

# TABLE OF CONTENTS

|   |      |
|---|------|
| PREFACE TO THE ENGLISH EDITION          | XV   |
| PREFACE                                 | XVII |
| M. PASCAL'S DIVISION MACHINE            | 1    |
| 0 FUNDAMENTAL STRUCTURES                | 7    |
| 1 Relations . . . . .                   | 9    |
| 2 Monoids . . . . .                     | 14   |
| 3 Words and languages . . . . .         | 18   |
| 4 Free monoids . . . . .                | 24   |
| 5 Semirings . . . . .                   | 27   |
| 6 Matrices . . . . .                    | 30   |
| 7 Lexicon of graph theory . . . . .     | 33   |
| 8 Complexity and decidability . . . . . | 34   |
| Solutions to the exercises . . . . .    | 39   |
| Notes & references . . . . .            | 46   |

## The three stages of rationality

|  |    |
|--|----|
| I THE SIMPLEST POSSIBLE MACHINE. . . . .   | 49 |
| 1 What is an 'automaton'? . . . . .  | 51 |
| 1.1 First definitions -- first examples . . . . .                                | 51 |
| <i>States, transitions, etc. -- Computations, recognised languages etc. --</i>   |    |
| <i>Transposition and left right duality</i>                                      |    |
| 1.2 Basic constructions, basic properties . . . . .                              | 60 |
| <i>Union -- Cartesian product -- Quotient (of a language)</i>                    |    |
| 1.3 The graph perspective . . . . .  | 66 |
| <i>Trim automata -- The empty and the infinite -- Criteria for recognisabil-</i> |    |
| <i>ity</i>   |    |
| 1.4 Some supplementary definitions . . . . .                                     | 74 |
| <i>Unambiguous automata -- Complete automata -- Deterministic au-</i>            |    |
| <i>tomata -- Automata with spontaneous transitions</i>                           |    |
| 2 Rational languages . . . . .   | 82 |
| 2.1 Rational operations . . . . .  | 82 |
| <i>Product of languages -- Star of a language -- Rational operations</i>         |    |
| 2.2 Rational languages . . . . .   | 86 |

|     |   |     |
|-----|---|-----|
| 2.3 | Rational is recognisable . . . . .  | 87  |
|     | <i>Normalised automata – Closure under product and star – Standard automata</i>   |     |
| 2.4 | Recognisable is rational . . . . .  | 94  |
|     | <i>The McNaughton–Yamada algorithm, or algorithm MNY– The state elimination method – Solving equations</i>  |     |
| 3   | The functional perspective . . . . .  | 101 |
| 3.1 | From transitions to the transition function . . . . .   | 102 |
| 3.2 | Deterministic automata . . . . .  | 104 |
|     | <i>Reformulation of the definition – Determinisation – The case of one-letter alphabets – Complement of recognisable languages</i>                      |     |
| 3.3 | Minimisation . . . . .  | 111 |
|     | <i>The automaton of quotients of a language. . . . . is minimal – Computation of the minimal automaton – Another minimisation method</i>                |     |
| 3.4 | Return to the Star Lemma . . . . .  | 118 |
|     | <i>Block iteration and block simplification – Ramsey’s Theorem – Proof of Theorem 3.3</i>   |     |
| 4   | Rational expressions . . . . .  | 123 |
| 4.1 | Rational expressions and languages . . . . .  | 124 |
|     | <i>Rational expressions over an alphabet – Rational expressions over a set of variables</i>   |     |
| 4.2 | Rational identities . . . . .   | 128 |
|     | <i>Classical identities – A formal computation</i>  |     |
| 4.3 | Expressions for the behaviour of a finite automaton . . . . .   | 133 |
|     | <i>The state elimination and equation solution methods – The BMC and MNY algorithms, identical orders – The BMC and MNY algorithms, distinct orders</i> |     |
| 4.4 | Derivation of expressions . . . . .   | 138 |
|     | <i>Derivatives of an expression – A theorem of J. Brzozowski – Derivative automaton</i>   |     |
| 5   | From expressions to automata . . . . .  | 145 |
| 5.1 | The standard automaton of an expression . . . . .   | 145 |
|     | <i>Direct construction – Thompson’s construction</i>  |     |
| 5.2 | The derived term automaton . . . . .  | 149 |
|     | <i>Derived terms – A theorem of V. Antimirov</i>  |     |
| 5.3 | String matching . . . . .   | 152 |
|     | <i>Automaton for finding a word – Searching by sliding window – Implementation with a default successor</i>   |     |
| 6   | Star height . . . . .   | 157 |
| 6.1 | Two heights and a degree . . . . .  | 158 |
|     | <i>Star height of an expression – Star height of a language – Loop complexity of an automaton</i>   |     |
| 6.2 | Eggan’s Theorem . . . . .   | 162 |
|     | <i>From expressions to automata – From automata to expressions: calculating the index– Not so fast</i>  |     |
| 6.3 | An infinite hierarchy . . . . .   | 167 |
| 6.4 | Generalised star height . . . . .   | 170 |
| 7   | A field of automata . . . . .   | 171 |

