

# CONTENTS

## I. INTRODUCTION

§ 1. The Subject-Matter . . . . .	1
§ 2. The Method of Presentation . . . . .	7
§ 3. The Point of View . . . . .	9

## II. THE SPECIAL THEORY OF RELATIVITY

### *Part I.* THE TWO POSTULATES AND THE LORENTZ TRANSFORMATION

§ 4. Introduction . . . . .	12
§ 5. The First Postulate of Relativity . . . . .	12
§ 6. The Second Postulate of Relativity . . . . .	15
§ 7. Necessity for Modifying Older Ideas as to Space and Time . . . . .	17
§ 8. The Lorentz Transformation Equations . . . . .	18
§ 9. Transformation Equations for Spatial and Temporal Intervals. Lorentz Contraction and Time Dilation . . . . .	22
§ 10. Transformation Equations for Velocity . . . . .	25
§ 11. Transformation Equation for the Lorentz Contraction Factor . . . . .	27
§ 12. Transformation Equations for Acceleration . . . . .	27

### *Part II.* TREATMENT OF SPECIAL RELATIVITY WITH THE HELP OF A FOUR-DIMENSIONAL GEOMETRY

§ 13. The Space-Time Continuum . . . . .	28
§ 14. The Three Plus One Dimensions of Space-Time . . . . .	29
§ 15. The Geometry Corresponding to Space-Time . . . . .	30
§ 16. The Signature of the Line Element and the Three Kinds of Interval . . . . .	31
§ 17. The Lorentz Rotation of Axes . . . . .	32
§ 18. The Transformation to Proper Coordinates . . . . .	33
§ 19. Use of Tensor Analysis in the Theory of Relativity . . . . .	34
§ 20. Simplification of Tensor Analysis in the Case of Special Relativity. Galilean Coordinates . . . . .	37
§ 21. Correspondence of Four-Dimensional Treatment with the Postulates of Special Relativity . . . . .	39

## III. SPECIAL RELATIVITY AND MECHANICS

### *Part I.* THE DYNAMICS OF A PARTICLE

§ 22. The Principles of the Conservation of Mass and Momentum . . . . .	42
§ 23. The Mass of a Moving Particle . . . . .	43
§ 24. The Transformation Equations for Mass . . . . .	45
§ 25. The Definition and Transformation Equations for Force . . . . .	45
§ 26. Work and Kinetic Energy . . . . .	47
§ 27. The Relations between Mass, Energy, and Momentum . . . . .	48
§ 28. Four-Dimensional Expression of the Mechanics of a Particle . . . . .	50
§ 29. Applications of the Dynamics of a Particle . . . . .	53
(a) The Mass of High-Velocity Electrons . . . . .	53
(b) The Relation between Force and Acceleration . . . . .	54
(c) Applications in Electromagnetic Theory . . . . .	55
(d) Tests of the Interrelation of Mass, Energy, and Momentum . . . . .	57

*Part II.* THE DYNAMICS OF A CONTINUOUS MECHANICAL MEDIUM

§ 30. The Principles Postulated . . . . .	59
§ 31. The Conservation of Momentum and the Components of Stress $t_{ij}$	60
§ 32. The Equations of Motion in Terms of the Stresses $t_{ij}$ . . . . .	60
§ 33. The Equation of Continuity . . . . .	62
§ 34. The Transformation Equations for the Stresses $t_{ij}$ . . . . .	62
§ 35. The Transformation Equations for the Densities of Mass and Momentum . . . . .	65
§ 36. Restatement of Results in Terms of the (Absolute) Stresses $p_{ij}$ .	69
§ 37. Four-Dimensional Expression of the Mechanics of a Continuous Medium . . . . .	70
§ 38. Applications of the Mechanics of a Continuous Medium . . . . .	73
(a) The Mass and Momentum of a Finite System . . . . .	74
(b) The Angular Momentum of a Finite System . . . . .	77
(c) The Right-Angled Lever as an Example . . . . .	79
(d) The Complete Static System . . . . .	80

