone set function on a lattice $\mathfrak{S}$. As before we define its **crude inner envelope** $\varphi_\star : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ to be

$$\varphi_\star(A) = \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset A\} \quad \text{for } A \subset X.$$

Likewise we define the **inner envelopes** $\varphi_\sigma, \varphi_\tau : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ as the counterparts of the respective outer formations to be

$$\begin{aligned}
\varphi_\sigma(A) &= \sup\{\lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } S_l \downarrow\subset A\} \quad \text{for } A \subset X,\\
\varphi_\tau(A) &= \sup\{\inf_{S\in\mathfrak{M}} \varphi(S) : \mathfrak{M} \text{ paving } \subset \mathfrak{S} \text{ with } \mathfrak{M} \downarrow\subset A\} \quad \text{for } A \subset X.
\end{aligned}$$

As before we have for $\bullet = \star\sigma\tau$ the common formula

$$\varphi_\bullet(A) = \sup\{\inf_{S\in\mathfrak{M}} \varphi(S) : \mathfrak{M} \text{ paving } \subset \mathfrak{S} \text{ of type } \bullet \text{ with } \mathfrak{M} \downarrow\subset A\}.$$

6.2. EXERCISE. $(\varphi_\bullet)\bot = (\varphi\bot)^\bullet$ for $\bullet = \star\sigma\tau$.

The upside-down transform method thus furnishes the inner counterparts of the respective properties proved in the outer situation. For convenience we list the basic ones.

6.3. PROPERTIES. 1) $\varphi_\star|\mathfrak{S} = \varphi$. 2) $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$. 3) *$\varphi_\bullet$ is isotone.* 4) *$\varphi_\bullet$ is inner regular $[\varphi_\bullet|\mathfrak{S}_\bullet > -\infty] \subset \mathfrak{S}_\bullet$.* 5) *Assume that φ is supermodular $\dot{+}$. Then $\varphi_\star$ is supermodular $\dot{+}$, and $\varphi_\bullet$ for $\bullet = \sigma\tau$ is supermodular $\dot{+}$ when either $\varphi < \infty$ or $\varphi_\bullet > -\infty$.*

6.4. EXERCISE. *Let* φ *be submodular* $\dot{+}$. *Assume that* $A, B \subset X$ *are* **coseparated** $\mathfrak{S}$ *in the sense that*

for each $M \in \mathfrak{S}$ *with* $M \subset A \cup B$
there exist $S, T \in \mathfrak{S}$ *with* $S \subset A$ *and* $T \subset B$ *such that* $M \subset S \cup T$.

Then $\varphi_\star(A \cup B) \dot{+} \varphi_\star(A \cap B) \leqq \varphi_\star(A) \dot{+} \varphi_\star(B)$.

6.5. PROPOSITION. *For an isotone set function* $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ *and* $\bullet = \sigma\tau$ *the following are equivalent.*

i) $\varphi_\bullet | \mathfrak{S} = \varphi$;
ii) φ *is downward* $\bullet$ *continuous.*

In this case we have furthermore

iii) $\varphi_\bullet | \mathfrak{S}_\bullet$ *is downward* $\bullet$ *continuous;*
iv) *if* $\{S \in \mathfrak{S}_\bullet : \varphi_\bullet(S) > -\infty\} \subset \mathfrak{S}$ *then* $\varphi_\bullet = \varphi_\star$.

6.6. LEMMA. *Let* $\mathfrak{M} \subset \mathfrak{S}_\bullet$ *be a paving of type* $\bullet$ *with* $\mathfrak{M} \downarrow A$. *Then of course* $A \in \mathfrak{S}_\bullet$. *Furthermore there exists a paving* $\mathfrak{N} \subset \mathfrak{S}$ *of type* $\bullet$ *with* $\mathfrak{N} \downarrow A$ *and* $\mathfrak{N} \subset (\sqsupset \mathfrak{M})$.

6.7. THEOREM. *Assume that* $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ *is isotone and supermodular* $\dot{+}$. *Then* φ_σ *and* φ_τ *are almost downward* σ *continuous.*

6.8. LEMMA. *Assume that* $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ *is isotone and supermodular* $\dot{+}$. *For* $P_1, \cdots, P_n, Q \in \mathfrak{S}$ *with* $\varphi(P_1), \cdots, \varphi(P_n), \varphi(Q) > -\infty$ *then* $\varphi(P_1 \cap \cdots \cap P_n \cap Q) > -\infty$ *and*

$$\varphi(P_1 \cap \cdots \cap P_n \cap Q) + \sum_{l=1}^{n} \varphi(P_l \cup Q) \geqq \sum_{l=1}^{n} \varphi(P_l) + \varphi(Q).$$

Next we define an isotone set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ to be **downward $\bullet$ essential** iff its upside-down transform $\varphi\perp : \mathfrak{S}\perp \to \overline{\mathbb{R}}$ is upward $\bullet$ essential. One verifies that this means that

$$\begin{aligned} \varphi_\bullet(A) &= \inf\{\varphi_\bullet(A \cup S) : S \in [\varphi > -\infty]\} \quad \text{for all } A \subset X \text{ with} \\ -\infty < \varphi_\bullet(A) &\leqq \inf\{\varphi_\bullet(A \cup S) : S \in [\varphi > -\infty]\} < \infty. \end{aligned}$$

We obtain the counterpart of the former result.

6.9. PROPOSITION. *Let* $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ *be isotone.* $\star$) φ *is downward* $\star$ *essential.* σ) *If* φ *is supermodular* $\dot{+}$ *then it is downward* σ *essential.* τ) *Assume that* φ *is supermodular* $\dot{+}$ *and such that each* $A \subset X$ *with* $\varphi_\tau(A) > -\infty$ *is downward enclosable* $[\varphi > -\infty]_\sigma$. *Then* φ *is downward* τ *essential.*

Therefore an isotone and supermodular $\dot{+}$ set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ will be called **downward essential** instead of downward τ essential.

After these transcriptions we conclude the subsection with some simple but important relations which involve envelopes of both kinds.

6.10. PROPOSITION. 1) *We have* $\varphi^\bullet \leqq \varphi_\star$ *on* $\mathfrak{S}^\bullet$. *Therefore* $\varphi^\bullet \leqq (\varphi_\star | \mathfrak{S}^\bullet)^\star$. 2) *The following are equivalent.* i) $\varphi = \varphi^\bullet | \mathfrak{S}$. ii) $\varphi^\bullet \geqq \varphi_\star$ *on* $\mathfrak{S}^\bullet$ *and hence* $\varphi^\bullet = \varphi_\star$ *on* $\mathfrak{S}^\bullet$. iii) $\varphi^\bullet \geqq \varphi_\star$. iv) $\varphi^\bullet \geqq (\varphi_\star | \mathfrak{S}^\bullet)^\star$ *and hence* $\varphi^\bullet = (\varphi_\star | \mathfrak{S}^\bullet)^\star$.

Proof. 1) Let $A \in \mathfrak{S}^\bullet$ and $\mathfrak{M} \subset \mathfrak{S}$ be a paving of type $\bullet$ with $\mathfrak{M} \uparrow A$. By definition then $\varphi(S) \leqq \varphi_\star(A)$ $\forall S \in \mathfrak{M}$ and hence

$$\varphi^\bullet(A) \leqq \sup_{S \in \mathfrak{M}} \varphi(S) \leqq \varphi_\star(A).$$

The second relation is then clear. 2) The implications iii)⇒iv)⇒ii)⇒i) are obvious. i)⇒iii) Let $A \subset X$. For $S \in \mathfrak{S}$ with $S \subset A$ then $\varphi(S) = \varphi^\bullet(S) \leqq \varphi^\bullet(A)$. Thus $\varphi_\star(A) \leqq \varphi^\bullet(A)$.

6.11. Exercise. 1) *We have* $\varphi_\bullet \geqq \varphi^\star$ *on* $\mathfrak{S}_\bullet$. *Therefore* $\varphi_\bullet \geqq (\varphi^\star|\mathfrak{S}_\bullet)_\star$. 2) *The following are equivalent.* i) $\varphi = \varphi_\bullet|\mathfrak{S}$. ii) $\varphi_\bullet \leqq \varphi^\star$ *on* $\mathfrak{S}_\bullet$ *and hence* $\varphi_\bullet = \varphi^\star$ *on* $\mathfrak{S}_\bullet$. iii) $\varphi_\bullet \leqq \varphi^\star$. iv) $\varphi_\bullet \leqq (\varphi^\star|\mathfrak{S}_\bullet)_\star$ *and hence* $\varphi_\bullet = (\varphi^\star|\mathfrak{S}_\bullet)_\star$.

In some earlier versions of the outer and inner extension theories the above formations

$$\varphi^{(\bullet)} := (\varphi_\star|\mathfrak{S}^\bullet)^\star \quad \text{and} \quad \varphi_{(\bullet)} := (\varphi^\star|\mathfrak{S}_\bullet)_\star$$

have been used in more or less explicit manner, in places where in the present text the envelopes $\varphi^\bullet$ and $\varphi_\bullet$ are the natural means. For some details we refer to the bibliographical annex to the chapter. Of course $\varphi^{(\star)} = \varphi^\star$ and $\varphi_{(\star)} = \varphi_\star$. For $\bullet = \sigma\tau$ the formations $\varphi^{(\bullet)}$ and $\varphi_{(\bullet)}$ are much more complicated than $\varphi^\bullet$ and $\varphi_\bullet$. For example, it is unclear whether beyond 6.10.2) and 6.11.2) they preserve semimodularity in the appropriate sense.

The Carathéodory Class in the Spirit of the Inner Theory

The second tool in the outer extension theory was the Carathéodory class $\mathfrak{C}(\cdot)$. Its definition and basic properties were not related to outer/inner aspects. Thus there is no reason for transcription. However, there was a subsequent subsection on the Carathéodory class in the spirit of the outer theory. For the present context this subsection consisted of intermediate results which need not be transcribed, so that the transcription could proceed to the next section on the main theorem. But the transcribed versions of some former results will be needed later, and therefore will be inserted at this point. For the transcription we refer to the earlier 4.15.

6.12. Remark. *Let* $\mathfrak{T}$ *be a paving in* X *and* $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ *be isotone and inner regular* $\mathfrak{T}$. *If* $A \subset X$ *satisfies*

$$\phi(P) \dot{+} \phi(Q) \leqq \phi(P|A|Q) \dot{+} \phi(P|A'|Q) \quad \textit{for all } P, Q \in \mathfrak{T},$$

then $A \in \mathfrak{C}(\phi, \dot{+})$.

6.13. Exercise. *Let* $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ *be a content* $\dot{+}$ *on an oval* $\mathfrak{A}$. *Then* $\mathfrak{A} \subset \mathfrak{C}(\alpha_\star, \dot{+})$.

6.14. Lemma. *Let* $P \subset Q \subset X$. *Assume that* $\varphi : P \sqsubset Q \to \mathbb{R}$ *is supermodular. If* $A \subset X$ *satisfies*

$$\varphi(P) + \varphi(Q) \leqq \varphi(P|A|Q) + \varphi(P|A'|Q),$$

then

$$\varphi(U) + \varphi(V) = \varphi(U|A|V) + \varphi(U|A'|V) \quad \textit{for all } U, V \in P \sqsubset Q.$$

6.15. Proposition. *Assume that* $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ *is an isotone set function. Let* $\mathfrak{P} \downarrow$ *and* $\mathfrak{Q} \uparrow$ *be pavings in* X *with nonvoid* $\mathfrak{P} \sqsubset \mathfrak{Q}$ *such that*

$\phi|\mathfrak{P}$ *and* $\phi|\mathfrak{Q}$ *are finite, and* $\phi|\mathfrak{P} \sqsubset \mathfrak{Q}$ *is supermodular.*

Furthermore let $\mathfrak{H} \downarrow$ *be a paving in* X *with* $\mathfrak{P} \subset \mathfrak{H}$ *such that*

ϕ *is inner regular* $\mathfrak{H} \sqsubset \mathfrak{Q}$,

$\phi(T) = \inf_{P \in \mathfrak{P}} \phi(T \cup P)$ *for all* $T \in \mathfrak{H} \sqsubset \mathfrak{Q}$.

If $A \subset X$ *satisfies*

$$\phi(P) + \phi(Q) \leqq \phi(P|A|Q) + \phi(P|A'|Q) \quad \text{for all } P \in \mathfrak{P} \text{ and } Q \in \mathfrak{Q} \text{ with } P \subset Q,$$

then $A \in \mathfrak{C}(\phi, \dot{+})$.

As before we have two addenda.

