Preface

This volume contains the courses delivered at the CIME meeting "Pseudo-differential Operators, Quantization and Signals" held in Cetraro, Italy, from June 19, 2006 to June 24, 2006 and includes the courses by H.-G. Feichtinger presenting new results for Gabor multipliers on modulation and Wiener amalgam spaces, by B. Helffer analyzing non-self-adjoint operators using microlocal techniques, by M. Lamoureux addressing applications of pseudo-differential operators in geophysics, and by N. Lerner applying the techniques of Wick quantization to problems on subellipticity and lower bounds. The lectures by J. Toft on Schatten–von Neumann classes of Weyl pseudo-differential operators are also included.

This introduction is written for non-specialists. We first recall the basic notions and give an account of some developments of pseudo-differential operators. Our starting point is the class of pseudo-differential operators studied in the 1965 seminal paper of Kohn and Nirenberg published in "Communications on Pure and Applied Mathematics." Then we give a brief overview of several pre-eminent ancestors and successors in the study of pseudo-differential operators before and after the Kohn–Nirenberg milestone. The connections with quantization envisaged by Hermann Weyl in his classic "Group Theory and Quantum Mechanics," first observed by Grossmann, Loupias and Stein in the 1968 paper "Annales de l'Institute Fourier (Grenoble)," will then be described in the context of Wigner transforms. These connections give new insights into the role of pseudo-differential operators in the analysis of signals and images in the perspectives of Gabor transforms and wavelet transforms. From these come the Stockwell transform that has numerous applications in geophysics and medical imaging. The recently developed mathematical underpinnings of the Stockwell transform will be highlighted.

1. Pseudo-differential Operators

The starting point is the class of classical pseudo-differential operators introduced by Kohn and Nirenberg [19] and modified almost immediately by Hörmander [16] about 40 years ago. To wit, let $m \in \mathbb{R}$. Then we let $S^m_{1,0}$ or simply S^m be the set of all C^∞ functions σ on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

for all x and ξ in $\mathbb{R}^n$. A function σ in S^m is called a symbol of order m.
Let $\sigma \in S^m$. Then we define the pseudo-differential operator T_σ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)\, d\xi$$

for all φ in $\mathcal{S}(\mathbb{R}^n)$ and all x in $\mathbb{R}^n$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx$$

for all ξ in $\mathbb{R}^n$. It is easy to prove that T_σ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ continuously. The most fundamental properties of pseudo-differential operators which are useful in the study of partial differential equations are listed as Theorems 1.1–1.3.

Theorem 1.1. *Let $\sigma \in S^0$. Then T_σ, initially defined on $\mathcal{S}(\mathbb{R}^n)$, can be uniquely extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.*

Theorem 1.2. *If $\sigma \in S^m$, then $T_\sigma^* = T_\tau$, where $\tau \in S^m$ and*

$$\tau \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma}.$$

Here, T_σ^* is the formal adjoint of T_σ.
To recall, the formal adjoint T_σ^* of T_σ is defined by

$$(T_\sigma\varphi, \psi)_{L^2(\mathbb{R}^n)} = (\varphi, T_\sigma^*\psi)_{L^2(\mathbb{R}^n)}$$

for all φ and ψ in $L^2(\mathbb{R}^n)$, where $(\,,\,)_{L^2(\mathbb{R}^n)}$ is the inner product in $L^2(\mathbb{R}^n)$. The asymptotic expansion $\tau \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma}$ means that

$$\tau - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma} \in S^{m-N}$$

for all positive integers N.

Theorem 1.3. *If $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$, then $T_\sigma T_\tau = T_\lambda$, where $\lambda \in S^{m_1+m_2}$ and*

$$\lambda \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau).$$

The asymptotic expansion

$$\lambda \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau)$$

means that

$$\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau) \in S^{m_1+m_2-N}$$

for all positive integers N.
All these results are very well known and can be found in the books [17] by Hörmander [20] by Kumano-go, [23] by Rodino, [29] by Wong and many others. We can see variants of these results in other settings in this presentation.