## IV. SPECIAL RELATIVITY AND ELECTRODYNAMICS

*Part I.* ELECTRON THEORY

§ 39. The Maxwell-Lorentz Field Equations . . . . .	84
§ 40. The Transformation Equations for $\mathbf{E}$ , $\mathbf{H}$ , and $\rho$ . . . . .	86
§ 41. The Force on a Moving Charge . . . . .	88
§ 42. The Energy and Momentum of the Electromagnetic Field . . . . .	89
§ 43. The Electromagnetic Stresses . . . . .	91
§ 44. Transformation Equations for Electromagnetic Densities and Stresses . . . . .	92
§ 45. Combined Result of Mechanical and Electromagnetic Actions . . . . .	93
§ 46. Four-Dimensional Expression of the Electron Theory . . . . .	95
(a) The Field Equations . . . . .	95
(b) Four-Dimensional Expression of Force on Moving Charge . . . . .	98
(c) Four-Dimensional Expression of Electromagnetic Energy-Momentum Tensor . . . . .	99
§ 47. Applications of the Electron Theory . . . . .	99

*Part II.* MACROSCOPIC THEORY

§ 48. The Field Equations for Stationary Matter . . . . .	101
§ 49. The Constitutive Equations for Stationary Matter . . . . .	102
§ 50. The Field Equations in Four-Dimensional Language . . . . .	102
§ 51. The Constitutive Equations in Four-Dimensional Language . . . . .	104
§ 52. The Field Equations for Moving Matter in Ordinary Vector Language . . . . .	105
§ 53. The Constitutive Equations for Moving Matter in Ordinary Vector Language . . . . .	108
§ 54. Applications of the Macroscopic Theory . . . . .	109
(a) The Conservation of Electric Charge . . . . .	109
(b) Boundary Conditions . . . . .	110
(c) The Joule Heating Effect . . . . .	112
(d) Electromagnetic Energy and Momentum . . . . .	113
(e) The Energy-Momentum Tensor . . . . .	115
(f) Applications to Experimental Observations . . . . .	116

## V. SPECIAL RELATIVITY AND THERMODYNAMICS

*Part I.* THE THERMODYNAMICS OF STATIONARY SYSTEMS

§ 55. Introduction . . . . .	118
§ 56. The First Law of Thermodynamics and the Zero Point of Energy Content . . . . .	120
§ 57. The Second Law of Thermodynamics and the Starting-Point for Entropy Content . . . . .	121
§ 58. Heat Content, Free Energy, and Thermodynamic Potential .	123
§ 59. General Conditions for Thermodynamic Change and Equilibrium	125
§ 60. Conditions for Change and Equilibrium in Homogeneous Systems	127
§ 61. Uniformity of Temperature at Thermal Equilibrium . . . . .	130
§ 62. Irreversibility and Rate of Change . . . . .	132
§ 63. Final State of an Isolated System . . . . .	134
§ 64. Energy and Entropy of a Perfect Monatomic Gas . . . . .	136
§ 65. Energy and Entropy of Black-Body Radiation . . . . .	139
§ 66. The Equilibrium between Hydrogen and Helium . . . . .	140
§ 67. The Equilibrium between Matter and Radiation . . . . .	146

*Part II.* THE THERMODYNAMICS OF MOVING SYSTEMS

§ 68. The Two Laws of Thermodynamics for a Moving System .	152
§ 69. The Lorentz Transformation for Thermodynamic Quantities .	153
(a) Volume and Pressure . . . . .	153
(b) Energy . . . . .	154
(c) Work . . . . .	156
(d) Heat . . . . .	156
(e) Entropy . . . . .	157
(f) Temperature . . . . .	158
§ 70. Thermodynamic Applications . . . . .	159
(a) Carnot Cycle Involving Change in Velocity . . . . .	159
(b) The Dynamics of Thermal Radiation . . . . .	161
§ 71. Use of Four-Dimensional Language in Thermodynamics .	162