6.16. Addendum. *Assume in addition that* $\phi|\mathfrak{P} \sqsubset \mathfrak{Q}$ *is downward* σ *continuous. Then* $\mathfrak{C}(\phi, \dot{+})$ *is a* σ *algebra.*

6.17. Addendum. *Assume in addition that* $\theta : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ *is an isotone set function with* $\theta \leqq \phi$ *such that*

$\theta|\mathfrak{P} = \phi|\mathfrak{P}$ *and* $\theta|\mathfrak{Q} = \phi|\mathfrak{Q}$,

$\theta(T) = \inf_{P \in \mathfrak{P}} \theta(T \cup P)$ *for all* $T \in \mathfrak{H} \sqsubset \mathfrak{Q}$.

Then $\phi|\mathfrak{C}(\phi, \dot{+})$ *is an extension of* $\theta|\mathfrak{C}(\theta, \dot{+})$.

The Inner Main Theorem

We assume that $\varphi : \mathfrak{S} \to [-\infty, \infty[$ is an isotone set function $\not\equiv -\infty$ on a lattice $\mathfrak{S}$ in X. The assertions which follow are immediate consequences of their outer counterparts via the upside-down transform method. We recall 6.1 and 6.2, and once more the earlier 4.15.

6.18. Theorem. *Assume that* $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ *is an inner* $\bullet$ *extension of* φ. *Then* α *is a restriction of* $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \dot{+})$.

6.19. Proposition. *Let* φ *be supermodular with* $\varphi_\bullet|\mathfrak{S} < \infty$, *and downward essential in case* $\bullet = \tau$. *Fix pavings*

$\mathfrak{P} \subset [\varphi > -\infty]$ *downward cofinal, and*

$\mathfrak{Q} \subset [\varphi > -\infty]$ *upward cofinal.*

If $A \subset X$ *satisfies*

$$\varphi_\bullet(P) + \varphi_\bullet(Q) \leqq \varphi_\bullet(P|A|Q) + \varphi_\bullet(P|A'|Q) \quad \text{for all } P \in \mathfrak{P} \text{ and } Q \in \mathfrak{Q} \text{ with } P \subset Q,$$

then $A \in \mathfrak{C}(\varphi_\bullet, \dot{+})$.

6.20. Addendum. *For* $\bullet = \sigma\tau$ *the class* $\mathfrak{C}(\varphi_\bullet, \dot{+})$ *is a* σ *algebra.*

The next assertion does not result from the upside-down technique but is an immediate consequence of 6.19 as before.

6.21. Exercise. *Let φ be supermodular with $\varphi_\bullet|\mathfrak{S} < \infty$, and downward essential in case $\bullet = \tau$. Assume that $\mathfrak{Q} \subset [\varphi > -\infty]$ is upward cofinal. Then*

$$\mathfrak{Q}\top\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}) \subset \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}).$$

After all the upside-down transform method furnishes the inner main theorem.

6.22. Theorem (Inner Main Theorem). *Let $\varphi : \mathfrak{S} \to [-\infty, \infty[$ be an isotone and supermodular set function $\not\equiv -\infty$ on a lattice $\mathfrak{S}$. Fix pavings*

$\mathfrak{P} \subset [\varphi > -\infty]$ *downward cofinal, and*
$\mathfrak{Q} \subset [\varphi > -\infty]$ *upward cofinal.*

Then the following are equivalent.

1) *There exist inner $\bullet$ extensions of φ, that is φ is an inner $\bullet$ premeasure.*

2) *$\varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$ is an inner $\bullet$ extension of φ. Furthermore*

$$\text{if } \bullet = \star \quad : \quad \varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}) \text{ is a content } \underset{\cdot}{+} \text{ on the algebra } \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}),$$
$$\text{if } \bullet = \sigma\tau \quad : \quad \varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}) \text{ is a measure } \underset{\cdot}{+} \text{ on the } \sigma \text{ algebra } \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}).$$

3) *$\varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$ is an extension of φ in the crude sense, that is $\mathfrak{S} \subset \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$ and $\varphi = \varphi_\bullet|\mathfrak{S}$.*

4) *$\varphi(U) + \varphi(V) = \varphi(M) \underset{\cdot}{+} \varphi_\bullet(U|M'|V)$ for all $U \subset M \subset V$ in $\mathfrak{S}$;* note that $M = U|M|V$. *In case $\bullet = \tau$ furthermore φ is downward essential.*

5) *$\varphi = \varphi_\bullet|\mathfrak{S}$; and $\varphi(P) + \varphi(Q) \leqq \varphi(M) + \varphi_\bullet(P|M'|Q)$ for all $P \subset M \subset Q$ with $P \in \mathfrak{P}$, $Q \in \mathfrak{Q}$, and $M \in \mathfrak{S}$ and hence $\in [\varphi > -\infty]$. In case $\bullet = \tau$ furthermore φ is downward essential.*

Note that 6.21 then implies $\mathfrak{Q}\top\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$.

6.23. Special Case (Traditional Type). *Assume that $\mathfrak{S}$ is an oval. Then condition* 5) *simplifies to*

5∘) *$\varphi = \varphi_\bullet|\mathfrak{S}$; and φ is submodular. In case $\bullet = \tau$ furthermore φ is downward essential.*

Comparison of the three Inner Theories

We obtain the inner counterparts of the respective outer results. In the present subsection let $\varphi : \mathfrak{S} \to [-\infty, \infty[$ be an isotone and supermodular set function $\not\equiv -\infty$ on a lattice $\mathfrak{S}$.

6.24. Proposition. *σ) In case $\varphi = \varphi_\sigma|\mathfrak{S}$ we have $\mathfrak{C}(\varphi_\star, \underset{\cdot}{+}) \subset \mathfrak{C}(\varphi_\sigma, \underset{\cdot}{+})$. τ) In case $\varphi = \varphi_\tau|\mathfrak{S}$ we have $\mathfrak{C}(\varphi_\sigma, \underset{\cdot}{+}) \subset \mathfrak{C}(\varphi_\tau, \underset{\cdot}{+})$.*

6.25. Proposition. *Assume that φ is modular. σ) In case $\varphi = \varphi_\sigma|\mathfrak{S}$ we have $\varphi_\star(A) = \varphi_\sigma(A)$ for all $A \in \mathfrak{C}(\varphi_\star, \underset{\cdot}{+})$ with $\varphi_\star(A) > -\infty$. τ) In case $\varphi = \varphi_\tau|\mathfrak{S}$ we have $\varphi_\sigma(A) = \varphi_\tau(A)$ for all $A \in \mathfrak{C}(\varphi_\sigma, \underset{\cdot}{+})$ with $\varphi_\sigma(A) > -\infty$.*

However, as before the three properties of φ to be an inner $\bullet$ premeasure for $\bullet = \star\sigma\tau$ are independent, except that as a consequence of 6.22.5) the combination $+ - +$ of these properties cannot occur.

Further Results on Nonsequential Continuity

In contrast to the ubiquitous σ continuity of measures the occurrence of τ continuity is restricted to particular and foremost situations and bound to severe limitations. The simplest illustration is as follows.

6.26. EXAMPLE. Let $\mathfrak{S}$ be a lattice in X which contains the finite subsets of X, and let $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ be isotone with $\varphi(S) = 0$ for all finite $S \subset X$. If φ is upward τ continuous then $\varphi = 0$.

In our extension theories the τ extensions are defined to possess a certain τ continuity, but it is restricted to the direct descendants of the initial domain $\mathfrak{S}$. In the sequel we deduce two further results on τ continuity. We restrict this topic to the inner situation where it is much more important. In the present subsection we assume $\varphi : \mathfrak{S} \to [-\infty, \infty[$ to be an isotone and supermodular set function $\not\equiv -\infty$ on a lattice $\mathfrak{S}$.

The main feature is the occurrence of the transporter $\mathfrak{S}\top\mathfrak{S}_\tau$, which of course is $= \mathfrak{S}_\tau\top\mathfrak{S}_\tau = (\mathfrak{S}_\tau)\top$ as well. As a rule the members of $\mathfrak{S}\top\mathfrak{S}_\tau$ can be much larger subsets of X than those of $\mathfrak{S}_\tau$. We note that for an inner τ premeasure φ we have

$$\mathfrak{S}\top\mathfrak{S}_\tau \subset [\varphi > -\infty]\top\mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau, \dot{+}),$$

and hence $(\mathfrak{S}\top\mathfrak{S}_\tau)\perp \subset \mathfrak{C}(\varphi_\tau, \dot{+})$ as well.

6.27. PROPOSITION. *Assume that $\varphi = \varphi_\tau|\mathfrak{S}$. Then the restriction $\varphi_\tau|\mathfrak{S}\top\mathfrak{S}_\tau$ is almost downward τ continuous.*

6.28. PROPOSITION. *Assume that φ is an inner τ premeasure. Then the restriction $\varphi_\tau|(\mathfrak{S}\top\mathfrak{S}_\tau)\perp$ is almost upward τ continuous.*

Proof of 6.27. Let $\mathfrak{M} \subset \mathfrak{S}\top\mathfrak{S}_\tau$ be a paving with $\varphi_\tau(M) < \infty\ \forall M \in \mathfrak{M}$ such that $\mathfrak{M} \downarrow H$; thus $H \in \mathfrak{S}\top\mathfrak{S}_\tau$. To be shown is

$$c := \inf_{M\in\mathfrak{M}} \varphi_\tau(M) \leqq \varphi_\tau(H).$$

We can assume that $c > -\infty$ and hence $c \in \mathbb{R}$. Then $\varphi_\tau(M) \in \mathbb{R}\ \forall M \in \mathfrak{M}$, but a priori $\varphi_\tau(H) = -\infty$ is possible. Let us fix real $\varepsilon > 0$ and $\lambda > \varphi_\tau(H)$. i) Fix $P \in \mathfrak{M}$. By 6.3.4) there exists $S \in \mathfrak{S}_\tau$ with $S \subset P$ such that $\varphi_\tau(S) > \varphi_\tau(P) - \varepsilon$. Note that $\varphi_\tau(S) \leqq \varphi_\tau(P) < \infty$ and hence $\varphi_\tau(S) \in \mathbb{R}$. ii) By assumption $\{M \cap S : M \in \mathfrak{M}\}$ is a paving $\subset \mathfrak{S}_\tau$ with $\downarrow H \cap S \in \mathfrak{S}_\tau$. By 6.5.iii) therefore

$$\inf_{M\in\mathfrak{M}} \varphi_\tau(M \cap S) = \varphi_\tau(H \cap S) \leqq \varphi_\tau(H) < \lambda.$$

Thus there exists $Q \in \mathfrak{M}$ such that $\varphi_\tau(Q \cap S) < \lambda$. Since $\mathfrak{M} \downarrow$ we can assume that $Q \subset P$, so that $Q \cup S \subset P$. iii) By 6.3.5) we then have

$$\begin{aligned} c + \varphi_\tau(S) &\leqq \varphi_\tau(Q) + \varphi_\tau(S) \leqq \varphi_\tau(Q \cup S) + \varphi_\tau(Q \cap S) \\ &\leqq \varphi_\tau(P) + \varphi_\tau(Q \cap S) < \varphi_\tau(S) + \varepsilon + \lambda, \end{aligned}$$

and hence $c < \varepsilon + \lambda$. The assertion follows.

Proof of 6.28. Let $\mathfrak{M} \subset (\mathfrak{S}\top\mathfrak{S}_\tau)\perp$ be a paving with $\varphi_\tau(M) > -\infty$ $\forall M \in \mathfrak{M}$ such that $\mathfrak{M} \uparrow H$; thus $H \in (\mathfrak{S}\top\mathfrak{S}_\tau)\perp$. To be shown is

$$c := \sup_{M\in\mathfrak{M}} \varphi_\tau(M) \geqq \varphi_\tau(H).$$

We can assume that $c < \infty$ and hence $c \in \mathbb{R}$. Then $\varphi_\tau(M) \in \mathbb{R}$ $\forall M \in \mathfrak{M}$, but a priori $\varphi_\tau(H) = \infty$ is possible. Let us fix real $\varepsilon > 0$ and $\lambda < \varphi_\tau(H)$. i) Fix $P \in \mathfrak{M}$. By 6.3.4) there exists $S \in \mathfrak{S}_\tau$ with $S \subset P$ such that $\varphi_\tau(S) > -\infty$ and hence $\varphi_\tau(S) \in \mathbb{R}$. ii) Once more by 6.3.4) there exists $T \in \mathfrak{S}_\tau$ with $T \subset H$ such that $\varphi_\tau(T) > \lambda$ and hence $\varphi_\tau(T) \in \mathbb{R}$. In view of $S \subset P \subset H$ we can assume that $S \subset T$. iii) By assumption $\{M' \cap T : M \in \mathfrak{M}\}$ is a paving $\subset \mathfrak{S}_\tau$ with $\downarrow H' \cap T = \varnothing$. Hence $\{(M' \cap T) \cup S : M \in \mathfrak{M}\}$ is a paving $\subset \mathfrak{S}_\tau$ with $\downarrow S \in \mathfrak{S}_\tau$. By 6.5.iii) therefore

$$\inf_{M\in\mathfrak{M}} \varphi_\tau((M' \cap T) \cup S) = \varphi_\tau(S).$$

Thus there exists $Q \in \mathfrak{M}$ such that $\varphi_\tau((Q' \cap T) \cup S) < \varphi_\tau(S) + \varepsilon$. Since $\mathfrak{M} \uparrow$ we can assume that $Q \supset P$. iv) We have $Q \in \mathfrak{C}(\varphi_\tau, \dot{+})$ and hence

$$\begin{aligned}
\varphi_\tau(S) + \lambda &< \varphi_\tau(S) + \varphi_\tau(T) = \varphi_\tau(S|Q|T) + \varphi_\tau(S|Q'|T) \\
&= \varphi_\tau(Q \cap T) + \varphi_\tau((Q' \cap T) \cup S) \\
&< \varphi_\tau(Q) + \varphi_\tau(S) + \varepsilon \leqq c + \varphi_\tau(S) + \varepsilon,
\end{aligned}$$

and therefore $\lambda < c + \varepsilon$. The assertion follows.

