Contents

Preface v

1 Matrices 1
1.1 Basic Terminology 1
1.2 Basic Operations 2
1.3 Some Basic Types of Matrices 5
Exercises 10

2 Submatrices and Partitioned Matrices 13
2.1 Some Terminology and Basic Results 13
2.2 Scalar Multiples, Transposes, Sums, and Products of Partitioned Matrices 17
2.3 Some Results on the Product of a Matrix and a Column Vector . . 19
2.4 Expansion of a Matrix in Terms of Its Rows, Columns, or Elements 20
Exercises 21

3 Linear Dependence and Independence 23
3.1 Definitions 23
3.2 Some Basic Results 23
Exercises 25

4 Linear Spaces: Row and Column Spaces 27
4.1 Some Definitions, Notation, and Basic Relationships and Properties 27
4.2 Subspaces 29
4.3 Bases 31
4.4 Rank of a Matrix 36
4.5 Some Basic Results on Partitioned Matrices and on Sums of Matrices 41
Exercises 46

5 Trace of a (Square) Matrix 49
5.1 Definition and Basic Properties 49
5.2 Trace of a Product 50
5.3 Some Equivalent Conditions 52
Exercises 52

6 Geometrical Considerations 55
6.1 Definitions: Norm, Distance, Angle, Inner Product, and Orthogonality 55
6.2 Orthogonal and Orthonormal Sets 61
6.3 Schwarz Inequality 62
6.4 Orthonormal Bases 63
Exercises 68

7 Linear Systems: Consistency and Compatibility 71
7.1 Some Basic Terminology 71
7.2 Consistency 72
7.3 Compatibility 73
7.4 Linear Systems of the Form $\mathbf{A}'\mathbf{A}\mathbf{X} = \mathbf{A}'\mathbf{B}$ 74
Exercise 77

8 Inverse Matrices 79
8.1 Some Definitions and Basic Results 79
8.2 Properties of Inverse Matrices 81
8.3 Premultiplication or Postmultiplication by a Matrix of Full Column or Row Rank 82
8.4 Orthogonal Matrices 84
8.5 Some Basic Results on the Ranks and Inverses of Partitioned Matrices 88
Exercises 103

9 Generalized Inverses 107
9.1 Definition, Existence, and a Connection to the Solution of Linear Systems 107
9.2 Some Alternative Characterizations 109
9.3 Some Elementary Properties 117
9.4 Invariance to the Choice of a Generalized Inverse 119
9.5 A Necessary and Sufficient Condition for the Consistency of a Linear System 120
9.6 Some Results on the Ranks and Generalized Inverses of Partitioned Matrices 121
9.7 Extension of Some Results on Systems of the Form $\mathbf{A}\mathbf{X} = \mathbf{B}$ to Systems of the Form $\mathbf{A}\mathbf{X}\mathbf{C} = \mathbf{B}$ 126
Exercises 126

10 Idempotent Matrices 133
10.1 Definition and Some Basic Properties 133
10.2 Some Basic Results 134
Exercises 136

11 Linear Systems: Solutions 139
11.1 Some Terminology, Notation, and Basic Results 139
11.2 General Form of a Solution 140
11.3 Number of Solutions 142
11.4 A Basic Result on Null Spaces 144
11.5 An Alternative Expression for the General Form of a Solution . . 144
11.6 Equivalent Linear Systems 145
11.7 Null and Column Spaces of Idempotent Matrices 146
11.8 Linear Systems With Nonsingular Triangular or Block-Triangular Coefficient Matrices 146
11.9 A Computational Approach 149
11.10 Linear Combinations of the Unknowns 150
11.11 Absorption 152
11.12 Extensions to Systems of the Form $\mathbf{AXC} = \mathbf{B}$ 157
Exercises 158

12 Projections and Projection Matrices 161
12.1 Some General Results, Terminology, and Notation 161
12.2 Projection of a Column Vector 163
12.3 Projection Matrices 166
12.4 Least Squares Problem 170
12.5 Orthogonal Complements 172
Exercises 177

13 Determinants 179
13.1 Some Definitions, Notation, and Special Cases 179
13.2 Some Basic Properties of Determinants 183
13.3 Partitioned Matrices, Products of Matrices, and Inverse Matrices 187
13.4 A Computational Approach 191
13.5 Cofactors 191
13.6 Vandermonde Matrices 195
13.7 Some Results on the Determinant of the Sum of Two Matrices . . 197
13.8 Laplace's Theorem and the Binet-Cauchy Formula 200
Exercises 205

14 Linear, Bilinear, and Quadratic Forms 209
14.1 Some Terminology and Basic Results 209
14.2 Nonnegative Definite Quadratic Forms and Matrices 212
14.3 Decomposition of Symmetric and Symmetric Nonnegative Definite Matrices 218

14.4 Generalized Inverses of Symmetric Nonnegative Definite Matrices . . . 222
14.5 LDU, U′DU, and Cholesky Decompositions . . . 223
14.6 Skew-Symmetric Matrices . . . 239
14.7 Trace of a Nonnegative Definite Matrix . . . 240
14.8 Partitioned Nonnegative Definite Matrices . . . 243
14.9 Some Results on Determinants . . . 247
14.10 Geometrical Considerations . . . 255
14.11 Some Results on Ranks and Row and Column Spaces and on Linear Systems . . . 259
14.12 Projections, Projection Matrices, and Orthogonal Complements . . . 260
Exercises . . . 277