## VI. THE GENERAL THEORY OF RELATIVITY

*Part I.* THE FUNDAMENTAL PRINCIPLES OF GENERAL RELATIVITY

§ 72. Introduction . . . . .	165
§ 73. The Principle of Covariance . . . . .	166
(a) Justification for the Principle of Covariance . . . . .	166
(b) Consequences of the Principle of Covariance . . . . .	167
(c) Method of Obtaining Covariant Expressions . . . . .	168
(d) Covariant Expression for Interval . . . . .	169
(e) Covariant Expression for the Trajectories of Free Particles and Light Rays . . . . .	171
§ 74. The Principle of Equivalence . . . . .	174
(a) Formulation of the Principle of Equivalence. Metric and Gravitation . . . . .	174
(b) Principle of Equivalence and Relativity of Motion . . . . .	176
(c) Justification for the Principle of Equivalence . . . . .	179

(d) Use of the Principle of Equivalence in Generalizing the Principles of Special Relativity. Natural and Proper Coordinates . . . . .	180
(e) Interval and Trajectory in the Presence of Gravitational Fields . . . . .	181
§ 75. The Dependence of Gravitational Field and Metric on the Distribution of Matter and Energy. Principle of Mach . . . . .	184
§ 76. The Field Corresponding to the Special Theory of Relativity. The Riemann-Christoffel Tensor . . . . .	185
§ 77. The Gravitational Field in Empty Space. The Contracted Riemann-Christoffel Tensor . . . . .	187
§ 78. The Gravitational Field in the Presence of Matter and Energy . . . . .	188

## *Part II.* ELEMENTARY APPLICATIONS OF GENERAL RELATIVITY

§ 79. Simple Consequences of the Principle of Equivalence . . . . .	192
(a) The Proportionality of Weight and Mass . . . . .	192
(b) Effect of Gravitational Potential on the Rate of a Clock . . . . .	192
(c) The Clock Paradox . . . . .	194
§ 80. Newton's Theory as a First Approximation . . . . .	198
(a) Motion of Free Particle in a Weak Gravitational Field . . . . .	198
(b) Poisson's Equation as an Approximation for Einstein's Field Equations . . . . .	199
§ 81. Units to be Used in Relativistic Calculations . . . . .	201
§ 82. The Schwarzschild Line Element . . . . .	202
§ 83. The Three Crucial Tests of Relativity . . . . .	205
(a) The Advance of Perihelion . . . . .	208
(b) The Gravitational Deflexion of Light . . . . .	209
(c) Gravitational Shift in Spectral Lines . . . . .	211

## VII. RELATIVISTIC MECHANICS

### *Part I.* SOME GENERAL MECHANICAL PRINCIPLES

§ 84. The Fundamental Equations of Relativistic Mechanics . . . . .	214
§ 85. The Nature of the Energy-Momentum Tensor. General Expression in the Case of a Perfect Fluid . . . . .	215
§ 86. The Mechanical Behaviour of a Perfect Fluid . . . . .	218
§ 87. Re-expression of the Equations of Mechanics in the Form of an Ordinary Divergence . . . . .	222
§ 88. The Energy-Momentum Principle for Finite Systems . . . . .	225
§ 89. The Densities of Energy and Momentum Expressed as Divergences . . . . .	229
§ 90. Limiting Values for Certain Quantities at a Large Distance from an Isolated System . . . . .	230
§ 91. The Mass, Energy and Momentum of an Isolated System . . . . .	232
§ 92. The Energy of a Quasi-Static Isolated System Expressed by an Integral Extending Only Over the Occupied Space . . . . .	234

### *Part II.* SOLUTIONS OF THE FIELD EQUATIONS

§ 93. Einstein's General Solution of the Field Equations in the Case of Weak Fields . . . . .	236
§ 94. Line Elements for Systems with Spherical Symmetry . . . . .	239

§ 95. Static Line Element with Spherical Symmetry . . .	241
§ 96. Schwarzschild's Exterior and Interior Solutions . . .	245
§ 97. The Energy of a Sphere of Perfect Fluid . . .	247
§ 98. Non-Static Line Elements with Spherical Symmetry . . .	250
§ 99. Birkhoff's Theorem . . . . .	252
§ 100. A More General Line Element . . . . .	253