The Conventional Inner Situation

The conventional inner situation is defined to be the specialization that $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$. Thus we consider a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and an isotone set function $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$. Although the full inner situation is known to be identical with the full outer situation, there are characteristic discrepancies between the two conventional situations, as it must be expected from traditional measure theory. Thus we have to assume this time that $\varphi < \infty$. As in the outer situation there are certain immediate simplifications: An inner $\bullet$ extension of φ is a ccontent $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A}$, with the further properties as above. Furthermore we have $\varphi_\bullet : \mathfrak{P}(X) \to [0, \infty]$ with $0 = \varphi(\varnothing) = \varphi_\star(\varnothing) \leqq \varphi_\sigma(\varnothing) \leqq \varphi_\tau(\varnothing)$. Thus as before we can write $\mathfrak{C}(\varphi_\bullet)$ instead of $\mathfrak{C}(\varphi_\bullet, \dot{+})$.

But there are two essential deviations from the conventional outer situation. One deviation is that this time all supermodular φ are downward essential. This is obvious from the definition. Therefore the respective condition can be deleted from the conventional inner results.

The other deviation from the conventional outer situation is that $\varphi_\bullet(\varnothing) = 0$ is a nontrivial condition when $\bullet = \sigma\tau$. This condition will be explored in the course of the present subsection. We shall see that it is much weaker than the full condition $\varphi_\bullet|\mathfrak{S} = \varphi$, and that its verification can be much easier and sometimes even trivial. Therefore it is desirable to have the conventional inner main theorem with a variant of condition 5) in which $\varphi_\bullet(\varnothing)=0$ occurs instead of $\varphi_\bullet|\mathfrak{S}=\varphi$. Of course then the subsequent partial condition in 5) has to be fortified. For this purpose we need the so-called satellites of the envelopes $\varphi_\bullet$ which will be defined next.

For fixed $\bullet=\star\sigma\tau$ and $B\in\mathfrak{S}$ we define the **satellite inner $\bullet$ envelopes** $\varphi_\bullet^B:\mathfrak{P}(X)\to[0,\infty[$ to be

$$\varphi_\bullet^B(A)=\sup\{\inf_{S\in\mathfrak{M}}\varphi(S) \;:\; \begin{array}{l}\mathfrak{M}\text{ paving }\subset\mathfrak{S}\text{ of type }\bullet\text{ with}\\ S\subset B\;\forall S\in\mathfrak{M}\text{ and }\mathfrak{M}\downarrow\subset A\}\end{array}\text{ for }A\subset X.$$

We list the basic properties of these satellites.

6.29. Properties. 1) $\varphi_\bullet^B\leqq\varphi(B)<\infty$. 2) $\varphi_\bullet^B$ *is isotone.* 3) *If* φ *is supermodular then* $\varphi_\bullet^B$ *is supermodular.* 4) *We have*

$$\varphi_\bullet(A)=\sup\{\varphi_\bullet^B(A):B\in\mathfrak{S}\}\quad\textit{for } A\subset X.$$

5) *Assume that* $\varphi=\varphi_\bullet|\mathfrak{S}$. *Then* $\varphi_\bullet(A)=\varphi_\bullet^B(A)$ *for* $A\subset B\in\mathfrak{S}$.

Proof. 1) and 2) are obvious, and 3) follows from 6.3.5) when one notes that $\varphi_\bullet^B=(\varphi|\{S\in\mathfrak{S}:S\subset B\})_\bullet$. 4) We have to prove $\leqq$. Fix $A\subset X$, and let $\mathfrak{M}$ be a paving $\subset\mathfrak{S}$ of type $\bullet$ such that $\mathfrak{M}\downarrow\subset A$. For fixed $B\in\mathfrak{M}$ then $\mathfrak{N}:=\{S\in\mathfrak{M}:S\subset B\}$ is a paving $\subset\mathfrak{S}$ of type $\bullet$ with $\mathfrak{N}\downarrow\subset A$ as well, and we have

$$\inf_{S\in\mathfrak{M}}\varphi(S)=\inf_{S\in\mathfrak{N}}\varphi(S)\leqq\varphi_\bullet^B(A).$$

The assertion follows. 5) Fix $A\subset B\in\mathfrak{S}$. We have to prove $\leqq$. Let $\mathfrak{M}$ be a paving $\subset\mathfrak{S}$ of type $\bullet$ such that $\mathfrak{M}\downarrow M\subset A$. For fixed $H\in\mathfrak{M}$ then $\{S\cup(H\cap B):S\in\mathfrak{M}\}$ is a paving $\subset\mathfrak{S}$ of type $\bullet$ with $\downarrow M\cup(H\cap B)=H\cap B\in\mathfrak{S}$. Hence by assumption and 6.5

$$\inf_{S\in\mathfrak{M}}\varphi(S)\leqq\inf_{S\in\mathfrak{M}}\varphi\big(S\cup(H\cap B)\big)=\varphi(H\cap B).$$

Now $\mathfrak{N}:=\{H\cap B:H\in\mathfrak{M}\}$ is a paving $\subset\mathfrak{S}$ of type $\bullet$ with $\mathfrak{N}\downarrow M\cap B=M$ as well. It follows that

$$\inf_{S\in\mathfrak{M}}\varphi(S)\leqq\inf_{S\in\mathfrak{N}}\varphi(S)\leqq\varphi_\bullet^B(A),$$

and hence the assertion.

The decisive fact on the satellite inner $\bullet$ envelopes is the next lemma.

6.30. Lemma. *Let* φ *be supermodular. Assume that* $\varphi_\bullet(\varnothing)=0$ *and that*

$$\varphi(B)\leqq\varphi(A)+\varphi_\bullet^B(B\setminus A)\quad\textit{for all } A\in\mathfrak{S}\textit{ and } B\in\mathfrak{Q}\textit{ with } A\subset B,$$

where $\mathfrak{Q}\subset\mathfrak{S}$ *is upward cofinal. Then* $\varphi=\varphi_\bullet|\mathfrak{S}$.

Proof. Fix $A\in\mathfrak{S}$. For $B\in\mathfrak{Q}$ with $B\supset A$ we combine the assumptions with 6.29.3)1) to obtain

$$\begin{aligned}\varphi_\bullet^B(A)+\varphi(B)&\leqq\varphi_\bullet^B(A)+\varphi(A)+\varphi_\bullet^B(B\setminus A)\\&\leqq\varphi(A)+\varphi_\bullet^B(B)+\varphi_\bullet^B(\varnothing)\leqq\varphi(A)+\varphi(B),\end{aligned}$$

and hence $\varphi_\bullet^B(A) \leqq \varphi(A)$. Since $\varphi_\bullet^B(\cdot)$ is isotone in $B \in \mathfrak{S}$ it follows from 6.29.4) that $\varphi_\bullet(A) \leqq \varphi(A)$.

As before it is natural to specialize 6.19 and 6.22 to $\mathfrak{P} = \{\varnothing\}$ and $\mathfrak{Q} = \mathfrak{S}$. We then obtain the conventional inner main theorem which follows.

6.31. THEOREM (Conventional Inner Main Theorem). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$, and $\varphi : \mathfrak{S} \to [0,\infty[$ be an isotone and supermodular set function with $\varphi(\varnothing) = 0$. Then the following are equivalent.*

1) *There exist inner $\bullet$ extensions of φ, that is φ is an inner $\bullet$ premeasure.*

2) *$\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ. Furthermore*

$$\textit{if } \bullet = \star \ : \ \varphi_\bullet|\mathfrak{C}(\varphi_\bullet) \textit{ is a ccontent on the algebra } \mathfrak{C}(\varphi_\bullet),$$
$$\textit{if } \bullet = \sigma\tau \ : \ \varphi_\bullet|\mathfrak{C}(\varphi_\bullet) \textit{ is a cmeasure on the } \sigma \textit{ algebra } \mathfrak{C}(\varphi_\bullet).$$

3) *$\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ in the crude sense, that is $\mathfrak{S} \subset \mathfrak{C}(\varphi_\bullet)$ and $\varphi = \varphi_\bullet|\mathfrak{S}$.*

4) *$\varphi(B) = \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

5) *$\varphi = \varphi_\bullet|\mathfrak{S}$; and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

5') *$\varphi_\bullet(\varnothing) = 0$; and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

Note that 6.21 then implies $\mathfrak{S}\top\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$.

Assume that $\mathfrak{S}$ is a lattice with $\varnothing \in \mathfrak{S}$. We define an isotone set function $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ to be **inner $\bullet$ tight** iff it fulfils

$$\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A) \quad \text{for all } A \subset B \text{ in } \mathfrak{S},$$

as it appears in condition 5') above. In case $\bullet = \star$ this means that

$$\varphi(B) \leqq \varphi(A) + \varphi_\star(B \setminus A) \quad \text{for all } A \subset B \text{ in } \mathfrak{S}.$$

It is obvious that

$$\text{inner } \star \text{ tight} \Rightarrow \text{inner } \sigma \text{ tight} \Rightarrow \text{inner } \tau \text{ tight}.$$

As before we show on the spot that both converses $\Leftarrow$ are false. The counterexamples will be isotone and modular set functions $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ which are downward τ continuous.

6.32. EXERCISE. Consider the situation of exercise 5.12. We have the equivalences

$$\varphi \text{ inner } \bullet \text{ tight} \iff \varphi_\bullet(\{a\}) = 1 \iff \{a\} \in (\mathrm{Op}(X))_\bullet.$$

Thus we obtain obvious counterexamples as announced above.

6.33. SPECIAL CASE (Traditional Type). *Assume that $\mathfrak{S}$ is a ring. Then conditions 5)5') simplify to*

5∘) *$\varphi = \varphi_\bullet|\mathfrak{S}$; and φ is submodular.*

5'∘) *$\varphi_\bullet(\varnothing) = 0$; and φ is submodular.*

We next consider the weakened condition $\varphi_\bullet(\varnothing) = 0$ which occurs in 6.30 and in the conventional inner main theorem. For this discussion we assume an isotone set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$. We define φ to be $\bullet$ **continuous at** $\varnothing$ iff

$$\inf_{S \in \mathfrak{M}} \varphi(S) = 0 \quad \text{for each paving } \mathfrak{M} \subset \mathfrak{S} \text{ of type } \bullet \text{ with } \mathfrak{M} \downarrow \varnothing,$$

and to be **almost** $\bullet$ **continuous at** $\varnothing$ iff this holds true whenever $\varphi(S) < \infty\ \forall S \in \mathfrak{M}$. It is clear that φ is $\bullet$ continuous at $\varnothing$ iff $\varphi_\bullet(\varnothing) = 0$. In order to obtain an obvious but famous criterion we define $\mathfrak{S}$ to be $\bullet$ **compact** iff each paving $\mathfrak{M} \subset \mathfrak{S}$ of type $\bullet$ with $\mathfrak{M} \downarrow \varnothing$ satisfies $\varnothing \in \mathfrak{M}$. The reason for this notion is obvious: In each Hausdorff topological space X the lattice $\mathrm{Comp}(X)$ of its compact subsets is τ compact. The next remark is then clear.

6.34. REMARK. *If $\mathfrak{S}$ is $\bullet$ compact then each isotone set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ is $\bullet$ continuous at $\varnothing$.*

It turns out that for $\bullet = \sigma\tau$ the condition $\varphi_\bullet(\varnothing) = 0$ is much weaker than $\varphi = \varphi_\bullet|\mathfrak{S}$. We shall present a dramatic example at the end of the subsection. The example will show in particular that the conventional inner main theorem becomes false when instead of 5)5') one forms the weaker condition which combines the first part of 5') with the second part of 5).

