

CONTENTS

1 FROM KEPLER TO SCHRÖDINGER . . . AND BEYOND	1
1.1 Classical Mechanics	2
1.1.1 Newton's Laws and Mach's Principle	
1.1.2 Mass, Force, and Momentum	
1.2 Symplectic Mechanics	6
1.2.1 Hamilton's Equations	
1.2.2 Gauge Transformations	
1.2.3 Hamiltonian Fields and Flows	
1.2.4 The "Symplectization of Science"	
1.3 Action and Hamilton-Jacobi's Theory	11
1.3.1 Action	
1.3.2 Hamilton-Jacobi's Equation	
1.4 Quantum Mechanics	13
1.4.1 Matter Waves	
1.4.2 "If There Is a Wave, There Must Be a Wave Equation!"	
1.4.3 Schrödinger's Quantization Rule and Geometric Quantization	
1.5 The Statistical Interpretation of Ψ	19
1.5.1 Heisenberg's Inequalities	
1.6 Quantum Mechanics in Phase Space	22
1.6.1 Schrödinger's "firefly" Argument	
1.6.2 The Symplectic Camel	
1.7 Feynman's "Path Integral"	25
1.7.1 The "Sum Over All Paths"	
1.7.2 The Metaplectic Group	
1.8 Bohmian Mechanics	27
1.8.1 Quantum Motion: The Bell-DGZ Theory	
1.8.2 Bohm's Theory	

1.9 Interpretations	31
1.9.1 Epistemology or Ontology?	
1.9.2 The Copenhagen Interpretation	
1.9.3 The Bohmian Interpretation	
1.9.4 The Platonic Point of View	
2 NEWTONIAN MECHANICS	37
2.1 Maxwell's Principle and the Lagrange Form	37
2.1.1 The Hamilton Vector Field	
2.1.2 Force Fields	
2.1.3 Statement of Maxwell's Principle	
2.1.4 Magnetic Monopoles and the Dirac String	
2.1.5 The Lagrange Form	
2.1.6 N -Particle Systems	
2.2 Hamilton's Equations	49
2.2.1 The Poincaré-Cartan Form and Hamilton's Equations	
2.2.2 Hamiltonians for N -Particle Systems	
2.2.3 The Transformation Law for Hamilton Vector Fields	
2.2.4 The Suspended Hamiltonian Vector Field	
2.3 Galilean Covariance	58
2.3.1 Inertial Frames	
2.3.2 The Galilean Group $\text{Gal}(3)$	
2.3.3 Galilean Covariance of Hamilton's Equations	
2.4 Constants of the Motion and Integrable Systems	65
2.4.1 The Poisson Bracket	
2.4.2 Constants of the Motion and Liouville's Equation	
2.4.3 Constants of the Motion in Involution	
2.5 Liouville's Equation and Statistical Mechanics	70
2.5.1 Liouville's Condition	
2.5.2 Marginal Probabilities	
2.5.3 Distributional Densities: An Example	
3 THE SYMPLECTIC GROUP	77
3.1 Symplectic Matrices and $Sp(n)$	77
3.2 Symplectic Invariance of Hamiltonian Flows	80
3.2.1 Notations and Terminology	
3.2.2 Proof of the Symplectic Invariance of Hamiltonian Flows	
3.2.3 Another Proof of the Symplectic Invariance of Flows*	
3.3 The Properties of $Sp(n)$	83

3.3.1 The Subgroups $U(n)$ and $O(n)$ of $Sp(n)$	
3.3.2 The Lie Algebra $\mathfrak{sp}(n)$	
3.3.3 $Sp(n)$ as a Lie Group	
3.4 Quadratic Hamiltonians	88
3.4.1 The Linear Symmetric Triatomic Molecule	
3.4.2 Electron in a Uniform Magnetic Field	
3.5 The Inhomogeneous Symplectic Group	92
3.5.1 Galilean Transformations and $ISp(n)$	
3.6 An Illuminating Analogy	94
3.6.1 The Optical Hamiltonian	
3.6.2 Paraxial Optics	
3.7 Gromov's Non-Squeezing Theorem	99
3.7.1 Liouville's Theorem Revisited	
3.7.2 Gromov's Theorem	
3.7.3 The Uncertainty Principle in Classical Mechanics	
3.8 Symplectic Capacity and Periodic Orbits	108
3.8.1 The Capacity of an Ellipsoid	
3.8.2 Symplectic Area and Volume	
3.9 Capacity and Periodic Orbits	113
3.9.1 Periodic Hamiltonian Orbits	
3.9.2 Action of Periodic Orbits and Capacity	
3.10 Cell Quantization of Phase Space	118
3.10.1 Stationary States of Schrödinger's Equation	
3.10.2 Quantum Cells and the Minimum Capacity Principle	
3.10.3 Quantization of the N -Dimensional Harmonic Oscillator	
4 ACTION AND PHASE	127
4.1 Introduction	127
4.2 The Fundamental Property of the Poincaré-Cartan Form	128
4.2.1 Helmholtz's Theorem: The Case $n = 1$	
4.2.2 Helmholtz's Theorem: The General Case	
4.3 Free Symplectomorphisms and Generating Functions	132
4.3.1 Generating Functions	
4.3.2 Optical Analogy: The Eikonal	
4.4 Generating Functions and Action	137
4.4.1 The Generating Function Determined by H	
4.4.2 Action vs. Generating Function	
4.4.3 Gauge Transformations and Generating Functions	

