

# Contents

Chapter I. Set Systems and Set Functions .....	1
1. Set Systems.....	1
Basic Notions and Notations.....	1
Inverse Images of Pavings.....	4
The Transporter .....	6
Complements for Ovals and $\sigma$ Ovals .....	9
2. Set Functions .....	10
Basic Properties of Set Functions .....	10
Contents and Measures .....	13
New versus Conventional Contents and Measures .....	15
The Main Example: The Volume in $\mathbb{R}^n$ .....	19
3. Some Classical Extension Theorems for Set Functions .....	22
The Classical Uniqueness Theorem .....	22
The Smiley-Horn-Tarski Theorem .....	23
Extensions of Set Functions to Lattices .....	27
Chapter II. The Extension Theories Based on Regularity .....	33
4. The Outer Extension Theory: Concepts and Instruments .....	33
The Basic Definition .....	33
The Outer Envelopes .....	34
Complements for the Nonsequential Situation .....	38
The Extended Carathéodory Construction .....	40
The Carathéodory Class in the Spirit of the Outer Theory ....	42
5. The Outer Extension Theory: The Main Theorem .....	45
The Outer Main Theorem .....	45
Comparison of the three Outer Theories .....	49
The Conventional Outer Situation .....	50
6. The Inner Extension Theory .....	53
The Basic Definition .....	54
The Inner Envelopes .....	54
The Carathéodory Class in the Spirit of the Inner Theory ....	56
The Inner Main Theorem .....	57
Comparison of the three Inner Theories .....	58
Further Results on Nonsequential Continuity .....	59
The Conventional Inner Situation .....	60

7. Complements to the Extension Theories .....	64
Comparison of the Outer and Inner Extension Theories .....	65
Lattices of Ringlike Types .....	68
Bibliographical Annex .....	72
Chapter III. Applications of the Extension Theories .....	79
8. Baire Measures .....	79
Basic Properties of Baire Measures .....	79
Inner Regularity in Separable Metric Spaces .....	83
Extension of Baire Measures to Borel Measures .....	84
The Hewitt-Yosida Theorem .....	85
9. Radon Measures .....	87
Radon Contents and Radon Measures .....	87
The Classical Example of a Non-Radon Borel Measure .....	91
The Notion of Support and the Decomposition Theorem .....	94
10. The Choquet Capacitability Theorem .....	98
Suslin and Co-Suslin Sets .....	98
The Extended Choquet Theorem .....	101
Combination with Basic Properties of the $\sigma$ Envelopes .....	104
The Measurability of Suslin and Co-Suslin Sets .....	105
Chapter IV. The Integral .....	109
11. The Horizontal Integral .....	109
Upper and Lower Measurable Functions .....	109
The Horizontal Integral .....	112
Regularity and Continuity of the Horizontal Integral .....	117
12. The Vertical Integral .....	121
Definition and Main Properties .....	121
Regularity and Continuity of the Vertical Integral .....	125
Comparison of the two Integrals .....	126
13. The Conventional Integral .....	128
Measurable Functions .....	128
Integrable Functions and the Integral .....	133
Integration over Subsets .....	137
Comparison with the Riemann Integral .....	139
Chapter V. The Daniell-Stone and Riesz Representation Theorems ...	143
14. Elementary Integrals on Lattice Cones .....	143
Introduction .....	143
Lattice Cones .....	146
Elementary Integrals .....	148
Representations of Elementary Integrals .....	151
15. The Continuous Daniell-Stone Theorem .....	154
Preparations on Lattice Cones .....	154
Preparations on Elementary Integrals .....	156
The New Envelopes for Elementary Integrals .....	156

	The Main Theorem .....	159
	An Extended Situation .....	163
16.	The Riesz Theorem .....	165
	Preliminaries .....	165
	The Main Theorem .....	167
	An Extended Situation .....	169
17.	The Non-Continuous Daniell-Stone Theorem .....	171
	Introduction .....	171
	The Maximality Lemma .....	172
	Subtight Sources .....	173
	The Main Theorem .....	176
Chapter VI. Transplantation of Contents and Measures .....		179
18.	Transplantation of Contents .....	179
	Introduction and Preparations .....	179
	The Existence Theorem .....	181
	Specializations of the Existence Theorem .....	183
	The Theorem of Łoś-Marczewski .....	186
	The Uniqueness Theorem .....	189
19.	Transplantation of Measures .....	190
	Preparations .....	190
	The Existence Theorem .....	191
	Specializations of the Existence Theorem .....	192
	The Uniqueness Theorem .....	194
	Extension of Baire Measures to Borel Measures .....	195
Chapter VII. Products of Contents and Measures .....		201
20.	The Traditional Product Formations .....	201
	The Basic Product Formation .....	201
	The Traditional Product Situation .....	205
	Product Measures .....	206
21.	The Product Formations Based on Inner Regularity .....	210
	Further Properties of the Basic Product Formation .....	210
	The Main Theorem .....	213
	The Sectional Representation .....	215
22.	The Fubini-Tonelli and Fubini Theorems .....	222
	Monotone Approximation of Functions .....	223
	The Fubini-Tonelli Theorems .....	225
	The Fubini Theorems .....	228
Chapter VIII. Applications of the New Contents and Measures .....		231
23.	The Jordan and Hahn Decomposition Theorems .....	231
	Introduction .....	231
	The Infimum Formation .....	234
	The Jordan Decomposition Theorem .....	236

The Existence of Minimal Sets .....	239
The Hahn Decomposition Theorem .....	241
24. The Lebesgue Decomposition and Radon-Nikodým Theorems ..	242
The Lebesgue Decomposition Theorem .....	243
The Radon-Nikodým Theorem .....	244
Bibliography .....	249
List of Symbols .....	255
Index .....	257
Subsequent Articles of the Author .....	261