The most important and simplest nontrivial example for the conventional inner situation is the familiar set function $\lambda : \mathfrak{K} = \mathrm{Comp}(\mathbb{R}^n) \to [0, \infty[$. It has been an example for the conventional outer situation in 5.14.

6.35. EXAMPLE. 1) λ is inner $\star$ tight and hence inner $\bullet$ tight for $\bullet = \star\sigma\tau$. In fact, we know that for $A \subset B$ in $\mathfrak{K}$ there exists a sequence $(K_l)_l$ in $\mathfrak{K}$ with $K_l \uparrow B \setminus A$ and hence $\lambda(K_l) \uparrow \lambda(B) - \lambda(A)$. This implies that $\lambda_\star(B \setminus A) = \lambda(B) - \lambda(A)$. 2) λ is $\bullet$ continuous at $\varnothing$ in view of 6.34; we even know from 2.25 that λ is downward $\bullet$ continuous. 3) Therefore the conventional inner main theorem shows that λ is an inner $\bullet$ premeasure for $\bullet = \star\sigma\tau$. 4) From 6.5.iv) we have $\lambda_\star = \lambda_\sigma = \lambda_\tau$. We shall see in 7.5 below that the common maximal inner $\bullet$ extension $\lambda_\bullet|\mathfrak{C}(\lambda_\bullet)$ coincides with $\Lambda := \lambda^\sigma|\mathfrak{C}(\lambda^\sigma)$.

We conclude with the example announced above. It is quite complicated.

6.36. EXERCISE. Construct an example of an isotone and modular set function $\varphi : \mathfrak{S} \to [0, \infty[$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $X = \bigcup_{S \in \mathfrak{S}} S \neq \varnothing$ such that

i) $\mathfrak{S}$ is σ compact and hence $\varphi_\sigma(\varnothing) = 0$, but
ii) $\varphi_\sigma(A) = \infty$ for all nonvoid $A \subset X$.

One can proceed as follows. 1) Let E be an infinite set and $X := E \times \mathbb{N}$. A subset $A \subset X$ is described via its sections $A(s) := \{m \in \mathbb{N} : (s, m) \in A\} \subset \mathbb{N}$ for $s \in E$. Define $\mathfrak{E}$ to consist of the subsets $A \subset X$ such that

1.i) $A(s) \subset \mathbb{N}$ is finite or cofinite for all $s \in E$;

1.ii) $A(s)=\varnothing$ for all $s\in E$ except a finite subset.

Then $\mathfrak{E}$ is a lattice with $\varnothing\in\mathfrak{E}$. Define $\phi:\mathfrak{E}\to[0,\infty[$ to be

$$\phi(A)=\#(\{s\in E: A(s)\subset\mathbb{N}\text{ is cofinite}\})\quad\text{for } A\in\mathfrak{E}.$$

Then ϕ is isotone and modular with $\phi(\varnothing)=0$. 2) If E is countable then there exists a function $\theta:X\to\mathfrak{P}(E)$ such that

2.i) $\theta(x)\subset E$ is infinite for all $x\in X$;
2.ii) for $u\neq v$ in X we have $\theta(u)\cap\theta(v)=\varnothing$;
2.iii) $E=\bigcup\limits_{x\in X}\theta(x)$;
2.iv) for $x=(s,m)\in X$ we have $s\notin\theta(x)$.

Hint: We can assume that $E=\mathbb{N}\cup(-\mathbb{N})$. Let $I:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be a bijection. Define $\theta:X\to\mathfrak{P}(E)$ to be

$$\begin{aligned}\theta(x)&=\{-\varepsilon I\big(n,I(p,q)\big):n\in\mathbb{N}\}\subset E\\&\quad\text{for } x=(\varepsilon p,q)\in X \text{ with } p,q\in\mathbb{N}\text{ and }\varepsilon\in\{-1,1\}.\end{aligned}$$

Then θ is as required. For the sequel we fix E and θ. 3) Define $\mathfrak{S}$ to consist of the subsets $S\in\mathfrak{E}$ such that

$$x\in X\setminus S\Rightarrow S(s)\subset\mathbb{N}\text{ is finite for all } s\in\theta(x).$$

Then $\mathfrak{S}$ is a lattice with $\varnothing\in\mathfrak{S}$. Define $\varphi:=\phi|\mathfrak{S}$. 4) $\mathfrak{S}$ is σ compact. Hint: Let $(S_l)_l$ be a sequence in $\mathfrak{S}$ with $S_l\downarrow\varnothing$. Fix a nonvoid finite subset $F\subset E$ such that $S_1(s)=\varnothing$ for all $s\in E\setminus F$. Then let $x_1,\cdots,x_r\in X$ such that $F\subset\bigcup\limits_{k=1}^{r}\theta(x_k)$. If $x_1,\cdots,x_r\notin S_l$, which is true for almost all $l\geqq 1$, then S_l is finite. This implies that $S_l=\varnothing$ for almost all $l\geqq1$. 5) Fix $a=(p,q)\in X$ and a nonvoid finite $F\subset\theta(a)\subset E$, and note that $p\notin\theta(a)$ and hence $p\notin F$. For $T\subset\mathbb{N}$ cofinite we define $S\subset X$ to be

5.i) $S(p):=\{q\}$;
5.ii) $S(s):=T$ for all $s\in F$;
5.iii) $S(s):=\varnothing$ for all other $s\in E$.

Then $S\in\mathfrak{S}$ with $a\in S$, and $\varphi(S)=\#(F)$. Deduce that $\varphi_\sigma(\{a\})\geqq\#(F)$. 6) It follows from 5) that $\varphi_\sigma(\{a\})=\infty$ for all $a\in X$ and hence $\varphi_\sigma(A)=\infty$ for all nonvoid $A\subset X$.

7. Complements to the Extension Theories

The present section has two independent themes. The first one is to compare the outer and inner extension theories. The other theme is to exhibit certain classes of lattices with $\varnothing$ on which the relevant tightness conditions will be automatic facts like on rings, but which are much more natural initial domains than rings. The idea has been used for the Lebesgue measure in both the outer and the inner situation. Its systematization will lead to an essential increase of the frame of applications. The section concludes with a bibliographical annex.

Comparison of the Outer and Inner Extension Theories

The main result of the present subsection is for a set function $\varphi : \mathfrak{S} \to \mathbb{R}$ on a lattice $\mathfrak{S}$ which is both an outer and an inner $\bullet$ premeasure. We restrict ourselves to $\bullet = \star\sigma$ since the case $\bullet = \tau$ is unrealistic. The result will be that the two maximal extensions $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$ and $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$ coincide to the extent which can be expected in view of the classical uniqueness theorem 3.1.

7.1. LEMMA. *Assume that* $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ *and* $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ *are isotone set functions on lattices* $\mathfrak{S}$ *and* $\mathfrak{A}$*, and that* α *extends* φ*.*

$\star$) $\varphi_\star \leqq \alpha \leqq \varphi^\star$ *on* $\mathfrak{A}$*, and* $\varphi_\star \leqq \varphi^\star$ *on* $\mathfrak{P}(X)$*.*

σ) *If* $\mathfrak{A}$ *is a* σ *lattice and* α *is almost upward and downward* σ *continuous then*

$$\begin{aligned} \varphi_\sigma(A) &\leqq \alpha(A) && \textit{for } A \in \mathfrak{A} \textit{ when } (+)\ \varphi < \infty \textit{ or } \varphi_\sigma(A) < \infty; \\ \alpha(A) &\leqq \varphi^\sigma(A) && \textit{for } A \in \mathfrak{A} \textit{ when } (-) -\infty < \varphi \textit{ or } -\infty < \varphi^\sigma(A); \\ \varphi_\sigma(A) &\leqq \varphi^\sigma(A) && \textit{for } A \subset X \textit{ when } (+) \textit{ and } (-). \end{aligned}$$

Proof. $\star$) is clear from the definitions. In σ) we can for the first assertion assume that $\varphi_\sigma(A) > -\infty$. Let $(S_l)_l$ be a sequence in $\mathfrak{S}$ with $S_l \downarrow$ some $U \subset A$. Then by definition $\lim\limits_{l\to\infty} \varphi(S_l) \leqq \varphi_\sigma(A)$, so that both times we can assume that $\varphi(S_l) < \infty\ \forall l \in \mathbb{N}$. It follows that $U \in \mathfrak{S}_\sigma \subset \mathfrak{A}$ and $\varphi(S_l) = \alpha(S_l) \downarrow \alpha(U) \leqq \alpha(A)$. Therefore $\varphi_\sigma(A) \leqq \alpha(A)$. The second assertion is proved in the same manner. In order to prove the third assertion assume that $\varphi^\sigma(A) < \varphi_\sigma(A)$ and fix a real c with $\varphi^\sigma(A) < c < \varphi_\sigma(A)$. By 4.1.4) and 6.3.4) there are

$$\begin{aligned} V &\in \mathfrak{S}^\sigma \subset \mathfrak{A} && \text{with } V \supset A \text{ and } \varphi^\sigma(V) < c, \\ U &\in \mathfrak{S}_\sigma \subset \mathfrak{A} && \text{with } U \subset A \text{ and } \varphi_\sigma(U) > c. \end{aligned}$$

We can apply the first assertion to U to obtain $\varphi_\sigma(U) \leqq \alpha(U)$, and the second assertion to V to obtain $\alpha(V) \leqq \varphi^\sigma(V)$. It follows that $\alpha(V) \leqq \varphi^\sigma(V) < c < \varphi_\sigma(U) \leqq \alpha(U)$ and hence a contradiction.

7.2. LEMMA. *Assume that* $\varphi : \mathfrak{S} \to \mathbb{R}$ *is an isotone set function on a lattice* $\mathfrak{S}$*, and that* $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ *is an extension of* φ *which is*

$$\begin{aligned} &\textit{for } \bullet = \star &&: \textit{ a content } \dot{\underset{\cdot}{+}} \textit{ on an oval } \mathfrak{A}; \\ &\textit{for } \bullet = \sigma &&: \textit{ a measure } \dot{\underset{\cdot}{+}} \textit{ on a } \sigma \textit{ oval } \mathfrak{A}. \end{aligned}$$

Then for $S, T \in \mathfrak{S}$ *and* $A \in \mathfrak{A}$ *we have*

$$\begin{aligned} \alpha(S|A|T) &= \varphi^\bullet(S|A|T) && \textit{when } A \in \mathfrak{C}(\varphi^\bullet, \dot{+}), \\ \alpha(S|A|T) &= \varphi_\bullet(S|A|T) && \textit{when } A \in \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}). \end{aligned}$$

Proof. Fix $S, T \in \mathfrak{S}$ and $A \in \mathfrak{A}$. Then $S|A|T \in \mathfrak{A}$ and $S|A'|T = T|A|S \in \mathfrak{A}$. From 7.1 we obtain

$$\varphi_\bullet(S|A|T) \leqq \alpha(S|A|T) \leqq \varphi^\bullet(S|A|T),$$

and the same for A'. Furthermore note that $S|A|T$ is between $S\cap T$ and $S\cup T$, and that in case $\bullet=\sigma$ the function φ is upward and downward σ continuous. Thus

$$\begin{aligned}\varphi^{\bullet}(S|A|T) &\leqq \varphi^{\bullet}(S\cup T)=\varphi(S\cup T)<\infty,\\ \varphi_{\bullet}(S|A|T) &\geqq \varphi_{\bullet}(S\cap T)=\varphi(S\cap T)>-\infty,\end{aligned}$$

so that the above values are all finite. The same is true for A'. Now we have on the one hand

$$\alpha(S|A|T)+\alpha(S|A'|T)=\alpha(S)+\alpha(T)=\varphi(S)+\varphi(T),$$

since by the modularity $\dot{+}$ of α both sides are $=\alpha(S\cup T)+\alpha(S\cap T)$. On the other hand we have

for $A\in\mathfrak{C}(\varphi^{\bullet},\dot{+}):\varphi^{\bullet}(S|A|T)+\varphi^{\bullet}(S|A'|T)=\varphi^{\bullet}(S)+\varphi^{\bullet}(T)=\varphi(S)+\varphi(T)$,
for $A\in\mathfrak{C}(\varphi_{\bullet},\underset{\cdot}{+}):\varphi_{\bullet}(S|A|T)+\varphi_{\bullet}(S|A'|T)=\varphi_{\bullet}(S)+\varphi_{\bullet}(T)=\varphi(S)+\varphi(T)$.
The combination furnishes the assertions.

The next result says that in a certain sense regularity can be turned around at the Carathéodory class.