2. Ancestors and Successors

Earliest sources of pseudo-differential operators can be traced to problems for n-dimensional singular integral equations. The first contributions to the theory of multi-dimensional singular integrals appear to be those of Tricomi [27] in 1928. To recall, let (r, θ) be the polar coordinates of a generic point $y = (y_1, y_2)$ in $\mathbb{R}^2$ and define for suitable functions φ on $\mathbb{R}^2$,

$$(P\varphi)(x) = \lim_{\varepsilon \to 0} \int_{r>\varepsilon} \frac{h(\theta)}{r^2} \varphi(x-y)\, dy, \quad x \in \mathbb{R}^2.$$

In general, the integral $\int_{\mathbb{R}^2} \frac{h(\theta)}{r^2} \varphi(x-y)\, dy$ is not absolutely convergent, but under the so-called Tricomi condition stipulating that

$$\int_0^{2\pi} h(\theta)\, d\theta = 0$$

and appropriate assumptions on h and φ, the limit exists and $(P\varphi)(x)$ is well defined for almost all x in $\mathbb{R}^2$. If we assume for simplicity that h is C^∞ on the unit circle S^1 with center at the origin, then P is a bounded linear operator from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$. Despite unsuccessful attempts by Tricomi in solving the equation

$$P\varphi = \psi$$

by finding another singular integral operator P^{-1} for which

$$P^{-1}P = I$$

and

$$PP^{-1} = I,$$

where I is the identity operator, we all know nowadays that this can be done using the Fourier transform. Indeed, P can be regarded as the convolution operator given by

$$P\varphi = K * \varphi,$$

where the singular kernel K given by

$$K(y) = \frac{h(\theta)}{r^2}, \quad y = (r, \theta) \in \mathbb{R}^2,$$

has to be suitably seen as a tempered distribution on $\mathbb{R}^2$. Applying the Fourier transform, we get

$$(P\varphi)^\wedge(\xi) = \sigma(\xi)\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^2,$$

where

$$\sigma(\xi) = 2\pi\hat{K}(\xi), \quad \xi \in \mathbb{R}^2.$$

In view of the Tricomi condition on $h \in C^\infty(S^1)$, σ turns out to be C^∞ and homogeneous of degree 0 on $\mathbb{R}^2 \setminus \{0\}$. Hence, apart from the singularity at the origin, σ is a symbol in S^0 depending on ξ only, and with the notation of the preceding section,

$$P = T_\sigma.$$

Furthermore, if σ is elliptic in the sense that there exists a positive constant C such that

$$|\sigma(\xi)| \geq C, \quad \xi \in \mathbb{R}^2,$$

then $\sigma^{-1} \in S^0$ and we can define P^{-1} to be $T_{\sigma^{-1}}$. Such applications of the Fourier transform were not known to Tricomi and it took almost 30 years for mathematicians to come to these simple conclusions. Milestones of the developments in this direction are the works of Giraud [13] in 1934, Calderón and Zygmund [4] in 1952 and Mihlin [21] in 1965. Additional references can be found in the introduction of [21] and the survey paper [24] of Seeley. In fact, the analysis has been extended to the case when h also depends on x, i.e., the kernel K is a function of x and y given by

$$K(x, y) = \frac{h(x, \theta)}{r^2},$$

where $y = (r, \theta)$. In the final formulation of these results in the setting of $\mathbb{R}^n$, the symbol $\sigma \in S^0$ is the Fourier transform with respect to y of the kernel $K(x, y)$ in terms of the singular integral given by

$$\sigma(x, \xi) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} e^{-iy\cdot\xi} K(x, y)\, dy, \quad x, \xi \in \mathbb{R}^n.$$

If σ is elliptic in the sense that there exists a positive constant C such that

$$|\sigma(x, \xi)| \geq C, \quad x, \xi \in \mathbb{R}^n,$$

then we still have

$$\sigma^{-1} \in S^0.$$

However, it is important to note that $T_{\sigma^{-1}}$ is no longer the inverse of P in this case. But, as in Theorem 1.3, we obtain

$$T_{\sigma^{-1}}P = I + K_1$$

and

$$PT_{\sigma^{-1}} = I + K_2,$$

where K_1 and K_2 are pseudo-differential operators of order -1. When we transfer the definition of P to a compact manifold M, the operators K_1 and K_2 are compact and P is then a Fredholm operator on $L^2(M)$. It is remarkable to note that this very rudimentary symbolic calculus with remainders of order -1 plays an important role in the proof of the Atiyah–Singer index formula in [1].

