

# Contents

|  |      |
|--|------|
| <b>Preface</b> .....   | vii  |
| <b>Acknowledgements</b> .....  | xi   |
| <b>Common Notations and Terminology</b> .....  | xvii |
| <b>1 Introduction</b> .....  | 1    |
| <b>2 Unwinding proofs ('Proof Mining')</b> .....   | 13   |
| 2.1 Introductory remark .....  | 13   |
| 2.2 Informal treatment of ineffective proofs .....   | 13   |
| 2.3 Herbrand's theorem and the no-counterexample interpretation ....                                       | 22   |
| 2.4 Exercises, historical comments and suggested further reading ....                                      | 38   |
| <b>3 Intuitionistic and classical arithmetic in all finite types</b> .....                                 | 41   |
| 3.1 Intuitionistic and classical predicate logic .....   | 41   |
| 3.2 Intuitionistic ('Heyting') arithmetic HA and Peano arithmetic PA .                                     | 44   |
| 3.3 Extensional intuitionistic ('Heyting') and classical ('Peano')<br>arithmetic in all finite types ..... | 46   |
| 3.4 Fragments of (W)E-HA <sup>ω</sup> and (W)E-PA <sup>ω</sup> .....                                       | 52   |
| 3.5 Fragments corresponding to the Grzegorzcyk hierarchy .....   | 54   |
| 3.6 Models of E-PA <sup>ω</sup> .....  | 67   |
| 3.7 Exercises, historical comments and suggested further reading ....                                      | 73   |
| <b>4 Representation of Polish metric spaces</b> .....  | 77   |
| 4.1 Representation of real numbers .....   | 77   |
| 4.2 Representation of complete separable metric ('Polish') spaces ....                                     | 81   |
| 4.3 Special representation of compact metric spaces .....  | 88   |
| 4.4 Fragments, exercises, historical comments and suggested further<br>reading .....                       | 94   |

|           |   |     |
|-----------|---|-----|
| <b>5</b>  | <b>Modified realizability</b> .....   | 97  |
| 5.1       | The soundness and program extraction theorems .....                           | 97  |
| 5.2       | Remarks on fragments of E-HA <sup>ω</sup> .....                               | 105 |
| 5.3       | Exercises, historical comments and suggested further reading ....             | 107 |
| <b>6</b>  | <b>Majorizability and the fan rule</b> .....                                  | 109 |
| 6.1       | A syntactic treatment of majorization and the fan rule .....                  | 109 |
| 6.2       | Exercises, historical comments and suggested further reading ....             | 114 |
| <b>7</b>  | <b>Semi-intuitionistic systems and monotone modified realizability</b> .....  | 115 |
| 7.1       | The soundness and bound extraction theorems .....                             | 115 |
| 7.2       | Fragments, exercises, historical comments and suggested further reading ..... | 123 |
| <b>8</b>  | <b>Gödel's functional ('Dialectica') interpretation</b> .....                 | 125 |
| 8.1       | Introduction .....  | 125 |
| 8.2       | The soundness and program extraction theorems .....                           | 129 |
| 8.3       | Fragments, exercises, historical comments and suggested further reading ..... | 138 |
| <b>9</b>  | <b>Semi-intuitionistic systems and monotone functional interpretation</b> ..  | 141 |
| 9.1       | The soundness and bound extraction theorems .....                             | 141 |
| 9.2       | Applications of monotone functional interpretation .....                      | 146 |
| 9.3       | Examples of axioms $\Delta$ : Weak König's lemma WKL .....                    | 149 |
| 9.4       | WKL as a universal sentence $\Delta$ .....                                    | 156 |
| 9.5       | Fragments, exercises, historical comments and suggested further reading ..... | 160 |
| <b>10</b> | <b>Systems based on classical logic and functional interpretation</b> .....   | 163 |
| 10.1      | The negative translation .....  | 163 |
| 10.2      | Combination of negative translation and functional interpretation .           | 165 |
| 10.3      | Application: Uniform weak König's lemma UWKL .....                            | 178 |
| 10.4      | Elimination of extensionality .....   | 180 |
| 10.5      | Fragments of (W)E-PA <sup>ω</sup> .....                                       | 188 |
| 10.6      | The computational strength of full extensionality .....                       | 191 |
| 10.7      | Exercises, historical comments and suggested further reading ....             | 195 |
| <b>11</b> | <b>Functional interpretation of full classical analysis</b> .....             | 199 |
| 11.1      | Functional interpretation of full comprehension .....                         | 199 |
| 11.2      | Functional interpretation of dependent choice .....                           | 206 |
| 11.3      | Functional interpretation of arithmetical comprehension .....                 | 209 |
| 11.4      | Functional interpretation of (IPP) by finite bar recursion .....              | 213 |
| 11.5      | Models of bar recursion .....   | 214 |
| 11.6      | Exercises, historical comments and suggested further reading ....             | 219 |