|                                   |  |     |
|-----------------------------------|--|-----|
| 7.1                               | The Rabin Scott model . . . . .  | 171 |
| 7.2                               | Two-way automaton . . . . .  | 172 |
| 7.3                               | Moore and Mealy machines . . . . .   | 174 |
| 8                                 | A crop of properties . . . . .   | 175 |
|                                   | Solutions to the exercises . . . . .   | 179 |
|                                   | Notes & references . . . . .   | 214 |
| II THE POWER OF ALGEBRA . . . . . |  | 217 |
| 1                                 | Automata and rational sets . . . . .   | 219 |
| 1.1                               | Automata over a monoid . . . . .   | 219 |
| 1.2                               | Rational sets . . . . .  | 220 |
|                                   | <i>The semiring <math>\mathfrak{P}(M)</math> – Rational operations and subsets – Rational expressions – Image under morphism – Intersection and inverse morphism</i> |     |
| 1.3                               | Behaviour of finite automata . . . . .   | 225 |
| 1.4                               | Unambiguous rational sets . . . . .  | 228 |
|                                   | <i>Definitions – The family URat</i>   |     |
| 2                                 | Actions and recognisable sets . . . . .  | 231 |
| 2.1                               | Actions on a set . . . . .   | 232 |
|                                   | <i>Definition – Matrix representation of actions – Subsets recognised by an action</i>   |     |
| 2.2                               | Recognisable here, recognisable there . . . . .  | 238 |
|                                   | <i>Consistency – Kleene's Theorem – Automaton of an action</i>   |     |
| 2.3                               | Elementary operations on recognisable subsets . . . . .  | 243 |
|                                   | <i>Boolean operations – Inverse morphism – Quotient – Morphism and product</i>   |     |
| 2.4                               | Minimisation . . . . .   | 246 |
|                                   | <i>Action morphisms – Minimal action – Syntactic congruence and monoid</i>   |     |
| 2.5                               | Algebra at work . . . . .  | 251 |
|                                   | <i>Two examples – Recognisable subsets included in a product</i>   |     |
| 3                                 | Morphisms and coverings . . . . .  | 255 |
| 3.1                               | Automata morphisms . . . . .   | 255 |
|                                   | <i>Definitions and examples – Conformal morphisms – Local properties</i>   |     |
| 3.2                               | Quotients of automata . . . . .  | 261 |
|                                   | <i>Out-surjective morphisms – Totally surjective morphisms – Moore's algorithm</i>   |     |
| 3.3                               | Automata coverings . . . . .   | 264 |
|                                   | <i>From local to global – Product of an automaton with an action – The Coloured Transition Lemma</i>   |     |
| 3.4                               | The Schützenberger covering . . . . .  | 270 |
| 4                                 | Universal automaton . . . . .  | 273 |