In addition to the obvious extension to an arbitrary order $m \in \mathbb{R}$, the most novel ideas of the Kohn–Nirenberg paper [19] in the context of the theory of singular integral operators are the precise asymptotic formulas articulated in Theorems 1.2 and 1.3. Almost immediately after the appearance of the work of Kohn and Nirenberg is the far-reaching calculus of Hörmander [16] concerning symbols σ of type (ρ, δ), $0 \leq \delta < \rho \leq 1$. Let us recall that a function σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$ and type (ρ, δ) if for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|}$$

for all x and ξ in $\mathbb{R}^n$. Since then, other generalizations and variants of pseudo-differential operators have appeared. Among many interesting classes is the very general class of pseudo-differential operators developed by Beals [2] in 1975 in which the Hörmander estimates are replaced by

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta}\lambda(x, \xi)\Psi(x, \xi)^{-|\beta|}\Phi(x, \xi)^{|\alpha|}$$

for all x and ξ in $\mathbb{R}^n$, where

$$\Psi(x, \xi) = (\Psi_1(x, \xi), \Psi_2(x, \xi), \dots, \Psi_n(x, \xi))$$

and

$$\Phi(x, \xi) = (\Phi_1(x, \xi), \Phi_2(x, \xi), \dots, \Phi_n(x, \xi))$$

are n-tuples of suitable weight functions, and $\lambda(x, \xi)$ is now the "order" of the corresponding pseudo-differential operator. Recasting the calculus of Beals, another achievement is due to Hörmander [16] using the Weyl expression for pseudo-differential operators. We refer the readers to [16] for a wide range of applications to linear partial differential equations. Weyl quantization is described in the next section, and for the sake of simplicity, we begin with a motivation based on symbols in S^m, i.e., Hörmander symbols with $\rho = 1$ and $\delta = 0$.

3. Weyl Transforms

Let $\sigma \in S^m$. Then we can associate to it the pseudo-differential operator T_σ, but T_σ is not the only operator that can be assigned to σ. To see what else can be done, let us note that for all φ in $\mathcal{S}(\mathbb{R}^n)$ and all x in $\mathbb{R}^n$,

$$\begin{aligned}(T_\sigma\varphi)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)\,d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)\varphi(y)\,dy\,d\xi,\end{aligned}$$

where the last integral is to be understood as an oscillatory integral in which the integral with respect to y has to be performed first. With this formula in hand, it requires a huge amount of ingenuity (certainly not logic) to see that we can associate to σ another useful linear operator W_σ on $\mathcal{S}$ defined by the same formula with $\sigma(x,\xi)$ replaced by $\sigma\left(\frac{x+y}{2},\xi\right)$. The linear operator W_σ can be traced back to the work [28] by Hermann Weyl and hence we call W_σ the Weyl transform associated to the symbol σ. In fact, we have the following connection between Weyl transforms and pseudo-differential operators.

Theorem 3.1. *Let $\sigma \in S^m$. Then there exists a symbol τ in S^m such that*

$$T_\sigma = W_\tau$$

and there exists a symbol κ in S^m such that

$$W_\sigma = T_\kappa.$$

Thus, there is a one-to-one correspondence between pseudo-differential operators and Weyl transforms. We have the following result, which can be thought of as the fundamental Theorem of pseudo-differential operators.

Theorem 3.2. *Let $\sigma \in S^m$, $m \in \mathbb{R}$. Then for all φ and ψ in $\mathcal{S}(\mathbb{R}^n)$,*

$$(W_\sigma\varphi,\psi)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma(x,\xi)W(\varphi,\psi)(x,\xi)\,dx\,d\xi,$$

where $W(\varphi,\psi)$ is the Wigner transform of φ and ψ defined by

$$W(\varphi,\psi)(x,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p}\varphi\left(x+\frac{p}{2}\right)\overline{\psi\left(x-\frac{p}{2}\right)}\,dp$$

for all x and ξ in $\mathbb{R}^n$.