4.4.4 Solving Hamilton's Equations with W	
4.4.5 The Cauchy Problem for Hamilton-Jacobi's Equation	
4.5 Short-Time Approximations to the Action	147
4.5.1 The Case of a Scalar Potential	
4.5.2 One Particle in a Gauge (\mathbf{A}, U)	
4.5.3 Many-Particle Systems in a Gauge (\mathbf{A}, U)	
4.6 Lagrangian Manifolds	156
4.6.1 Definitions and Basic Properties	
4.6.2 Lagrangian Manifolds in Mechanics	
4.7 The Phase of a Lagrangian Manifold	161
4.7.1 The Phase of an Exact Lagrangian Manifold	
4.7.2 The Universal Covering of a Manifold*	
4.7.3 The Phase: General Case	
4.7.4 Phase and Hamiltonian Motion	
4.8 Keller-Maslov Quantization	168
4.8.1 The Maslov Index for Loops	
4.8.2 Quantization of Lagrangian Manifolds	
4.8.3 Illustration: The Plane Rotator	
5 SEMI-CLASSICAL MECHANICS	179
5.1 Bohmian Motion and Half-Densities	179
5.1.1 Wave-Forms on Exact Lagrangian Manifolds	
5.1.2 Semi-Classical Mechanics	
5.1.3 Wave-Forms: Introductory Example	
5.2 The Leray Index and the Signature Function*	186
5.2.1 Cohomological Notations	
5.2.2 The Leray Index: $n = 1$	
5.2.3 The Leray Index: General Case	
5.2.4 Properties of the Leray Index	
5.2.5 More on the Signature Function	
5.2.6 The Reduced Leray Index	
5.3 De Rham Forms	201
5.3.1 Volumes and their Absolute Values	
5.3.2 Construction of De Rham Forms on Manifolds	
5.3.3 De Rham Forms on Lagrangian Manifolds	
5.4 Wave-Forms on a Lagrangian Manifold	212
5.4.1 Definition of Wave Forms	
5.4.2 The Classical Motion of Wave-Forms	

5.4.3. The Shadow of a Wave-Form	
6 THE METAPLECTIC GROUP AND THE MASLOV INDEX	221
6.1 Introduction	221
6.1.1 Could Schrödinger have Done it Rigorously?	
6.1.2 Schrödinger's Idea	
6.1.3 $Sp(n)$'s "Big Brother" $Mp(n)$	
6.2 Free Symplectic Matrices and their Generating Functions	225
6.2.1 Free Symplectic Matrices	
6.2.2 The Case of Affine Symplectomorphisms	
6.2.3 The Generators of $Sp(n)$	
6.3 The Metaplectic Group $Mp(n)$	231
6.3.1 Quadratic Fourier Transforms	
6.3.2 The Operators $M_{L,m}$ and V_P	
6.4 The Projections Π and Π^ε	237
6.4.1 Construction of the Projection Π	
6.4.2 The Covering Groups $Mp^\varepsilon(n)$	
6.5 The Maslov Index on $Mp(n)$	242
6.5.1 Maslov Index: A "Simple" Example	
6.5.2 Definition of the Maslov Index on $Mp(n)$	
6.6 The Cohomological Meaning of the Maslov Index*	247
6.6.1 Group Cocycles on $Sp(n)$	
6.6.2 The Fundamental Property of $m(\cdot)$	
6.7 The Inhomogeneous Metaplectic Group	253
6.7.1 The Heisenberg Group	
6.7.2 The Group $IMp(n)$	
6.8 The Metaplectic Group and Wave Optics	258
6.8.1 The Passage from Geometric to Wave Optics	
6.9 The Groups $Symp(n)$ and $Ham(n)^*$	260
6.9.1 A Topological Property of $Symp(n)$	
6.9.2 The Group $Ham(n)$ of Hamiltonian Symplectomorphisms	
6.9.3 The Groenewold-Van Hove Theorem	
7 SCHRÖDINGER'S EQUATION AND THE METATRON	267
7.1 Schrödinger's Equation for the Free Particle	267
7.1.1 The Free Particle's Phase	
7.1.2 The Free Particle Propagator	
7.1.3 An Explicit Expression for G	

7.1.4 The Metaplectic Representation of the Free Flow	
7.1.5 More Quadratic Hamiltonians	
7.2 Van Vleck's Determinant	277
7.2.1 Trajectory Densities	
7.3 The Continuity Equation for Van Vleck's Density	280
7.3.1 A Property of Differential Systems	
7.3.2 The Continuity Equation for Van Vleck's Density	
7.4 The Short-Time Propagator	284
7.4.1 Properties of the Short-Time Propagator	
7.5 The Case of Quadratic Hamiltonians	288
7.5.1 Exact Green Function	
7.5.2 Exact Solutions of Schrödinger's Equation	
7.6 Solving Schrödinger's Equation: General Case	290
7.6.1 The Short-Time Propagator and Causality	
7.6.2 Statement of the Main Theorem	
7.6.3 The Formula of Stationary Phase	
7.6.4 Two Lemmas — and the Proof	
7.7 Metatrons and the Implicate Order	300
7.7.1 Unfolding and Implicate Order	
7.7.2 Prediction and Retrodiction	
7.7.3 The Lie-Trotter Formula for Flows	
7.7.4 The "Unfolded" Metatron	
7.7.5 The Generalized Metaplectic Representation	
7.8 Phase Space and Schrödinger's Equation	313
7.8.1 Phase Space and Quantum Mechanics	
7.8.2 Mixed Representations in Quantum Mechanics	
7.8.3 Complementarity and the Implicate Order	
A Symplectic Linear Algebra	323
B The Lie-Trotter Formula for Flows	327
C The Heisenberg Groups	331
D The Bundle of s-Densities	335
E The Lagrangian Grassmannian	339
BIBLIOGRAPHY	343
INDEX	353