7.3. PROPOSITION. *Assume that $\phi:\mathfrak{P}(X)\to\overline{\mathbb{R}}$ is isotone and submodular $\dot{+}$. Let $\mathfrak{T}$ be a paving in X such that ϕ is outer regular $\mathfrak{T}$. If $A\in\mathfrak{C}(\phi,\dot{+})$ is such that there exists $T\in\mathfrak{T}$ with $T\subset A$ and $-\infty<\phi(T)\leqq\phi(A)<\infty$ then*

$$\phi(A)=\sup\{\phi(P):P\in\mathrm{O}(\mathfrak{T})\ \textit{with}\ P\subset A\}.$$

Proof. Fix $\varepsilon>0$. i) By assumption there exists $S\in\mathfrak{T}$ with $S\supset A$ such that $\phi(S)\leqq\phi(A)+\varepsilon$. Thus $\phi(S)\in\mathbb{R}$. Also fix $T\in\mathfrak{T}$ as described above. From $A\in\mathfrak{C}(\phi,\dot{+})$ we conclude that

$$\phi(T)\dot{+}\phi(S)=\phi(T|A|S)\dot{+}\phi(T|A'|S)=\phi(A)\dot{+}\phi(S|A|T).$$

Therefore $\phi(S|A|T)$ is finite and $\leqq\phi(T)+\varepsilon$. ii) By assumption there exists $H\in\mathfrak{T}$ with $H\supset S|A|T$ such that

$$\phi(H)\leqq\phi(S|A|T)+\varepsilon\leqq\phi(T)+2\varepsilon.$$

Thus $\phi(H)\in\mathbb{R}$. Note that

$$\begin{aligned}&H\supset S|A|T\supset T\cap A=T \text{ since } T\subset A,\\ &S\cap H'\subset S\cap(S'|A|T')=S\cap T'\cap A\subset A.\end{aligned}$$

iii) Now define $P:=S|H|T\in\mathrm{O}(\mathfrak{T})$. The last inclusions show that $T\subset P\subset A$. Furthermore

$$P\cap H=T\cap H=T \text{ and } P\cup H=(S\cap H')\cup H=S\cup H\supset S\supset A.$$

Since ϕ is submodular $\dot{+}$ this implies that

$$\phi(A)\dot{+}\phi(T)\leqq\phi(P\cup H)\dot{+}\phi(P\cap H)\leqq\phi(P)\dot{+}\phi(H),$$

and we know that all terms are finite. From this and from ii) it follows that $\phi(A)+\phi(T)\leqq\phi(P)+\phi(T)+2\varepsilon$ or $\phi(P)\geqq\phi(A)-2\varepsilon$. Thus we have the assertion.

We shall also need the upside-down counterpart.

7.4. Exercise. *Assume that* $\phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ *is isotone and supermodular* $\underset{\cdot}{+}$. *Let* $\mathfrak{T}$ *be a paving in* X *such that* ϕ *is inner regular* $\mathfrak{T}$. *If* $A \in \mathfrak{C}(\phi, \underset{\cdot}{+})$ *is such that there exists* $T \in \mathfrak{T}$ *with* $A \subset T$ *and* $-\infty < \phi(A) \leqq \phi(T) < \infty$ *then*

$$\phi(A) = \inf\{\phi(P) : P \in \mathrm{O}(\mathfrak{T}) \text{ with } P \supset A\}.$$

We can now obtain the desired comparison theorem.

7.5. Theorem. *Let* $\mathfrak{S}$ *be a lattice and* $\bullet = \star\sigma$. *Assume that the set function* $\varphi : \mathfrak{S} \to \mathbb{R}$ *is both an outer and an inner* $\bullet$ *premeasure. Then* $\mathfrak{C}(\varphi^\bullet, \dot{+}) = \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}) =: \mathfrak{C}$. *Furthermore* $\varphi^\bullet(A) = \varphi_\bullet(A)$ *for all* $A \in \mathfrak{C} \cap (\mathfrak{S}_\bullet \sqsubset \mathfrak{S}^\bullet)$, *except that in case* $\bullet = \sigma$ *one has also to admit that* $\varphi^\sigma(A) = \infty$ *and* $\varphi_\sigma(A) = -\infty$.

Note that in the conventional situation $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the latter exceptional case cannot occur.

Proof. 1) φ is isotone and modular and fulfils $\varphi = \varphi^\bullet|\mathfrak{S} = \varphi_\bullet|\mathfrak{S}$. Thus 7.1 and 7.2 can be applied to both $\alpha := \varphi^\bullet|\mathfrak{C}(\varphi^\bullet, \dot{+})$ and $\alpha := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$. Each time it follows that $\varphi_\bullet \leqq \varphi^\bullet$ on $\mathfrak{P}(X)$, and that

$$\varphi^\bullet(S|A|T) = \varphi_\bullet(S|A|T) \quad \text{for } S, T \in \mathfrak{S} \text{ and } A \in \mathfrak{C}(\varphi^\bullet, \dot{+}) \cap \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}).$$

In view of $S \cap T \subset S|A|T \subset S \cup T$ the common value is finite. 2) We claim that

$$\varphi^\bullet(S|A|T) = \varphi_\bullet(S|A|T) \quad \text{for } S, T \in \mathfrak{S} \text{ and } A \in \mathfrak{C}(\varphi^\bullet, \dot{+}).$$

To see this one applies 7.3 to $\phi := \varphi^\bullet$ and $\mathfrak{T} := \mathfrak{S}^\bullet$, and to the subset $S|A|T \in \mathfrak{C}(\varphi^\bullet, \dot{+})$; note that $S \cap T \in \mathfrak{S} \subset \mathfrak{S}^\bullet$ is as required in 7.3. It follows that

$$\begin{aligned}\varphi^\bullet(S|A|T) &= \sup\{\varphi^\bullet(P) : P \in \mathrm{O}(\mathfrak{S}^\bullet) \text{ with } P \subset S|A|T\} \\ &= \sup\{\varphi^\bullet(P) : P \in \mathrm{O}(\mathfrak{S}^\bullet) \text{ with } S \cap T \subset P \subset S|A|T\}.\end{aligned}$$

Now for the $P \in \mathrm{O}(\mathfrak{S}^\bullet)$ of the last kind $P = S \cap T|P|S \cup T$. Furthermore we have $\mathfrak{S}^\bullet \subset \mathfrak{C}(\varphi^\bullet, \dot{+}) \cap \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$ and hence $P \in \mathrm{O}(\mathfrak{S}^\bullet) \subset \mathfrak{C}(\varphi^\bullet, \dot{+}) \cap \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$. Thus 1) implies that

$$\varphi^\bullet(P) = \varphi^\bullet(S \cap T|P|S \cup T) = \varphi_\bullet(S \cap T|P|S \cup T) = \varphi_\bullet(P).$$

It follows that $\varphi^\bullet(S|A|T) \leqq \varphi_\bullet(S|A|T)$ and hence $= \varphi_\bullet(S|A|T)$. Note that the common value is finite as before. 2') Likewise we have

$$\varphi^\bullet(S|A|T) = \varphi_\bullet(S|A|T) \quad \text{for } S, T \in \mathfrak{S} \text{ and } A \in \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}).$$

The proof is as in 2), but with 7.4 instead of 7.3.

3) We next prove that $\mathfrak{C}(\varphi^\bullet, \dot{+}) \subset \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$. In fact, let $A \in \mathfrak{C}(\varphi^\bullet, \dot{+})$. For $S, T \in \mathfrak{S}$ we obtain from 2)

$$\begin{aligned}\varphi(S) + \varphi(T) = \varphi^\bullet(S) + \varphi^\bullet(T) &= \varphi^\bullet(S|A|T) + \varphi^\bullet(S|A'|T) \\ &= \varphi_\bullet(S|A|T) + \varphi_\bullet(S|A'|T).\end{aligned}$$

Thus 6.19 implies that $A \in \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+})$. 3') We obtain $\mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}) \subset \mathfrak{C}(\varphi^\bullet, \dot{+})$ as in 3), but based on 2') and 5.2 instead of 2) and 6.19. 4) So far we have proved that $\mathfrak{C}(\varphi^\bullet, \dot{+}) = \mathfrak{C}(\varphi_\bullet, \underset{\cdot}{+}) =: \mathfrak{C}$, and furthermore that

$$\varphi^\bullet(S|A|T) = \varphi_\bullet(S|A|T) \in \mathbb{R} \quad \text{for } S, T \in \mathfrak{S} \text{ and } A \in \mathfrak{C}.$$

5) We finish the case $\bullet = \star$. If $A \in \mathfrak{C} \cap (\mathfrak{S}_\star \sqsubset \mathfrak{S}^\star) = \mathfrak{C} \cap (\mathfrak{S} \sqsubset \mathfrak{S})$ then $S \subset A \subset T$ for some $S, T \in \mathfrak{S}$. It follows that $S|A|T = A$ and hence $\varphi^\star(A) = \varphi_\star(A)$.

6) We turn to the case $\bullet = \sigma$. Fix $A \in \mathfrak{C} \cap (\mathfrak{S}_\sigma \sqsubset \mathfrak{S}^\sigma)$. Thus $P \subset A \subset Q$ where $P_l \downarrow P$ and $Q_l \uparrow Q$ for some sequences $(P_l)_l$ and $(Q_l)_l$ in $\mathfrak{S}$. Furthermore fix $S \in \mathfrak{S}$. From 4) we obtain

$$\begin{aligned} P_l|A|S \downarrow P|A|S = S \cap A \quad &\text{and hence} \quad \varphi^\sigma(S \cap A) = \varphi_\sigma(S \cap A) < \infty, \\ S|A|Q_l \uparrow S|A|Q = S \cup A \quad &\text{and hence} \quad \varphi^\sigma(S \cup A) = \varphi_\sigma(S \cup A) > -\infty. \end{aligned}$$

Now 4.12.4) implies that

$$\begin{aligned} \varphi(S) + \varphi^\sigma(A) &= \varphi^\sigma(S \cup A) \dot{+} \varphi^\sigma(S \cap A), \\ \varphi(S) + \varphi_\sigma(A) &= \varphi_\sigma(S \cup A) \underset{\cdot}{+} \varphi_\sigma(S \cap A). \end{aligned}$$

Thus if $\varphi^\sigma(A) \neq \varphi_\sigma(A)$ then we must have $\varphi^\sigma(S \cup A) = \varphi_\sigma(S \cup A) = \infty$ and $\varphi^\sigma(S \cap A) = \varphi_\sigma(S \cap A) = -\infty$, and hence $\varphi^\sigma(A) = \infty$ and $\varphi_\sigma(A) = -\infty$. We shall see in exercise 7.7 below that this indeed can happen.

7.6. Example. For $\lambda : \mathfrak{K} = \operatorname{Comp}(\mathbb{R}^n) \to [0, \infty[$ the maximal inner σ extension $\lambda_\sigma|\mathfrak{C}(\lambda_\sigma)$, which is the common maximal inner $\bullet$ extension $\lambda_\bullet|\mathfrak{C}(\lambda_\bullet)$, coincides with $\Lambda := \lambda^\sigma|\mathfrak{C}(\lambda^\sigma)$. This fact has been announced in 6.35.

We conclude with the example announced in connection with 7.5.

7.7. Exercise. Construct an example which shows that in 7.5 it can happen that $\varphi^\sigma(A) = \infty$ and $\varphi_\sigma(A) = -\infty$. Hint: On $X = \mathbb{R}$ define the paving $\mathfrak{S}$ to consist of all $S \in \operatorname{Bor}(X)$ such that S is bounded above and S' is bounded below. Note that $\mathfrak{S}$ is an oval. We write $R := [0, \infty[$ and $L :=]-\infty, 0]$, and define $\varphi : \mathfrak{S} \to \mathbb{R}$ to be

$$\varphi(S) = \Lambda(S \cap R) - \Lambda(S' \cap L) \text{ for } S \in \mathfrak{S}.$$

Then show that $\varphi^\sigma(R) = \infty$ and $\varphi_\sigma(R) = -\infty$.

Lattices of Ringlike Types

We restrict ourselves to the conventional outer and inner situations. The most unfamiliar notion considered so far is that of tightness. Therefore it is desirable to have transparent assumptions which ensure the relevant tightness conditions. The simplest assumption of this type is that the initial domain be a ring. However, the previous theories allow to work with certain weaker assumptions which are much more realistic.

Let $\mathfrak{S}$ be a lattice in a nonvoid set X and $\bullet = \star\sigma\tau$. We define $\mathfrak{S}$ to be

upward $\bullet$ full iff $B \setminus A \in \mathfrak{S}^\bullet$ for each pair $A \subset B$ in $\mathfrak{S}$,
downward $\bullet$ full iff $B \setminus A \in \mathfrak{S}_\bullet$ for each pair $A \subset B$ in $\mathfrak{S}$.