The Wigner transform is a very well-behaved bilinear form on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and it satisfies the so-called Moyal identity or the Plancherel formula to the effect that

$$\|W(\varphi,\psi)\|_{L^2(\mathbb{R}^{2n})} = \|\varphi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}$$

for all φ and ψ in $L^2(\mathbb{R}^n)$.
A *tour de force* from Theorems 3.1 and 3.2 shows that we can now define pseudo-differential operators with nonsmooth symbols not in the Hörmander class S^m. To be specific, we look at symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ only.
Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we define the Weyl transform W_σ on $L^2(\mathbb{R}^n)$ by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi)\, W(f,g)(x,\xi)\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$. Then we have the following analogs of Theorems 1.1–1.3.

Theorem 3.3. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert–Schmidt operator.*

Theorem 3.4. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the adjoint W_σ^* of W_σ is given by*

$$W_\sigma^* = W_{\overline{\sigma}}.$$

Theorem 3.5. *Let σ and τ be symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$W_\sigma W_\tau = W_\lambda,$$

where $\lambda \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and is given by

$$\hat{\lambda} = (2\pi)^{-n}(\hat{\sigma} *_{1/4} \hat{\tau}).$$

Theorem 3.5, which is attributed to Grossmann, Loupias and Stein [15], tells us that the product of two Weyl transforms with symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is again a Weyl transform with symbol in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and is given by a twisted convolution. Let us recall that the twisted convolution $f *_{1/4} g$ of two measurable functions f and g on $\mathbb{C}^n (= \mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$(f *_{1/4} g)(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{i[z,w]/4}dw$$

for all z in $\mathbb{C}^n$, where $[z,w]$ is the symplectic form of z and w given by

$$[z,w] = 2\,\mathrm{Im}(z \cdot \overline{w}).$$

See the books [3] by Boggiatto, Buzano and Rodino, [12] by Folland, [25] by Stein and [30] by Wong for details and related topics.

4. Gabor Transforms

If we make a change of variables in the definition of the Wigner transform, then we get for all f and g in $L^2(\mathbb{R}^n)$, and all x and ξ in $\mathbb{R}^n$,

$$W(f,g)(x,\xi) = 2^n e^{2ix\cdot\xi}(G_{\tilde{g}}f)(2x,2\xi),$$

where

$$\tilde{g}(x) = g(-x)$$

for all x in $\mathbb{R}^n$ and $G_{\tilde{g}}f$ is the well-known Gabor transform or the short-time Fourier transform of f with window $\tilde{g}$ given by

$$(G_{\tilde{g}}f)(x,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-it\cdot\xi}f(t)\overline{\tilde{g}(t-x)}\,dt$$

for all x and ξ in $\mathbb{R}^n$. In image analysis, we can think of $(G_{\tilde{g}}f)(x,\xi)$ as the spectral content of the image f with frequency ξ at the point x.

Let us now fix a window φ in $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$. Then the Gabor transform $G_\varphi f$ of f is given by

$$(G_\varphi f)(x,\xi) = (2\pi)^{-n/2}(f, M_\xi T_{-x}\varphi)_{L^2(\mathbb{R}^n)}$$

for all x and ξ in $\mathbb{R}^n$, where M_ξ and T_{-x} are the modulation operator and the translation operator given by

$$(M_\xi h)(t) = e^{it\cdot\xi}h(t)$$

and

$$(T_{-x}h)(t) = h(t-x)$$

for all measurable functions h on $\mathbb{R}^n$ and all t in $\mathbb{R}^n$. Now, for all x and ξ in $\mathbb{R}^n$, we define the function $\varphi_{x,\xi}$ on $\mathbb{R}^n$ by

$$\varphi_{x,\xi} = M_\xi T_{-x}\varphi.$$

We call the functions $\varphi_{x,\xi}$, $x,\xi\in\mathbb{R}^n$, the Gabor wavelets generated from the Gabor mother wavelet φ by translations and modulations.

The usefulness of the Gabor wavelets in signal and image analysis is enhanced by the following resolution of the identity formula, which allows the reconstruction of a signal or an image from its Gabor spectrum.