## VIII. RELATIVISTIC ELECTRODYNAMICS

*Part I.* THE COVARIANT GENERALIZATION OF ELECTRICAL THEORY

§ 101. Introduction . . . . .	258
§ 102. The Generalized Lorentz Electron Theory. The Field Equations . . . . .	258
§ 103. The Motion of a Charged Particle . . . . .	259
§ 104. The Energy-Momentum Tensor . . . . .	261
§ 105. The Generalized Macroscopic Theory . . . . .	261

*Part II.* SOME APPLICATIONS OF RELATIVISTIC ELECTRODYNAMICS

§ 106. The Conservation of Electric Charge . . . . .	264
§ 107. The Gravitational Field of a Charged Particle . . . . .	265
§ 108. The Propagation of Electromagnetic Waves . . . . .	267
§ 109. The Energy-Momentum Tensor for Disordered Radiation . . . . .	269
§ 110. The Gravitational Mass of Disordered Radiation . . . . .	271
§ 111. The Energy-Momentum Tensor Corresponding to a Directed Flow of Radiation . . . . .	272
§ 112. The Gravitational Field Corresponding to a Directed Flow of Radiation . . . . .	273
§ 113. The Gravitational Action of a Pencil of Light . . . . .	274
(a) The Line Element in the Neighbourhood of a Limited Pencil of Light . . . . .	274
(b) Velocity of a Test Ray of Light in the Neighbourhood of the Pencil . . . . .	275
(c) Acceleration of a Test Particle in the Neighbourhood of the Pencil . . . . .	277
§ 114. The Gravitational Action of a Pulse of Light . . . . .	279
(a) The Line Element in the Neighbourhood of the Limited Track of a Pulse of Light . . . . .	279
(b) Velocity of a Test Ray of Light in the Neighbourhood of the Pulse . . . . .	281
(c) Acceleration of a Test Particle in the Neighbourhood of the Pulse . . . . .	282
§ 115. Discussion of the Gravitational Interaction of Light Rays and Particles . . . . .	285
§ 116. The Generalized Doppler Effect . . . . .	288

## IX. RELATIVISTIC THERMODYNAMICS

*Part I.* THE EXTENSION OF THERMODYNAMICS TO GENERAL RELATIVITY

§ 117. Introduction . . . . .	291
§ 118. The Relativistic Analogue of the First Law of Thermodynamics . . . . .	292
§ 119. The Relativistic Analogue of the Second Law of Thermodynamics . . . . .	293

§ 120. On the Interpretation of the Relativistic Second Law of Thermodynamics . . . . .	296
§ 121. On the Interpretation of Heat in Relativistic Thermodynamics . . . . .	297
§ 122. On the Use of Co-Moving Coordinates in Thermodynamic Considerations . . . . .	301

### *Part II.* APPLICATIONS OF RELATIVISTIC THERMODYNAMICS

§ 123. Application of the First Law to Changes in the Static State of a System . . . . .	304
§ 124. Application of the Second Law to Changes in the Static State of a System . . . . .	306
§ 125. The Conditions for Static Thermodynamic Equilibrium . . . . .	307
§ 126. Static Equilibrium in the Case of a Spherical Distribution of Fluid . . . . .	308
§ 127. Chemical Equilibrium in a Gravitating Sphere of Fluid . . . . .	311
§ 128. Thermal Equilibrium in a Gravitating Sphere of Fluid . . . . .	312
§ 129. Thermal Equilibrium in a General Static Field . . . . .	315
§ 130. On the Increased Possibility in Relativistic Thermodynamics for Reversible Processes at a Finite Rate . . . . .	319
§ 131. On the Possibility for Irreversible Processes without Reaching a Final State of Maximum Entropy . . . . .	326
§ 132. Conclusion . . . . .	330