Thus we have

$$\begin{aligned} \mathfrak{S} \text{ ring} \quad &\Leftrightarrow \quad \mathfrak{S} \text{ upward } \star \text{ full} \\ &\Rightarrow \quad \mathfrak{S} \text{ upward } \sigma \text{ full } \Rightarrow \mathfrak{S} \text{ upward } \tau \text{ full;} \\ \mathfrak{S} \text{ ring} \quad &\Leftrightarrow \quad \mathfrak{S} \text{ downward } \star \text{ full} \\ &\Rightarrow \quad \mathfrak{S} \text{ downward } \sigma \text{ full } \Rightarrow \mathfrak{S} \text{ downward } \tau \text{ full .} \end{aligned}$$

If $\mathfrak{S}$ is upward $\bullet$ full then $\varnothing \in \mathfrak{S}$, but trivial examples show that this need not be true if $\mathfrak{S}$ is downward $\bullet$ full.

7.8. EXAMPLES (for $\bullet = \sigma$). 1) Let X be a topological space. For the sublattices $\mathrm{CCl}(X)$ and $\mathrm{COp}(X)$ of $\mathrm{Baire}(X)$ defined in 1.6.4) we refer to 8.1 below. 2) Let X be a semimetrizable topological space. One verifies that $\mathrm{Cl}(X) \subset (\mathrm{Op}(X))_\sigma$ and hence that $\mathrm{Op}(X) \subset (\mathrm{Cl}(X))^\sigma$. It follows that $\mathrm{Cl}(X)$ is upward σ full and $\mathrm{Op}(X)$ is downward σ full. 3) Let X be a metrizable topological space. Then 2) implies that $\mathrm{Comp}(X)$ is upward σ full. This has been used for $X = \mathbb{R}^n$ in 5.14.1) and 6.35.

7.9. EXERCISE. Let X be a Hausdorff topological space. Prove that

$\mathrm{Cl}(X)$ is always upward τ full, and downward τ full iff X is discrete;
$\mathrm{Op}(X)$ is always downward τ full, and upward τ full iff X is discrete.

This makes clear that the case $\bullet = \tau$ is much less important than the cases $\bullet = \star\sigma$.

We come to the decisive point. We start with the upward fullness conditions.

7.10. PROPOSITION. *Assume that $\mathfrak{S}$ is upward $\bullet$ full. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and modular with $\varphi(\varnothing) = 0$. 1) φ is outer $\bullet$ tight. 2) If $\varphi < \infty$ and $\varphi = \varphi^\bullet|\mathfrak{S}$ then φ is inner $\star$ tight.*

Proof. Fix $A \subset B$ in $\mathfrak{S}$. By 1.4.1) there exists a paving $\mathfrak{M} \subset \mathfrak{S}$ of type $\bullet$ with $\mathfrak{M} \uparrow B \setminus A$. 1) To be shown is $\varphi(A) + \varphi^\bullet(B \setminus A) \leqq \varphi(B)$. We can assume that $\varphi(B) < \infty$. For $S \in \mathfrak{M}$ we have $\varphi(A) + \varphi(S) = \varphi(A \cup S) \leqq \varphi(B)$. It follows that

$$\varphi(A) + \varphi^\bullet(B \setminus A) \leqq \varphi(A) + \sup_{S \in \mathfrak{M}} \varphi(S) \leqq \varphi(B).$$

2) To be shown is $\varphi(B) \leqq \varphi(A) + \varphi_\star(B \setminus A)$. Now $\{A \cup S : S \in \mathfrak{M}\} \subset \mathfrak{S}$ is a paving of type $\bullet$ with $\uparrow B$, so that by assumption $\sup\limits_{S \in \mathfrak{M}} \varphi(A \cup S) = \varphi(B)$. From $\varphi(A \cup S) = \varphi(A) + \varphi(S)$ for $S \in \mathfrak{M}$ we obtain

$$\varphi(B) = \varphi(A) + \sup_{S \in \mathfrak{M}} \varphi(S) \leqq \varphi(A) + \varphi_\star(B \setminus A).$$

7.11. REMARK. We emphasize that

σ) if $\mathfrak{S}$ is upward σ full then φ need not be outer $\star$ tight;

τ) if $\mathfrak{S}$ is upward τ full then φ need not be outer σ tight.

Thus assertion 1) cannot be improved in this respect. For counterexamples we refer to 5.12: In a Hausdorff topological space X let $a \in X$ and $\psi := \delta_a|\mathrm{Cl}(X)$. σ) If X is metrizable then $\mathrm{Cl}(X)$ is upward σ full by 7.8.2). But if a is not an isolated point of X, that is if $\{a\} \notin \mathrm{Op}(X)$, then ψ is not outer $\star$ tight by 5.12.2). τ) $\mathrm{Cl}(X)$ is always upward τ full by 7.9. But if $\{a\} \notin (\mathrm{Op}(X))_\sigma$ then ψ is not outer σ tight by 5.12.2).

The main consequence which follows will be restricted to the case $\bullet = \sigma$. The case $\bullet = \star$ would be contained in the earlier 5.13 and 6.33, and the case $\bullet = \tau$ would be more involved and seems to be without substantial applications.

7.12. THEOREM. *Assume that $\mathfrak{S}$ is upward σ full. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and modular with $\varphi(\varnothing) = 0$ and $\varphi = \varphi^\sigma|\mathfrak{S}$.* 1) *$\varphi$ is an outer σ premeasure.* 2) *If $\varphi < \infty$ then φ is an inner σ premeasure.*

Let us add at once that for $\varphi < \infty$ it follows from 7.5 that $\mathfrak{C}(\varphi^\sigma) = \mathfrak{C}(\varphi_\sigma) =: \mathfrak{C}$ and $\varphi^\sigma(A) = \varphi_\sigma(A)$ for all $A \in \mathfrak{C} \cap (\sqsubset \mathfrak{S}^\sigma)$.

Proof. 1) is clear from 7.10.1) and the conventional outer main theorem 5.11. 2) By 7.10.2) and the conventional inner main theorem 6.31 we have to prove that $\varphi_\sigma(\varnothing) = 0$. By 1) φ is an outer σ premeasure. Now consider a countable paving $\mathfrak{M} \subset \mathfrak{S}$ with $\mathfrak{M} \downarrow \varnothing$. To be shown is $\inf\limits_{S \in \mathfrak{M}} \varphi(S) = 0$. We fix $E \in \mathfrak{M}$. Then likewise $\mathfrak{M}(E) := \{S \in \mathfrak{M} : S \subset E\} \subset \mathfrak{S}$ is a countable paving with $\downarrow \varnothing$. To be shown is of course $\inf\limits_{S \in \mathfrak{M}(E)} \varphi(S) = 0$. By assumption we have $\{E \setminus S : S \in \mathfrak{M}(E)\} \subset \mathfrak{S}^\sigma$, and this is a countable paving with $\uparrow E \in \mathfrak{S} \subset \mathfrak{S}^\sigma$. Thus we have

$$\sup_{S \in \mathfrak{M}(E)} \varphi^\sigma(E \setminus S) = \varphi^\sigma(E) = \varphi(E).$$

In view of $\varphi^\sigma(E \setminus S) = \varphi(E) - \varphi(S)$ for $S \in \mathfrak{M}(E)$ this is the assertion.

7.13. EXERCISE. The above theorem becomes false in both parts when instead of $\varphi = \varphi^\sigma|\mathfrak{S}$ one assumes that $\varphi_\sigma(\varnothing) = 0$. In fact, we shall construct set functions $\varphi : \mathfrak{S} \to [0, \infty[$ on upward σ full lattices which are isotone and modular with $\varphi(\varnothing) = \varphi_\sigma(\varnothing) = 0$ but do not fulfil $\varphi = \varphi^\sigma|\mathfrak{S}$.

Let X be a Hausdorff topological space and $a \in X$ such that $\{a\}$ is not open but $\in (\mathrm{Op}(X))_\sigma$. 1) Construct a set function $\varphi : \mathrm{Cl}(X) \to [0, \infty[$ which is isotone and modular with $\varphi(\varnothing) = 0$ but not upward σ continuous. Hint: Consider on the real vector space $\mathrm{B}(X, \mathbb{R})$ of the bounded functions $X \to \mathbb{R}$ the sublinear functional $\vartheta : \mathrm{B}(X, \mathbb{R}) \to \mathbb{R}$, defined to be

$$\vartheta(f) = \limsup_{x \to a} f(x) := \inf\{\sup(f|U \setminus \{a\}) : a \in U \text{ open } \subset X\}.$$

Let after Hahn-Banach $\phi : \mathrm{B}(X, \mathbb{R}) \to \mathbb{R}$ be a linear functional $\leqq \vartheta$. Note that $\phi \leqq \vartheta \leqq \sup$. Then define $\varphi : \mathrm{Cl}(X) \to [0, \infty[$ to be $\varphi(A) = \phi(\chi_A)$ for

$A \in \mathrm{Cl}(X)$. Consider for a sequence of open $U_l \downarrow \{a\}$ the sequence of the closed subsets $U_l' \cup \{a\}$. 2) If in particular X is compact then φ must be σ and even τ continuous at $\varnothing$.

Let us reformulate the last theorem in order that it looks like the classical Carathéodory extension theorem. The latter theorem is the upper closed path under the assumption that $\mathfrak{S}$ be a ring, and likewise the earlier 5.13 for $\bullet = \sigma$.

7.14. REFORMULATION. *Assume that $\mathfrak{S}$ is upward σ full. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and modular with $\varphi(\varnothing) = 0$. Then we have the implications as shown below* (the simple arrows are obvious implications).

φ *can be extended to cmeasure on σ algebra which is outer regular $\mathfrak{S}^\sigma$*

$\Uparrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

φ *is upward σ continuous* $\leftarrow$ φ *can be extended to cmeasure on σ algebra*

$\Downarrow$ $\varphi < \infty$ $\qquad\qquad\qquad\qquad$ $\uparrow$

φ *can be extended to cmeasure on σ algebra which is inner regular $\mathfrak{S}_\sigma$.*

We turn to the counterparts for the downward fullness conditions.

7.15. PROPOSITION. *Assume that $\mathfrak{S}$ is downward $\bullet$ full with $\varnothing \in \mathfrak{S}$. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and modular with $\varphi(\varnothing) = 0$. 1) If $\varphi < \infty$ then φ is inner $\bullet$ tight. 2) If φ is almost $\bullet$ continuous at $\varnothing$ then φ is outer $\star$ tight.*

Proof. Fix $A \subset B$ in $\mathfrak{S}$. By 1.4.1) there exists a paving $\mathfrak{M} \subset \mathfrak{S}$ of type $\bullet$ with $\mathfrak{M} \downarrow B \setminus A$. We can assume that $S \subset B \ \ \forall S \in \mathfrak{M}$. 1) To be shown is $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$. For $S \in \mathfrak{M}$ we have $A \cup S = B$ and hence $\varphi(B) \leqq \varphi(B) + \varphi(A \cap S) = \varphi(A \cup S) + \varphi(A \cap S) = \varphi(A) + \varphi(S)$. It follows that

$$\varphi(B) \leqq \varphi(A) + \inf_{S \in \mathfrak{M}} \varphi(S) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A).$$

2) To be shown is $\varphi(A) + \varphi^\star(B \setminus A) \leqq \varphi(B)$. We can assume that $\varphi(B) < \infty$. Then $\{A \cap S : S \in \mathfrak{M}\} \subset \mathfrak{S}$ is a paving of type $\bullet$ with $\downarrow \varnothing$, and all its members have $\varphi(\cdot) < \infty$. Hence by assumption $\inf\limits_{S \in \mathfrak{M}} \varphi(A \cap S) = 0$. For $S \in \mathfrak{M}$ now

$$\begin{aligned} \varphi(B) + \varphi(A \cap S) &= \varphi(A \cup S) + \varphi(A \cap S) \\ &= \varphi(A) + \varphi(S) \geqq \varphi(A) + \varphi^\star(B \setminus A). \end{aligned}$$

The assertion follows.

7.16. THEOREM. *Assume that $\mathfrak{S}$ is downward σ full with $\varnothing \in \mathfrak{S}$. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and modular with $\varphi(\varnothing) = 0$, and almost σ continuous at $\varnothing$. 1) If $\varphi < \infty$ then φ is an inner σ premeasure. 2) If φ is semifinite above then φ is an outer σ premeasure.*

Let us add as before that for $\varphi < \infty$ we obtain from 7.5 that $\mathfrak{C}(\varphi^\sigma) = \mathfrak{C}(\varphi_\sigma) =: \mathfrak{C}$ and $\varphi^\sigma(A) = \varphi_\sigma(A)$ for all $A \in \mathfrak{C} \cap (\sqsubset \mathfrak{S}^\sigma)$.