Theorem 4.1. For all f in $L^2(\mathbb{R}^n)$,

$$f = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f,\varphi_{x,\xi})_{L^2(\mathbb{R}^n)}\varphi_{x,\xi}dx\,d\xi.$$

Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we define the Gabor multiplier $G_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$(G_{\sigma,\varphi} f, g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi)(G_\varphi f)(x,\xi)\overline{(G_\varphi g)(x,\xi)}\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$. Using the Gabor wavelets, we see that $G_{\sigma,\varphi} f$ is equal to

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi)(f, \varphi_{x,\xi})_{L^2(\mathbb{R}^n)} \varphi_{x,\xi}\, dx\, d\xi$$

for all f in $L^2(\mathbb{R}^n)$.
Gabor multipliers are also known as localization operators, Daubechies operators, anti-Wick quantization and Wick quantization. The following results are the analogs of Theorems 1.1–1.3 for Gabor multipliers.

Theorem 4.2. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Gabor multiplier $G_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert–Schmidt operator.*

Theorem 4.3. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the adjoint $G^*_{\sigma,\varphi}$ of $G_{\sigma,\varphi}$ is given by*

$$G^*_{\sigma,\varphi} = G_{\overline{\sigma},\varphi}.$$

Theorem 4.4. *Let σ and τ be functions in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$G_{\sigma,\varphi} G_{\tau,\varphi} = G_{\lambda,\varphi},$$

where

$$\hat{\lambda} = (2\pi)^{-n}(\hat{\sigma} *^{1/2} \hat{\tau}).$$

In Theorem 4.4, we have a new twisted convolution. To wit, the new twisted convolution $f *^{1/2} g$ of two measurable functions f and g on $\mathbb{C}^n$, is defined by

$$(f *^{1/2} g)(z) = \int_{\mathbb{C}^n} f(z-w)\, g(w) e^{(z\cdot\overline{w} - |w|^2)/2} dw$$

for all z in $\mathbb{C}^n$ provided that the integral exists. Theorem 4.4 can be found in the 2000 paper [10] by Du and Wong.
The interesting feature with Theorem 4.4 is that the new twisted convolution $f *^{1/2} g$ of two functions f and g in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ need not be in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. This phenomenon is the motivation for many interesting research papers on the product of Gabor multipliers. It suffices to mention the works [5] by Coburn, [7] by Cordero and Gröchenig and [8] by Cordero and Rodino.
What is a Gabor multiplier? Is it something already well known to us? The answer is yes.

Theorem 4.5. *Let* $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$*. Then*

$$G_{\sigma,\varphi} = W_{\sigma * V(\varphi,\varphi)},$$

where

$$V(\varphi,\varphi)^\wedge = W(\varphi,\varphi).$$

References for the materials in this section are the books [9] by Daubechies, [14] by Gröchenig, [31] by Wong and many others.

5. Wavelet Transforms

Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$ and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Then we call φ a mother wavelet and φ is said to satisfy the admissibility condition.

Let φ be a mother wavelet. Then for all b in $\mathbb{R}$ and a in $\mathbb{R}\backslash\{0\}$, we can define the wavelet $\varphi_{b,a}$ by

$$\varphi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \varphi\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}.$$

We call $\varphi_{b,a}$ the affine wavelet generated from the mother wavelet φ by translation and dilation. To put things in perspective, let $b \in \mathbb{R}$ and let $a \in \mathbb{R}\backslash\{0\}$. Then we let T_b be the translation operator as before and D_a be the dilation operator defined by

$$(D_a f)(x) = \sqrt{|a|} f(ax)$$

for all x in $\mathbb{R}$ and all measurable functions f on $\mathbb{R}$. So, the wavelet $\varphi_{b,a}$ can be expressed as

$$\varphi_{b,a} = T_{-b} D_{1/a} \varphi.$$

Let φ be a mother wavelet. Then the wavelet transform $\Omega_\varphi f$ of a function f in $L^2(\mathbb{R})$ is defined to be the function on $\mathbb{R} \times \mathbb{R}\backslash\{0\}$ by

$$(\Omega_\varphi f)(b,a) = (f, \varphi_{b,a})_{L^2(\mathbb{R})}$$

for all b in $\mathbb{R}$ and a in $\mathbb{R}\backslash\{0\}$. At the heart of the analysis of the wavelet transform is the following resolution of the identity formula.

