

# Contents

<b>1</b>	<b>Markov Chains</b>	1
1.1	Introduction	2
1.2	Probabilities of Sample Paths	5
1.3	Construction of Markov Chains	8
1.4	Examples	10
1.5	Stopping Times and Strong Markov Property	16
1.6	Classification of States	19
1.7	Hitting and Absorbtion Probabilities	26
1.8	Branching Processes	30
1.9	Stationary Distributions	33
1.10	Limiting Distributions	40
1.11	Regenerative Property and Cycle Costs	42
1.12	Strong Laws of Large Numbers	45
1.13	Examples of Limiting Averages	50
1.14	Optimal Design of Markovian Systems	53
1.15	Closed Network Model	55
1.16	Open Network Model	59
1.17	Reversible Markov Chains	61
1.18	Markov Chain Monte Carlo	68
1.19	Markov Chains on Subspaces	71
1.20	Limit Theorems via Coupling	73
1.21	Criteria for Positive Recurrence	76
1.22	Review of Conditional Probabilities	81
1.23	Exercises	84
<b>2</b>	<b>Renewal and Regenerative Processes</b>	99
2.1	Renewal Processes	99
2.2	Strong Laws of Large Numbers	104
2.3	The Renewal Function	107

2.4	Future Expectations . . . . .	114
2.5	Renewal Equations . . . . .	114
2.6	Blackwell's Theorem . . . . .	116
2.7	Key Renewal Theorem . . . . .	118
2.8	Regenerative Processes . . . . .	121
2.9	Limiting Distributions for Markov Chains . . . . .	126
2.10	Processes with Regenerative Increments . . . . .	126
2.11	Average Sojourn Times in Regenerative Processes . . . . .	129
2.12	Batch-Service Queueing System . . . . .	132
2.13	Central Limit Theorems . . . . .	135
2.14	Terminating Renewal Processes . . . . .	139
2.15	Stationary Renewal Processes . . . . .	144
2.16	Refined Limit Laws . . . . .	148
2.17	Proof of the Key Renewal Theorem* . . . . .	151
2.18	Proof of Blackwell's Theorem* . . . . .	153
2.19	Stationary-Cycle Processes* . . . . .	155
2.20	Exercises . . . . .	156
<b>3</b>	<b>Poisson Processes . . . . .</b>	<b>169</b>
3.1	Poisson Processes on $\mathbb{R}_+$ . . . . .	170
3.2	Characterizations of Classical Poisson Processes . . . . .	173
3.3	Location of Points . . . . .	176
3.4	Functions of Point Locations . . . . .	179
3.5	Poisson Processes on General Spaces . . . . .	181
3.6	Integrals and Laplace Functionals of Poisson Processes . . . . .	183
3.7	Poisson Processes as Sample Processes . . . . .	188
3.8	Deterministic Transformations of Poisson Processes . . . . .	190
3.9	Marked and Space-Time Poisson Processes . . . . .	194
3.10	Partitions and Translations of Poisson Processes . . . . .	196
3.11	Markov/Poisson Processes . . . . .	201
3.12	Poisson Input-Output Systems . . . . .	203
3.13	Network of $M_t/G_t/\infty$ Stations . . . . .	206
3.14	Cox Processes . . . . .	211
3.15	Compound Poisson Processes . . . . .	214
3.16	Poisson Law of Rare Events . . . . .	216
3.17	Poisson Convergence Theorems* . . . . .	218
3.18	Exercises . . . . .	225
<b>4</b>	<b>Continuous-Time Markov Chains . . . . .</b>	<b>241</b>
4.1	Introduction . . . . .	242
4.2	Examples . . . . .	245
4.3	Markov Properties . . . . .	247
4.4	Transition Probabilities and Transition Rates . . . . .	251
4.5	Existence of CTMCs . . . . .	253
4.6	Uniformization, Travel Times and Transition Probabilities . . . . .	255

4.7	Stationary and Limiting Distributions . . . . .	258
4.8	Regenerative Property and Cycle Costs . . . . .	263
4.9	Ergodic Theorems . . . . .	264
4.10	Expectations of Cost and Utility Functions . . . . .	269
4.11	Reversibility . . . . .	272
4.12	Modeling of Reversible Phenomena . . . . .	277
4.13	Jackson Network Processes . . . . .	282
4.14	Multiclass Networks . . . . .	287
4.15	Poisson Transition Times . . . . .	291
4.16	Palm Probabilities . . . . .	299
4.17	PASTA at Poisson Transitions . . . . .	303
4.18	Relating Palm and Ordinary Probabilities . . . . .	306
4.19	Stationarity Under Palm Probabilities . . . . .	310
4.20	$G/G/1$ , $M/G/1$ and $G/M/1$ Queues . . . . .	314
4.21	Markov-Renewal Processes* . . . . .	321
4.22	Exercises . . . . .	323
<b>5</b>	<b>Brownian Motion . . . . .</b>	<b>341</b>
5.1	Definition and Strong Markov Property . . . . .	342
5.2	Brownian Motion as a Gaussian Process . . . . .	345
5.3	Maximum Process and Hitting Times . . . . .	349
5.4	Special Random Times . . . . .	352
5.5	Martingales . . . . .	354
5.6	Optional Stopping of Martingales . . . . .	358
5.7	Hitting Times for Brownian Motion with Drift . . . . .	361
5.8	Limiting Averages and Law of the Iterated Logarithm . . . . .	364
5.9	Donsker's Functional Central Limit Theorem . . . . .	368
5.10	Regenerative and Markov FCLTs . . . . .	373
5.11	Peculiarities of Brownian Sample Paths . . . . .	377
5.12	Brownian Bridge Process . . . . .	379
5.13	Geometric Brownian Motion . . . . .	383
5.14	Multidimensional Brownian Motion . . . . .	385
5.15	Brownian/Poisson Particle System . . . . .	387
5.16	$G/G/1$ Queues in Heavy Traffic . . . . .	389
5.17	Brownian Motion in a Random Environment . . . . .	393
5.18	Exercises . . . . .	394
<b>6</b>	<b>Appendix . . . . .</b>	<b>405</b>
6.1	Probability Spaces and Random Variables . . . . .	405
6.2	Table of Distributions . . . . .	407
6.3	Random Elements and Stochastic Processes . . . . .	409
6.4	Expectations as Integrals . . . . .	410
6.5	Functions of Stochastic Processes . . . . .	412

6.6 Independence .....	415
6.7 Conditional Probabilities and Expectations .....	417
6.8 Existence of Stochastic Processes .....	419
6.9 Convergence Concepts .....	421
<b>Bibliographical Notes .....</b>	<b>427</b>
<b>References .....</b>	<b>429</b>
<b>Notation .....</b>	<b>435</b>
<b>Index .....</b>	<b>437</b>