Proof. 1) In view of $\varphi < \infty$ we have $\varphi_\sigma(\varnothing) = 0$. Hence the assertion is clear from 7.15.1) and the conventional inner main theorem 6.31. 2) By

7.15.2) and the conventional outer main theorem 5.11 we have to prove that $\varphi = \varphi^\sigma|\mathfrak{S}$. Since $\varphi^\sigma|\mathfrak{S} \leqq \varphi$ by 4.1.1)2) we have to show that $\varphi \leqq \varphi^\sigma|\mathfrak{S}$; and since φ is assumed to be semifinite above it suffices to show that $\varphi(A) \leqq \varphi^\sigma(A)$ for all $A \in \mathfrak{S}$ with $\varphi(A) < \infty$. To achieve this we pass from $\mathfrak{S}$ to $\mathfrak{T} := [\varphi < \infty] \subset \mathfrak{S}$ which is a lattice and downward σ full with $\varnothing \in \mathfrak{T}$ as well. Also $\psi := \varphi|\mathfrak{T} < \infty$ is isotone and modular with $\psi(\varnothing) = 0$, and σ continuous at $\varnothing$. By 1) therefore ψ is an inner σ premeasure. Now fix $A \in \mathfrak{S}$ with $\varphi(A) < \infty$, that is $A \in \mathfrak{T}$. We have to show that $\sup\limits_{S \in \mathfrak{M}} \varphi(S) = \sup\limits_{S \in \mathfrak{M}} \psi(S)$ is $\geqq \varphi(A) = \psi(A)$ for each countable paving $\mathfrak{M} \subset \mathfrak{S}$ with $\mathfrak{M} \uparrow A$, which implies that $\mathfrak{M} \subset \mathfrak{T}$. By assumption we have $\{A \setminus S : S \in \mathfrak{M}\} \subset \mathfrak{T}_\sigma$, and this is a countable paving with $\downarrow \varnothing$. Hence $\inf\limits_{S \in \mathfrak{M}} \psi_\sigma(A \setminus S) = 0$. In view of $\psi_\sigma(A \setminus S) = \psi(A) - \psi(S)$ for $S \in \mathfrak{M}$ this means that $\sup\limits_{S \in \mathfrak{M}} \psi(S) = \psi(A)$. The proof is complete.

7.17. Exercise. Assertion 2) becomes false without the assumption that φ be semifinite above, even if $\mathfrak{S}$ is a ring. Hint for a counterexample: Let X be an infinite countable set, and let $\mathfrak{S}$ consist of its finite and cofinite subsets. Define $\varphi : \mathfrak{S} \to [0, \infty]$ to be $\varphi(A) = 0$ if A is finite and $\varphi(A) = \infty$ if A is cofinite.

As before we conclude with an obvious but useful reformulation.

7.18. Reformulation. *Assume that $\mathfrak{S}$ is downward σ full with $\varnothing \in \mathfrak{S}$. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and modular with $\varphi(\varnothing) = 0$. Then we have the implications as shown below* (the simple arrows are obvious implications).

φ can be extended to cmeasure on σ algebra which is outer regular $\mathfrak{S}^\sigma$

$\Uparrow$ *φ semifinite above* $\qquad\qquad\qquad \downarrow$

φ is almost σ cont at $\varnothing$ $\quad\leftarrow\quad$ *φ can be extended to cmeasure on σ algebra*

$\Downarrow$ $\varphi < \infty$ $\qquad\qquad\qquad \uparrow$

φ can be extended to cmeasure on σ algebra which is inner regular $\mathfrak{S}_\sigma$.

Bibliographical Annex

The present subsection attempts to describe the development of the extension theories for contents and measures on the basis of lattices and of outer and inner regularity. We shall restrict ourselves to the conventional outer and inner situations in the above sense, because we know of no prior work in the full situations of isotone set functions with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$. To be sure, there has been extensive work devoted to set functions with values in complete abelian Hausdorff topological groups, after the model of Sion [1969]. But in these papers the words isotone and regular do not occur, or at least attain different characters. Therefore we consider this work to be a domain on its own, and specialize its results to isotone set functions with values in $[0, \infty[\subset \mathbb{R}$. In compensation, the results will be considered to include regularity in the relevant sense whenever this can be read from the context.

Most of the papers to be discussed fall into the frame of the outer and inner $\bullet$ extensions for $\bullet = \star\sigma\tau$, as defined at the outset in sections 4 and 6 above. The exceptions are the paper of Pettis [1951] cited in the introduction, and the extension procedures which follow the traditional two-step model of topological measure theory, in short from compact subsets via open subsets to arbitrary subsets. These contributions culminate in the work of Sapounakis-Sion [1983][1987] which will be discussed hereafter.

At present we start to formulate a scheme in order to describe the results of the former papers. The scheme is shaped after the conventional outer and inner main theorems 5.11 and 6.31, except that their properties 5) and 5)5') will be dropped and incorporated into 4). Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$. Assume that

in the outer situation (=:out): $\varphi : \mathfrak{S} \to [0,\infty]$ is isotone and submodular with $\varphi(\varnothing) = 0$,
in the inner situation (=:inn): $\varphi : \mathfrak{S} \to [0,\infty[$ is isotone and supermodular with $\varphi(\varnothing) = 0$.

For fixed out/inn and $\bullet = \star\sigma\tau$ we consider the properties of φ which follow.

(1) *φ is an outer/inner $\bullet$ premeasure, that is φ has outer/inner $\bullet$ extensions.* It is equivalent to require that *φ has an outer/inner $\bullet$ extension which is*

$$\begin{aligned} &\textit{for } \bullet = \star &:\quad &\textit{a ccontent on an algebra,}\\ &\textit{for } \bullet = \sigma\tau &:\quad &\textit{a cmeasure on a } \sigma \textit{ algebra.}\end{aligned}$$

The other properties of φ are with respect to a further isotone set function $\phi : \mathfrak{P}(X) \to [0,\infty]$. The formation $\mathfrak{C}(\phi)$ is as defined above.

(2 for ϕ) *$\phi|\mathfrak{C}(\phi)$ is an outer/inner $\bullet$ extension of φ which is*

$$\begin{aligned} &\textit{for } \bullet = \star &:\quad &\textit{a ccontent on an algebra,}\\ &\textit{for } \bullet = \sigma\tau &:\quad &\textit{a cmeasure on a } \sigma \textit{ algebra.}\end{aligned}$$

(3 for ϕ) *$\phi|\mathfrak{C}(\phi)$ is an extension of φ in the crude sense.*

(4 for ϕ) *$\varphi(B) = \varphi(A) + \phi(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

We consider one more condition for φ with respect to ϕ.

(U for ϕ) *Each outer/inner $\bullet$ extension of φ is a restriction of $\phi|\mathfrak{C}(\phi)$.*

We note the obvious implications

$$\begin{array}{ccccc} (2 \text{ for } \phi) & \Longrightarrow & (1) & & \\ \| & & \Downarrow (\text{U for } \phi) & & \\ (2 \text{ for } \phi) & \Longrightarrow & (3 \text{ for } \phi) & \Longrightarrow & (4 \text{ for } \phi). \end{array}$$

The most important of the above properties for φ is of course (1). For a subordinate set function ϕ the most valuable properties are (2 for ϕ) and (U for ϕ), because their combination means that ϕ dominates the set function φ in the formation of extensions of the respective kind. On the other hand the most direct and simplest of the properties of φ relative to ϕ is of course (4 for ϕ). Therefore the most needed implications are (4 for ϕ) $\Rightarrow \cdots$, in order to obtain sufficient conditions for (1), and (1) $\Rightarrow \cdots$, in particular (U for ϕ), in order to have necessary conditions for (1).

Before we describe the historical development we recall that the present conventional outer main theorem 5.11 asserts that in the outer cases $\bullet = \star\sigma\tau$ the properties (1) and (2 for $\varphi^\bullet$), (3 for $\varphi^\bullet$), (4 for $\varphi^\bullet$) are equivalent, provided that in case $\bullet = \tau$ one adds to (4 for φ^τ) the requirement that φ be upward essential. Furthermore 5.1 says that (U for $\varphi^\bullet$) holds true. The present conventional inner main theorem 6.31 combined with 6.18 asserts the same in the inner cases $\bullet = \star\sigma\tau$ with respect to $\varphi_\bullet$, this time without addendum in case $\bullet = \tau$. A provisional announcement of these facts was in König [1992c]. We do not have to come back to the outer case $\bullet = \tau$, because it has not been treated before.

In the outer and inner cases $\bullet = \star$ the results have been in the literature for quite some time in more or less comprehensive versions. See for example Topsøe [1970b] theorem 4.1 and Adamski [1984b] section 2. But the author has not seen the complete formulations before König [1992b] theorem A13.

We turn to the outer and inner cases $\bullet = \sigma$. We have to restrict ourselves to the basic achievements of the individual papers, perhaps with small simplifications. As the earlierst paper we mention Choksi [1958], because it comprised several previous results. Its theorem 1 asserts that

inn: (4 for $\varphi_\star$) and $\mathfrak{S}$ σ compact $\Rightarrow$ (1).

The leap forward around 1970 started in Topsøe [1970a] theorem 1 and [1970b] section 2 (and notes to section 5) with the results

inn: (4 for $\varphi_\star$) and φ σ continuous at $\varnothing$ $\Rightarrow$ (2 for $\varphi_\star$) and (U for $\varphi_\star$), when $\mathfrak{S}$ fulfils $\cap\sigma$,

inn: (4 for $\varphi_\star$) and φ σ continuous at $\varnothing$ $\Rightarrow$ (2 for $\varphi_{(\sigma)}$),

with $\varphi_{(\sigma)}$ and its relatives as defined after 6.10 and 6.11. Kelley-Srinivasan [1971] proved in corollary 2 that

out: (4 for φ°) $\Rightarrow$ (2 for φ°) and hence $\Leftrightarrow$ (2 for φ°),

for the Carathéodory outer measure φ° as defined in the introduction. Thus of course (4 for φ°) $\Rightarrow$ (1). The authors claimed without proof that even (4 for φ°) $\Leftrightarrow$ (1), but the present author cannot see this. In propositions 8 and 9 they proved via φ° that

out: (4 for $\varphi^\star$) and φ upward σ continuous $\Leftrightarrow$ (1), when $\mathfrak{S}$ fulfils $\cup\sigma$,
inn: (4 for $\varphi_\star$) and φ σ continuous at $\varnothing$ $\Leftrightarrow$ (1), when $\mathfrak{S}$ fulfils $\cap\sigma$.

Ridder [1971][1973] proved the last implications $\Rightarrow$ under the assumption that $\mathfrak{S}$ fulfils both $\cup\sigma$ and $\cap\sigma$. Then Kelley-Nayak-Srinivasan [1973] obtained an independent proof of the result of Topsøe [1970b] that

inn: (4 for $\varphi_\star$) and φ σ continuous at $\varnothing$ $\Rightarrow$ (2 for $\varphi_{(\sigma)}$).

The conditions $\cup\sigma$ and $\cap\sigma$ for $\mathfrak{S}$ are of course severe restrictions which often are not fulfilled.

From the present text we know that beyond these restrictions the converses $\cdots \Rightarrow$ (4 for $\varphi^\star$) and $\cdots \Rightarrow$ (4 for $\varphi_\star$) of the above assertions are all false. From 5.14 we see that in the outer situation even $\lambda : \mathfrak{K} = \text{Comp}(\mathbb{R}^n) \to [0, \infty[$ is a counterexample. This expresses the basic inadequacy of the formations $\varphi^\star$ and $\varphi_\star$ for the treatment of $\bullet = \sigma\tau$.

To this line of papers we add the work of Lipecki [1974], who in the frame of abstract-valued set functions as described above proved an extended version of the last-mentioned result.

At last we quote from Adamski [1982] the two results

out: (4 for $\varphi^\star$) and φ upward σ continuous $\Rightarrow$ (2 for φ°),
inn: (4 for $\varphi_\star$) and φ σ continuous at $\varnothing \Rightarrow$ (2 for $\varphi_{(\sigma)}$),

declared as direct counterparts. These results are contained in the former ones, the first one since its hypothesis implies at once (4 for φ°). We quote the results in their combination as an example for the odd kind of monopoly which the Carathéodory outer measure held in the outer situation, in spite of what we have said in the introduction. Another example is a note in the recent book of Kelley-Srinivasan [1988] page 20 which says that, in a certain sense, the properties (4 for φ°) and (4 for $\varphi_\star$) are dual to each other.

We remain in the outer and inner cases $\bullet = \sigma$. The next papers were essential improvements, because of results in which $\varphi^\star$ and $\varphi_\star$ as well as φ° did no more occur. The main results in Fox-Morales [1983] theorems 3.16 and 3.10 were

out: (4 for $\varphi^{(\sigma)}$) and φ upward σ continuous $\Rightarrow$ (1),
inn: (4 for $\varphi_{(\sigma)}$) and φ downward σ continuous $\Rightarrow$ (1).