Theorem 5.1. *Let* φ *be a mother wavelet. Then for all functions* f *and* g *in* $L^2(\mathbb{R})$,

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\Omega_\varphi f)(b,a) \overline{(\Omega_\varphi g)(b,a)} \frac{db\, da}{a^2},$$

where

$$c_\varphi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi.$$

The resolution of the identity formula leads to the reconstruction formula which says that

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} \varphi_{b,a} \frac{db\, da}{a^2}$$

for all f in $L^2(\mathbb{R})$. In other words, we have a reconstruction formula for the signal f from a knowledge of its time-scale spectrum.
Let φ be a mother wavelet and let $\sigma \in L^2(\mathbb{R} \times \mathbb{R})$. Then we define the wavelet multiplier $\Omega_{\sigma,\varphi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\Omega_{\sigma,\varphi} f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b,a)(f, \varphi_{b,a})_{L^2(\mathbb{R})} \varphi_{b,a} \frac{db\, da}{a^2}$$

for all f in $L^2(\mathbb{R})$.
As in the case of the Gabor multipliers, we have the following results.

Theorem 5.2. *The wavelet multiplier*

$$\Omega_{\sigma,\varphi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a Hilbert–Schmidt operator.

Theorem 5.3. *The adjoint $\Omega_{\sigma,\varphi}^*$ of the wavelet multiplier $\Omega_{\sigma,\varphi}$ is given by*

$$\Omega_{\sigma,\varphi}^* = \Omega_{\overline{\sigma},\varphi}.$$

What is the product of two wavelet multipliers? The answer is not so simple and seems to depend on the availability of a useful formula for a wavelet multiplier. Some technical information in this direction can be found in the paper [32] by Wong. If

$$\sigma(b,a) = \alpha(a)\beta(b)$$

for all b in $\mathbb{R}$ and all a in $\mathbb{R}\backslash\{0\}$, then $\Omega_{\sigma,\varphi}$ is a paracommutator in the sense of Janson and Peetre [18], and Peng and Wong [22]. If σ is a function of a only, then $\Omega_{\sigma,\varphi}$ is a paraproduct in the sense of Coifman and Meyer [6]. If σ is a function of b only, then $\Omega_{\sigma,\varphi}$ is a Fourier multiplier.

6. Stockwell Transforms

Let us recall that for a signal f in $L^2(\mathbb{R})$, the Gabor transform $(G_\varphi f)(x,\xi)$ with respect to the window φ gives the time–frequency content of f at time x and frequency ξ by using the window φ at time x. The drawback here is that a window of fixed width is used for all time x. It is more accurate if we can have an adaptive window that gives a wide window for low frequency and a narrow window for high frequency. That this can be done comes from our experiences with the wavelet transform. Indeed, we see that the window $\varphi_{b,a}$ is narrow if the scale a is small and the window is wide when the scale is big.

Now, the Stockwell transform $S_\varphi f$ with window φ of a signal f is defined by

$$(S_\varphi f)(x,\xi) = (2\pi)^{-1/2}|\xi| \int_{-\infty}^{\infty} e^{-it\xi} f(t)\overline{\varphi(\xi(t-x))}\, dt$$

for all x and ξ in $\mathbb{R}$. Formally, we note that for all f in $L^2(\mathbb{R})$, all x in $\mathbb{R}$ and all ξ in $\mathbb{R}\backslash\{0\}$,

$$(S_\varphi f)(x,\xi) = (f, \varphi^{x,\xi})_{L^2(\mathbb{R})},$$

where

$$\varphi^{x,\xi} = (2\pi)^{-1/2} M_\xi T_{-x} \tilde{D}_\xi \varphi.$$

Here, the dilation operator $\tilde{D}_\xi$ is defined by

$$(\tilde{D}_\xi f)(t) = |\xi| f(\xi t)$$

for all t in $\mathbb{R}$ and all measurable functions f on $\mathbb{R}$. Besides the modulation, a notable feature in the Stockwell transform is the normalizing factor in the dilation operator, which is $|\cdot|$ and not $|\cdot|^{1/2}$ as in the case of the wavelet transforms. These features distinguish the Stockwell transform from the wavelet transforms.