15 Matrix Differentiation . . . 289
15.1 Definitions, Notation, and Other Preliminaries . . . 290
15.2 Differentiation of (Scalar-Valued) Functions: Some Elementary Results . . . 296
15.3 Differentiation of Linear and Quadratic Forms . . . 298
15.4 Differentiation of Matrix Sums, Products, and Transposes (and of Matrices of Constants) . . . 300
15.5 Differentiation of a Vector or (Unrestricted or Symmetric) Matrix With Respect to Its Elements . . . 303
15.6 Differentiation of a Trace of a Matrix . . . 304
15.7 The Chain Rule . . . 306
15.8 First-Order Partial Derivatives of Determinants and Inverse and Adjoint Matrices . . . 308
15.9 Second-Order Partial Derivatives of Determinants and Inverse Matrices . . . 312
15.10 Differentiation of Generalized Inverses . . . 314
15.11 Differentiation of Projection Matrices . . . 319
15.12 Evaluation of Some Multiple Integrals . . . 324
Exercises . . . 327
Bibliographic and Supplementary Notes . . . 335

16 Kronecker Products and the Vec and Vech Operators . . . 337
16.1 The Kronecker Product of Two or More Matrices: Definition and Some Basic Properties . . . 337
16.2 The Vec Operator: Definition and Some Basic Properties . . . 343
16.3 Vec-Permutation Matrix . . . 347
16.4 The Vech Operator . . . 354
16.5 Reformulation of a Linear System . . . 367
16.6 Some Results on Jacobian Matrices . . . 368
Exercises . . . 371
Bibliographic and Supplementary Notes . . . 377

17 Intersections and Sums of Subspaces 379
17.1 Definitions and Some Basic Properties 379
17.2 Some Results on Row and Column Spaces and on the Ranks of Partitioned Matrices 385
17.3 Some Results on Linear Systems and on Generalized Inverses of Partitioned Matrices 392
17.4 Subspaces: Sum of Their Dimensions Versus Dimension of Their Sum 396
17.5 Some Results on the Rank of a Product of Matrices 398
17.6 Projections Along a Subspace 402
17.7 Some Further Results on the Essential Disjointness and Orthogonality of Subspaces and on Projections and Projection Matrices 409
Exercises 411
Bibliographic and Supplementary Notes 417

18 Sums (and Differences) of Matrices 419
18.1 Some Results on Determinants 419
18.2 Some Results on Inverses and Generalized Inverses and on Linear Systems 423
18.3 Some Results on Positive (and Nonnegative) Definiteness 437
18.4 Some Results on Idempotency 439
18.5 Some Results on Ranks 444
Exercises 450
Bibliographic and Supplementary Notes 458

19 Minimization of a Second-Degree Polynomial (in *n* Variables) Subject to Linear Constraints 459
19.1 Unconstrained Minimization 460
19.2 Constrained Minimization 463
19.3 Explicit Expressions for Solutions to the Constrained Minimization Problem 468
19.4 Some Results on Generalized Inverses of Partitioned Matrices . . 476
19.5 Some Additional Results on the Form of Solutions to the Constrained Minimization Problem 483
19.6 Transformation of the Constrained Minimization Problem to an Unconstrained Minimization Problem 489
19.7 The Effect of Constraints on the Generalized Least Squares Problem 491
Exercises 492
Bibliographic and Supplementary Notes 495

20 The Moore-Penrose Inverse 497
20.1 Definition, Existence, and Uniqueness (of the Moore-Penrose Inverse) 497
20.2 Some Special Cases 499

20.3 Special Types of Generalized Inverses 500
20.4 Some Alternative Representations and Characterizations 507
20.5 Some Basic Properties and Relationships 508
20.6 Minimum Norm Solution to the Least Squares Problem 512
20.7 Expression of the Moore-Penrose Inverse as a Limit 512
20.8 Differentiation of the Moore-Penrose Inverse 514
Exercises 517
Bibliographic and Supplementary Notes 519

21 Eigenvalues and Eigenvectors 521
21.1 Definitions, Terminology, and Some Basic Results 522
21.2 Eigenvalues of Triangular or Block-Triangular Matrices and of Diagonal or Block-Diagonal Matrices 528
21.3 Similar Matrices 530
21.4 Linear Independence of Eigenvectors 534
21.5 Diagonalization 537
21.6 Expressions for the Trace and Determinant of a Matrix 545
21.7 Some Results on the Moore-Penrose Inverse of a Symmetric Matrix 546
21.8 Eigenvalues of Orthogonal, Idempotent, and Nonnegative Definite Matrices 548
21.9 Square Root of a Symmetric Nonnegative Definite Matrix 550
21.10 Some Relationships 551
21.11 Eigenvalues and Eigenvectors of Kronecker Products 554
21.12 Singular Value Decomposition 556
21.13 Simultaneous Diagonalization 566
21.14 Generalized Eigenvalue Problem 569
21.15 Differentiation of Eigenvalues and Eigenvectors 571
21.16 An Equivalence (Involving Determinants and Polynomials) 574
Appendix: Some Properties of Polynomials 580
Exercises 582
Bibliographic and Supplementary Notes 588

22 Linear Transformations 589
22.1 Some Definitions, Terminology, and Basic Results 589
22.2 Scalar Multiples, Sums, and Products of Linear Transformations 595
22.3 Inverse Transformations and Isomorphic Linear Spaces 598
22.4 Matrix Representation of a Linear Transformation 601
22.5 Terminology and Properties Shared by a Linear Transformation and Its Matrix Representation 609
22.6 Linear Functionals and Dual Transformations 612
Exercises 616

References 621

Index 625