Then Găină [1986] proved

out: (4 for $\varphi^{(\sigma)}$) and φ upward σ continuous $\Leftrightarrow$ (2 for $\varphi^{(\sigma)}$),
inn: (4 for $\varphi_{(\sigma)}$) and φ downward σ continuous $\Leftrightarrow$ (2 for $\varphi_{(\sigma)}$).

Both papers were in the frame of abstract-valued set functions, the first one still based on Sion [1969]. The independent work of König [1985] theorems 3.3 with 3.1 and 3.4 with 3.2 obtained

out: (4 for φ^σ) $\Leftrightarrow$ (2 for φ^σ), and furthermore (U for φ^σ),
inn: (4 for φ_σ) $\Leftrightarrow$ (2 for φ_σ), and furthermore (U for φ_σ),

and hence the full results as in the present text. We emphasize that the important fortified counterparts 7.14 and 7.18 of the classical extension theorem can be deduced from the last three papers, but not from the earlier ones. Their essence is in König [1985] theorems 3.8 and 3.9.

In the outer and inner cases $\bullet = \sigma$ it remains to review the work of Glazkov [1988] which stands somewhat apart. It assumed an arbitrary paving $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and an arbitrary set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$, and defined besides φ° the somewhat brutal inner counterpart $\varphi_\circ : \mathfrak{P}(X) \to [0, \infty]$ to be

$$\varphi_\circ(A) = \sup\{\sum_{l=1}^{r} \varphi(S_l) : S_1, \cdots, S_r \in \mathfrak{S} \text{ pairwise disjoint } \subset A\}.$$

As far as the present author knows, this formation had been considered in earlier decades, but was later abandoned because of severe unsymmetries with φ°. Nevertheless the paper obtained some notable results, based on appropriate definitions of outer and inner tightness. The outer result says that $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$, which is known to be a cmeasure on a σ algebra, is an extension of φ iff φ is outer tight. However, the inner counterpart on $\varphi_\circ|\mathfrak{C}(\varphi_\circ)$ is not an equivalence assertion but restricted to certain sufficient conditions which, except the requirement that φ be inner tight, do not look adequate for an equivalence assertion. Thus there is not much hope for symmetry based on the formations φ° and $\varphi_\circ$.

The review of the inner case $\bullet = \tau$ is short. Prior to the present text we quote the work of Topsøe [1970ab], also reproduced in Pollard-Topsøe [1975]. It was our model in that it aimed at a uniform treatment of the three cases $\bullet = \star\sigma\tau$. Thus Pollard-Topsøe [1975] theorem B asserts that

inn: (4 for $\varphi_\star$) and φ $\bullet$ continuous at $\varnothing \Rightarrow$ (2 for $\varphi_{(\bullet)}$).

The converse $\Leftarrow$ is false for $\bullet = \tau$ as it has been for $\bullet = \sigma$. There are also parts of the present comparison theorems 6.24 and 6.25 in Topsøe [1970b] theorem 5.1, and of the present τ continuity theorem 6.27 inTopsøe [1970b] lemma 2.3. At last Topsøe [1970a] lemma 1 seems to be the ancestor of the results like the present lemma 6.30.

At the end of the subsection we want to discuss the work of Sapounakis-Sion [1983][1987] as announced above. The concern here is Sapounakis-Sion [1987] part I with the fundamental theorem 1.1 and its corollaries. We shall later comment on certain applications. The reproduction will be a free one in certain minor points.

The situation is that of a two-step extension procedure. Assume that $\mathfrak{S}$ and $\mathfrak{T}$ are lattices in X which contain $\varnothing$ and fulfil $\mathfrak{S} \subset \mathfrak{T}\top\perp$, and let $\varphi : \mathfrak{S} \to [0,\infty[$ be an isotone set function with $\varphi(\varnothing) = 0$. We form $\psi := \varphi_\star|\mathfrak{T}$, so that $\psi : \mathfrak{T} \to [0,\infty]$ is an isotone set function with $\psi(\varnothing) = 0$. The aim is to obtain a cmeasure $\alpha : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A}$ with the properties

I) $\mathfrak{A} \supset \mathfrak{S}$, and $\mathfrak{A} \supset \mathfrak{T}$ and hence $\mathfrak{A} \supset \mathfrak{T}^\sigma$;

II) $\alpha|\mathfrak{S} = \varphi$, and α is inner regular $\mathfrak{S}$ at $\mathfrak{T}$ and outer regular $\mathfrak{T}^\sigma$.

Although this task seems to be quite different from those in the present text, we shall see that it can be incorporated into our extension theories. We do this with the next theorem which is based on the main results of the present chapter.

7.19. THEOREM. *Assume that $\mathfrak{S}$ is upward enclosable $[\psi < \infty] = [\varphi_\star|\mathfrak{T} < \infty]$. Then there exists a cmeasure $\alpha : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A}$ with the above properties* I)II) *iff*

i) *φ is supermodular and inner $\star$ tight, and*

ii) *$\psi = \varphi_\star|\mathfrak{T}$ is submodular and upward σ continuous.*

In this case ψ is an outer σ and $\star$ premeasure, and $\psi^\sigma|\mathfrak{C}(\psi^\sigma)$ is as required. Furthermore each cmeasure α which is as required is a restriction of $\psi^\sigma|\mathfrak{C}(\psi^\sigma)$.

This theorem can serve as a substitute for Sapounakis-Sion [1987] theorem 1.1 and some of the subsequent results. The main differences are that these authors on the one hand postulate $\alpha := \psi^\circ|\mathfrak{C}(\psi^\circ)$ from the start, and on the other hand do not present equivalence theorems in concrete terms like the above one, but are content with sufficient conditions.

Proof of the theorem. We first assume that $\alpha : \mathfrak{A} \to [0,\infty]$ is a cmeasure on a σ algebra $\mathfrak{A}$ with the properties I)II). Then $\alpha|\mathfrak{T}$ is an outer σ premeasure with $\alpha|\mathfrak{T} = (\alpha|\mathfrak{S})_\star|\mathfrak{T} = \varphi_\star|\mathfrak{T} = \psi$. Thus ψ is an outer σ premeasure, and hence in particular fulfils ii). We see from 5.1 that α is a restriction of $\psi^\sigma|\mathfrak{C}(\psi^\sigma)$. Thus $\psi^\sigma|\mathfrak{C}(\psi^\sigma)$ fulfils I)II) as well. Now we have to prove i). Since $\varphi = \alpha|\mathfrak{S}$ is modular it remains to show that it is inner $\star$ tight. To see this fix $A \subset B$ in $\mathfrak{S}$, and then $T \in \mathfrak{T}$ with $\alpha(T) = \psi(T) < \infty$ such that $B \subset T$. In view of $\mathfrak{S} \subset \mathfrak{T}\top\perp$ we have $T \setminus A \in \mathfrak{T}$, of course with $\alpha(T\setminus A) < \infty$. We fix $\varepsilon > 0$ and then $K \in \mathfrak{S}$ with $K \subset T \setminus A$ and $\alpha(T \setminus A) \leqq \alpha(K) + \varepsilon$. Now

$$\alpha(B \setminus A) + \alpha(T \setminus B) = \alpha(T \setminus A) \leqq \alpha(K) + \varepsilon = \alpha(K \cap B) + \alpha(K \cap B') + \varepsilon,$$

with all terms finite. On the other hand

$$K \cap B \in \mathfrak{S} \text{ with } K \cap B \subset A' \cap B = B \setminus A,$$
$$K \cap B' \subset T \cap B' = T \setminus B.$$

It follows that $\alpha(B \setminus A) \leqq \alpha(K \cap B) + \varepsilon$. Therefore

$$\begin{aligned} \varphi(B) - \varphi(A) &= \alpha(B) - \alpha(A) = \alpha(B \setminus A) \leqq \alpha(K \cap B) + \varepsilon \\ &= \varphi(K \cap B) + \varepsilon \leqq \varphi_\star(B \setminus A) + \varepsilon, \end{aligned}$$

and hence the assertion.

We next assume that φ and ψ fulfil i)ii). We first prove

(0) $\quad \varphi_\star(B \cap A) + \psi^\star(B \cap A') \leqq \psi(B) = \varphi_\star(B) \quad$ for $A \subset X$ and $B \in \mathfrak{T}$.

We can assume that $\psi(B) = \varphi_\star(B) < \infty$ and hence $\varphi_\star(B \cap A) < \infty$. We fix $\varepsilon > 0$ and then $S \in \mathfrak{S}$ with $S \subset B \cap A$ such that $\varphi_\star(B \cap A) \leqq \varphi(S) + \varepsilon$. Now

$$\varphi_\star(B) = \varphi_\star(B) + \varphi_\star(\varnothing) \geqq \varphi_\star(B \cap S) + \varphi_\star(B \cap S'),$$

since $\varphi_\star$ is supermodular by 6.3.5). Here we have on the one hand

$$B \cap S = S \text{ and hence } \varphi_\star(B \cap S) = \varphi(S) \geqq \varphi_\star(B \cap A) - \varepsilon.$$

On the other hand $B \cap S' \in \mathfrak{T}$ in view of $\mathfrak{S} \subset \mathfrak{T}\top\perp$; furthermore

$$B \cap S' \supset B \cap A' \text{ and hence } \varphi_\star(B \cap S') = \psi(B \cap S') \geqq \psi^\star(B \cap A').$$

It follows that $\varphi_\star(B) \geqq \varphi_\star(B \cap A) - \varepsilon + \psi^\star(B \cap A')$. Thus (0) is proved.

Now (0) will be applied three times. 1) From (0) for $A \subset B$ in $\mathfrak{T}$ we see that ψ is inner $\star$ tight and hence inner σ tight. Therefore ψ is an inner $\star$ and σ premeasure. From 5.8.σ) we know that $\mathfrak{C}(\psi^\star) \subset \mathfrak{C}(\psi^\sigma)$, and from 5.9.$\sigma$) that $\psi^\star(M) = \psi^\sigma(M)$ for all $M \in \mathfrak{C}(\psi^\star)$ with $\psi^\star(M) < \infty$. Thus $\alpha := \psi^\sigma|\mathfrak{C}(\psi^\sigma)$ is a cmeasure on a σ algebra $\supset \mathfrak{T}$ which is outer regular $\mathfrak{T}^\sigma$.

Furthermore we know from 5.11 that $\mathfrak{S}\perp \subset \mathfrak{T}\top = \mathfrak{T}\top\mathfrak{T} \subset \mathfrak{C}(\psi^\star) \subset \mathfrak{C}(\psi^\sigma)$ and hence $\mathfrak{S} \subset \mathfrak{C}(\psi^\star) \subset \mathfrak{C}(\psi^\sigma)$.

It remains to prove $\alpha|\mathfrak{S} = \varphi$. In fact, then we have also $(\alpha|\mathfrak{S})_\star|\mathfrak{T} = \varphi_\star|\mathfrak{T} = \psi = \alpha|\mathfrak{T}$, that is α is inner regular $\mathfrak{S}$ at $\mathfrak{T}$. To see the assertion fix $A \in \mathfrak{S}$. By assumption there exists $B \in [\psi < \infty] \subset \mathfrak{T}$ with $A \subset B$. 2) From (0) and ii) with 4.1.5) we have

$$\begin{aligned}\varphi_\star(B \cap A) + \psi^\star(B \cap A') &\leqq \psi(B) = \psi^\star(B) + \psi^\star(\varnothing)\\ &\leqq \psi^\star(B \cap A) + \psi^\star(B \cap A'),\end{aligned}$$

so that $B \cap A = A$ furnishes $\varphi(A) \leqq \psi^\star(A)$. 3) From (0) applied to A' and B and from $A \in \mathfrak{S} \subset \mathfrak{C}(\varphi_\star)$ in view of i) we obtain

$$\varphi_\star(B \cap A') + \psi^\star(B \cap A) \leqq \varphi_\star(B) = \varphi_\star(B \cap A') + \varphi_\star(B \cap A),$$

with all terms finite. Thus $B \cap A = A$ furnishes $\psi^\star(A) \leqq \varphi(A)$. Therefore we have $\varphi(A) = \psi^\star(A) < \infty$ and hence $\varphi(A) = \psi^\sigma(A) = \alpha(A)$. The proof is complete.

We mention at last that in Sapounakis-Sion [1983][1987] one requires but $\cup$ for $\mathfrak{S}$ and $\cap$ for $\mathfrak{T}$, instead of $\mathfrak{S}$ and $\mathfrak{T}$ to be lattices as above. Maurice Sion has pointed out to the author that it is a benefit of the Carathéodory outer measure that it permits to start with pavings which fulfil $\cap$ but need not be lattices. This aspect can of course become relevant, but it is expected that the last subsection of the present section 3 will be able to take care of it as well.