## X. APPLICATIONS TO COSMOLOGY

### *Part I.* STATIC COSMOLOGICAL MODELS

§ 133. Introduction . . . . .	331
§ 134. The Three Possibilities for a Homogeneous Static Universe . . . . .	333
§ 135. The Einstein Line Element . . . . .	335
§ 136. The de Sitter Line Element . . . . .	335
§ 137. The Special Relativity Line Element . . . . .	336
§ 138. The Geometry of the Einstein Universe . . . . .	337
§ 139. Density and Pressure of Material in the Einstein Universe . . . . .	339
§ 140. Behaviour of Particles and Light Rays in the Einstein Universe . . . . .	341
§ 141. Comparison of Einstein Model with Actual Universe . . . . .	344
§ 142. The Geometry of the de Sitter Universe . . . . .	346
§ 143. Absence of Matter and Radiation from the de Sitter Universe . . . . .	348
§ 144. Behaviour of Test Particles and Light Rays in the de Sitter Universe . . . . .	349
(a) The Geodesic Equations . . . . .	349
(b) Orbits of Particles . . . . .	351
(c) Behaviour of Light Rays in the de Sitter Universe . . . . .	353
(d) Doppler Effect in the de Sitter Universe . . . . .	354
§ 145. Comparison of de Sitter Model with Actual Universe . . . . .	359

### *Part II.* THE APPLICATION OF RELATIVISTIC MECHANICS TO NON-STATIC HOMOGENEOUS COSMOLOGICAL MODELS

§ 146. Reasons for Changing to Non-Static Models . . . . .	361
§ 147. Assumption Employed in Deriving Non-Static Line Element . . . . .	362

§ 148. Derivation of Line Element from Assumption of Spatial Isotropy . . . . .	364
§ 149. General Properties of the Line Element . . . . .	370
(a) Different Forms of Expression for the Line Element . . . . .	370
(b) Geometry Corresponding to Line Element . . . . .	371
(c) Result of Transfer of Origin of Coordinates . . . . .	372
(d) Physical Interpretation of Line Element . . . . .	375
§ 150. Density and Pressure in Non-Static Universe . . . . .	376
§ 151. Change in Energy with Time . . . . .	379
§ 152. Change in Matter with Time . . . . .	381
§ 153. Behaviour of Particles in the Model . . . . .	383
§ 154. Behaviour of Light Rays in the Model . . . . .	387
§ 155. The Doppler Effect in the Model . . . . .	389
§ 156. Change in Doppler Effect with Distance . . . . .	392
§ 157. General Discussion of Dependence on Time for Closed Models . . . . .	394
(a) General Features of Time Dependence, $R$ real, $\rho_{00} \geq 0$ , $p_0 \geq 0$ . . . . .	395
(b) Curve for the Critical Function of $R$ . . . . .	396
(c) Monotonic Universes of Type $M_1$ , for $\Lambda > \Lambda_E$ . . . . .	399
(d) Asymptotic Universes of Types $A_1$ and $A_2$ , for $\Lambda = \Lambda_E$ . . . . .	400
(e) Monotonic Universes of Type $M_2$ and Oscillating Universes of Types $O_1$ and $O_2$ , for $0 < \Lambda < \Lambda_E$ . . . . .	401
(f) Oscillating Universes of Type $O_3$ , for $\Lambda \leq 0$ . . . . .	402
§ 158. General Discussion of Dependence on Time for Open Models . . . . .	403
§ 159. On the Instability of the Einstein Static Universe . . . . .	405
§ 160. Models in Which the Amount of Matter is Constant . . . . .	407
§ 161. Models Which Expand from an Original Static State . . . . .	409
§ 162. Ever Expanding Models Which do not Start from a Static State . . . . .	412
§ 163. Oscillating Models ( $\Lambda = 0$ ) . . . . .	412
§ 164. The Open Model of Einstein and de Sitter ( $\Lambda = 0, R_0 = \infty$ ) . . . . .	415
§ 165. Discussion of Factors which were Neglected in Studying Special Models . . . . .	416