The Stockwell transform has recently been successfully used in seismic waves [26] by Stockwell, Mansinha and Lowe and in medical imaging [34] by Zhu and others. An attempt in understanding the mathematical underpinnings of the Stockwell transform is underway by Wong and Zhu. See [33] in this direction and we describe some of the results therein.

Theorem 6.1. *Let φ be a window with*

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1.$$

Then for all f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} (S_\varphi f)(x,\xi)\, dx = \hat{f}(\xi)$$

for all ξ in $\mathbb{R}$.

See Fig. 1 for an illustration of Theorem 6.1. In view of Theorem 6.1, we have a reconstruction formula for a signal f in terms of its Stockwell spectrum, which says that

$$f = \mathcal{F}^{-1} A S_\varphi f,$$

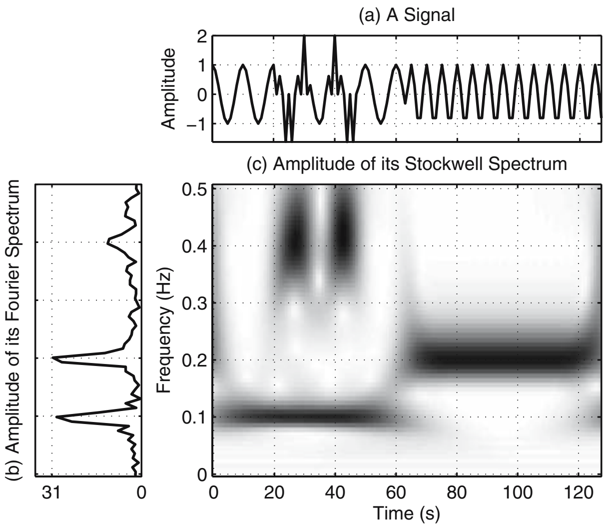

Fig. 1 Time–frequency representation of the Stockwell transform: (**a**) a signal consisting of multiple frequency components (**b**) the amplitude of the corresponding Fourier spectrum, i.e., $|(\mathcal{F}f)(k)|$ (**c**) the contour plotting the amplitude of the corresponding Stockwell transform, i.e., $|(Sf)(\tau, k)|$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform and A is the time average operator given by

$$(AF)(\xi) = \int_{-\infty}^{\infty} F(x, \xi)\, dx$$

for all ξ in $\mathbb{R}$ and all measurable functions F on $\mathbb{R} \times \mathbb{R}$.
For the second result, we let M be the set of all measurable functions F on $\mathbb{R} \times \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} F(x, \xi)\, dx \right|^2 d\xi < \infty.$$

Then M is an indefinite Hilbert space in which the indefinite inner product $(\ ,\ )_M$ is given by

$$(F, G)_M = (AF, AG)_{L^2(\mathbb{R})}$$

for all F and G in M.
Then we have a characterization of the Stockwell spectra given by the following theorem.

Theorem 6.2. $\{S_\varphi f : f \in L^2(\mathbb{R})\} = M/Z$, where

$$Z = \{F : \mathbb{R} \times \mathbb{R} \to \mathbb{C} : AF = 0\}.$$

Can we reconstruct a signal from its Stockwell spectrum? The answer is yes provided that we choose the right window. To do this, we say that a function φ in $L^2(\mathbb{R})$ satisfies the admissibility condition if and only if

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi < \infty.$$

For a function in $L^2(\mathbb{R})$ satisfying the admissibility condition, we define the constant c_φ by

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi.$$

Theorem 6.3. Let φ be a function in $L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$ satisfying the admissibility condition. Then for all f in $L^2(\mathbb{R})$,

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{x,\xi})_{L^2(\mathbb{R})} \varphi^{x,\xi} \frac{dx\, d\xi}{|\xi|}.$$

Remark: It is important to note that an admissible wavelet φ for the Stockwell transform has to satisfy the condition

$$\hat{\varphi}(-1) = 0.$$

So, the Gaussian window that has been used exclusively for the Stockwell transform in the literature is not admissible.
This formula and its discretization can be found in the paper [11] by Du, Wong and Zhu.

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