Contents

Preface ix

Part I Basic Techniques

1 Hilbert Space Techniques

1.1 The projection on a closed convex set 3
1.2 The Riesz representation theorem 6
1.3 The Lax–Milgram theorem 8
1.4 Convergence techniques 10
Exercises 11

2 A Survey of Essential Analysis

2.1 L^p-techniques 13
2.2 Introduction to distributions 18
2.3 Sobolev Spaces 22
Exercises 32

3 Weak Formulation of Elliptic Problems

3.1 Motivation 35
3.2 The weak formulation 38
Exercises 41

4 Elliptic Problems in Divergence Form

4.1 Weak formulation 43
4.2 The weak maximum principle 49
4.3 Inhomogeneous problems 53
Exercises 54

5 Singular Perturbation Problems

5.1 A prototype of a singular perturbation problem 57
5.2 Anisotropic singular perturbation problems 61
Exercises 69

6 Problems in Large Cylinders
6.1 A model problem . . . 73
6.2 Another type of convergence . . . 79
6.3 The general case . . . 82
6.4 An application . . . 86
Exercises . . . 89

7 Periodic Problems
7.1 A general theory . . . 93
7.2 Some additional remarks . . . 101
Exercises . . . 103

8 Homogenization
8.1 More on periodic functions . . . 106
8.2 Homogenization of elliptic equations . . . 109
8.2.1 The one-dimensional case . . . 109
8.2.2 The n-dimensional case . . . 112
Exercises . . . 119

9 Eigenvalues
9.1 The one-dimensional case . . . 121
9.2 The higher-dimensional case . . . 123
9.3 An application . . . 127
Exercises . . . 128

10 Numerical Computations
10.1 The finite difference method . . . 129
10.2 The finite element method . . . 135
Exercises . . . 147

Part II More Advanced Theory

11 Nonlinear Problems
11.1 Monotone methods . . . 153
11.2 Quasilinear equations . . . 160
11.3 Nonlocal problems . . . 166
11.4 Variational inequalities . . . 170
Exercises . . . 174

12 L^∞-estimates
12.1 Some simple cases . . . 177
12.2 A more involved estimate . . . 180
12.3 The Sobolev–Gagliardo–Nirenberg inequality . . . 183
12.4 The maximum principle on small domains . . . 187
Exercises . . . 188

13 Linear Elliptic Systems
13.1 The general framework . . . 191
13.2 Some examples . . . 197
Exercises . . . 202

14 The Stationary Navier–Stokes System
14.1 Introduction . . . 203
14.2 Existence and uniqueness result . . . 205
Exercises . . . 208

15 Some More Spaces
15.1 Motivation . . . 211
15.2 Essential features of the Sobolev spaces $W^{k,p}$ . . . 212
15.3 An application . . . 215
Exercises . . . 216

16 Regularity Theory
16.1 Introduction . . . 217
16.2 The translation method . . . 221
16.3 Regularity of functions in Sobolev spaces . . . 224
16.4 The bootstrap technique . . . 227
Exercises . . . 229

17 The p-Laplace Equation
17.1 A minimization technique . . . 231
17.2 A weak maximum principle and its consequences . . . 237
17.3 A generalization of the Lax–Milgram theorem . . . 239
Exercises . . . 245

18 The Strong Maximum Principle
18.1 A first version of the maximum principle . . . 247
18.2 The Hopf maximum principle . . . 252
18.3 Application: the moving plane technique . . . 255
Exercises . . . 259

19 Problems in the Whole Space
19.1 The harmonic functions, Liouville theorem 261
19.2 The Schrödinger equation 267
Exercises 273

Appendix: Fixed Point Theorems
A.1 The Brouwer fixed point theorem 275
A.2 The Schauder fixed point theorem 279
Exercises 280

Bibliography 281

Index 287