*Part III.* THE APPLICATION OF RELATIVISTIC THERMODYNAMICS TO NON-STATIC HOMOGENEOUS COSMOLOGICAL MODELS

§ 166. Application of the Relativistic First Law . . . . .	420
§ 167. Application of the Relativistic Second Law . . . . .	421
§ 168. The Conditions for Thermodynamic Equilibrium in a Static Einstein Universe . . . . .	423
§ 169. The Conditions for Reversible and Irreversible Changes in Non-Static Models . . . . .	424
§ 170. Model Filled with Incoherent Matter Exerting No Pressure as an Example of Reversible Behaviour . . . . .	426
§ 171. Model Filled with Black-Body Radiation as an Example of Reversible Behaviour . . . . .	427
§ 172. Discussion of Failure to Obtain Periodic Motions without Singular States . . . . .	429
§ 173. Interpretation of Reversible Expansions by an Ordinary Observer . . . . .	432

§ 174. Analytical Treatment of a Succession of Expansions and Contractions for a Closed Model with $\Lambda = 0$ . . . . .	435
(a) The Upper Boundary of Expansion . . . . .	436
(b) Time Necessary to Reach Maximum . . . . .	436
(c) Time Necessary to Complete Contraction . . . . .	437
(d) Behaviour at Lower Limit of Contraction . . . . .	438
§ 175. Application of Thermodynamics to a Succession of Irreversible Expansions and Contractions . . . . .	439

*Part IV. CORRELATION OF PHENOMENA IN THE ACTUAL UNIVERSE WITH THE HELP OF NON-STATIC HOMOGENEOUS MODELS*

§ 176. Introduction . . . . .	445
§ 177. The Observational Data . . . . .	446
(a) The Absolute Magnitudes of the Nearer Nebulae . . . . .	446
(b) The Corrected Apparent Magnitudes for more Distant Nebulae . . . . .	448
(c) Nebular Distances Calculated from Apparent Magnitudes . . . . .	453
(d) Relation of Observed Red-Shift to Magnitude and Distance . . . . .	454
(e) Relation of Apparent Diameter to Magnitude and Distance . . . . .	457
(f) Actual Diameters and Masses of Nebulae . . . . .	458
(g) Distribution of Nebulae in Space . . . . .	459
(h) Density of Matter in Space . . . . .	461
§ 178. The Relation between Coordinate Position and Luminosity . . . . .	462
§ 179. The Relation between Coordinate Position and Astronomically Determined Distance . . . . .	465
§ 180. The Relation between Coordinate Position and Apparent Diameter . . . . .	467
§ 181. The Relation between Coordinate Position and Counts of Nebular Distribution . . . . .	468
§ 182. The Relation between Coordinate Position and Red-shift . . . . .	469
§ 183. The Relation of Density to Spatial Curvature and Cosmological Constant . . . . .	473
§ 184. The Relation between Red-shift and Rate of Disappearance of Matter . . . . .	475
§ 185. Summary of Correspondences between Model and Actual Universe . . . . .	478
§ 186. Some General Remarks Concerning Cosmological Models . . . . .	482
(a) Homogeneity . . . . .	482
(b) Spatial Curvature . . . . .	483
(c) Temporal Behaviour . . . . .	484
§ 187. Our Neighbourhood as a Sample of the Universe as a Whole . . . . .	486

*Appendix I. SYMBOLS FOR QUANTITIES*

Scalar Quantities . . . . .	489
Vector Quantities . . . . .	490
Tensors . . . . .	490
Tensor Densities . . . . .	491



<i>Appendix II.</i> SOME FORMULAE OF VECTOR ANALYSIS . . . . .	491
<i>Appendix III.</i> SOME FORMULAE OF TENSOR ANALYSIS	
(a) General Notation . . . . .	493
(b) The Fundamental Metrical Tensor and its Properties . . . . .	494
(c) Tensor Manipulations . . . . .	495
(d) Miscellaneous Formulae . . . . .	496
(e) Formulae Involving Tensor Densities . . . . .	496
(f) Four-Dimensional Volume. Proper Spatial Volume . . . . .	496
<i>Appendix IV.</i> USEFUL CONSTANTS . . . . .	497
<i>Subject Index</i> . . . . .	499
<i>Name Index</i> . . . . .	502