|           |   |     |
|-----------|---|-----|
| <b>12</b> | <b>A non-standard principle of uniform boundedness</b> . . . . .                            | 223 |
| 12.1      | The $\Sigma_1^0$ -boundedness principle . . . . .   | 223 |
| 12.2      | Applications of $\Sigma_1^0$ -boundedness . . . . .   | 232 |
| 12.3      | Remarks on the fragments $E\text{-}G_nA^\omega$ . . . . .                                   | 238 |
| 12.4      | Exercises, historical comments and suggested further reading . . . . .                      | 241 |
| <b>13</b> | <b>Elimination of monotone Skolem functions</b> . . . . .                                   | 243 |
| 13.1      | Skolem functions of type degree 1 in fragments of finite type arithmetic . . . . .          | 243 |
| 13.2      | Elimination of Skolem functions for monotone formulas . . . . .                             | 247 |
| 13.3      | The principle of convergence for bounded monotone sequences of real numbers (PCM) . . . . . | 262 |
| 13.4      | $\Pi_1^0\text{-CA}$ and $\Pi_1^0\text{-AC}$ . . . . .                                       | 265 |
| 13.5      | The Bolzano-Weierstraß property for bounded sequences in $\mathbb{R}^d$ . . . . .           | 269 |
| 13.6      | Exercises, historical comments and suggested further reading . . . . .                      | 272 |
| <b>14</b> | <b>The Friedman A-translation</b> . . . . .   | 273 |
| 14.1      | The A-translation . . . . .   | 273 |
| 14.2      | Historical comments and suggested further reading . . . . .                                 | 277 |
| <b>15</b> | <b>Applications to analysis: general metatheorems I</b> . . . . .                           | 279 |
| 15.1      | A general metatheorem for Polish spaces . . . . .   | 279 |
| 15.2      | Applications to uniqueness proofs . . . . .   | 284 |
| 15.3      | Applications to monotone convergence theorems . . . . .                                     | 291 |
| 15.4      | Applications to proofs of contractivity . . . . .   | 292 |
| 15.5      | Remarks on fragments of $\mathcal{S}^\omega$ . . . . .                                      | 293 |
| 15.6      | Historical comments and suggested further reading . . . . .                                 | 295 |
| <b>16</b> | <b>Case study I: Uniqueness proofs in approximation theory</b> . . . . .                    | 297 |
| 16.1      | Uniqueness proofs in best approximation theory . . . . .                                    | 297 |
| 16.2      | Best Chebycheff approximation I . . . . .   | 303 |
| 16.3      | Best Chebycheff approximation II . . . . .  | 328 |
| 16.4      | Best $L_1$ -approximation . . . . .   | 348 |
| 16.5      | Exercises, historical comments and suggested further reading . . . . .                      | 376 |
| <b>17</b> | <b>Applications to analysis: general metatheorems II</b> . . . . .                          | 377 |
| 17.1      | Introduction . . . . .  | 377 |
| 17.2      | Main results in the metric and hyperbolic case . . . . .                                    | 391 |
| 17.3      | The case of normed spaces . . . . .   | 410 |
| 17.4      | Proofs of theorems 17.35, 17.52 and 17.69 . . . . .   | 420 |
| 17.5      | Further variations . . . . .  | 431 |
| 17.6      | Treatment of several metric or normed spaces $X_1, \dots, X_n$ simultaneously . . . . .     | 435 |
| 17.7      | A generalized uniform boundedness principle $\exists\text{-UB}^X$ . . . . .                 | 436 |
| 17.8      | Applications of $\exists\text{-UB}^X$ . . . . .   | 441 |
| 17.9      | Fragments of $\mathcal{S}^\omega[\dots]$ . . . . .  | 450 |

17.10 Exercises, historical comments and suggested further reading . . . . 452

**18 Case study II: Applications to the fixed point theory of nonexpansive mappings . . . . . 455**

18.1 General facts . . . . . 455

18.2 Applications of the metatheorems from chapter 17 . . . . . 461

18.3 Logical analysis of the proof of the Borwein-Reich-Shafrir theorem . . . . . 468

18.4 Asymptotically nonexpansive mappings . . . . . 496

18.5 Applications of proof mining in ergodic theory . . . . . 499

18.6 Exercises, historical comments and suggested further reading . . . . 501

**19 Final comments . . . . . 503**

**References . . . . . 507**

**List of formal systems and term classes . . . . . 525**

**List of axioms and rules . . . . . 527**

**Index . . . . . 529**