|     |   |     |
|-----|---|-----|
| 4.1 | Factorisations . . . . .  | 275 |
|     | <i>2-factorisations – Sub-factorisations and factorisations – Morphisms and factorisations</i>  |     |
| 4.2 | Universal automata of a subset . . . . .  | 279 |
|     | <i>Definitions and examples – Properties – Universal automaton relative to a generating set – Universality of universal automata</i>  |     |
| 4.3 | Construction of the universal automaton . . . . .   | 286 |
|     | <i>Expansion of a deterministic automaton – Extraction of the universal automaton</i>   |     |
| 4.4 | Language approximations . . . . .   | 291 |
| 5   | The importance of being well ordered . . . . .  | 293 |
| 5.1 | Well quasi-orderings . . . . .  | 293 |
| 5.2 | Derivations . . . . .   | 297 |
|     | <i>Preparations – Proof of Theorem 5.4</i>  |     |
| 6   | Rationals in the free group . . . . .   | 301 |
| 6.1 | Recognisable and rational in groups . . . . .   | 301 |
|     | <i>Recognisable subsets – Rational subgroups – Fatou property</i>   |     |
| 6.2 | Description of the free group . . . . .   | 305 |
|     | <i>Dyck congruence and Dyck words – Shamir congruence and parenthetic words – Simplifications – Reduction associated with a simplification – Unambiguous factorisation induced by a reduction</i> |     |
| 6.3 | Rationals of the free group . . . . .   | 314 |
|     | <i>Rationals of simplification monoids – Return to the free group</i>   |     |
| 6.4 | Büchi systems . . . . .   | 319 |
| 7   | Rationals in commutative monoids . . . . .  | 323 |
| 7.1 | The natural order on $A^{\oplus}$ . . . . .   | 323 |
|     | <i>The free commutative monoid – Dickson's Lemma</i>  |     |
| 7.2 | The lexicographic order on $\mathbb{N}^k$ . . . . .   | 326 |
|     | <i>Congruences of <math>\mathbb{N}^k</math> – Lexicographic decomposition</i>   |     |
| 7.3 | Subtractive submonoids and affine sets . . . . .  | 330 |
| 7.4 | Semi-linear and semi-simple sets . . . . .  | 333 |
| 7.5 | Rationals of $\mathbb{N}^k$ . . . . .   | 335 |
|     | <i>The Freedom Lemma – Positive solutions of Diophantine linear systems – Semi-simple subsets of <math>\mathbb{Z}^k</math> – Proof of Theorems 7.3 and 7.4</i>                                    |     |
| 7.6 | Rationals of commutative monoids . . . . .  | 341 |
| 8   | Star height of group languages . . . . .  | 342 |
|     | Solutions to the exercises . . . . .  | 348 |
|     | Notes & references . . . . .  | 372 |
| III | THE PERTINENCE OF ENUMERATION . . . . .   | 375 |
| 1   | Formal power series on a graded monoid . . . . .  | 379 |
| 1.1 | Formal power series over $M$ with coefficients in $\mathbb{K}$ . . . . .  | 379 |
|     | <i>Operations on <math>\mathbb{K}\langle\langle M \rangle\rangle</math> – Support of a series – characteristic series – Hadamard product – Scalar product</i>                                     |     |
| 1.2 | Graded monoids . . . . .  | 383 |

|     |  |     |
|-----|--|-----|
| 1.3 | Topology on $\mathbb{K}\langle\langle M \rangle\rangle$ . . . . .  | 385 |
|     | <i>Distance · Distance on <math>\mathbb{K}\langle\langle M \rangle\rangle</math> · Summable families · Continuous morphisms</i>  |     |
| 2   | $\mathbb{K}$ -automata and $\mathbb{K}$ -rational power series . . . . .   | 392 |
| 2.1 | Star of a power series . . . . .   | 393 |
|     | <i>Star in a topological semiring – Star of a proper series · Star of an arbitrary series</i>  |     |
| 2.2 | $\mathbb{K}$ -rational series . . . . .  | 398 |
|     | <i><math>\mathbb{K}</math>-rational operations · Rational <math>\mathbb{K}</math>-expressions · Star of a matrix</i>   |     |
| 2.3 | Weighted automaton in a semiring . . . . .   | 402 |
|     | <i><math>\mathbb{K}</math>-automaton over <math>M</math> – Behaviour of a <math>\mathbb{K}</math>-automaton – Notes – Some other definitions and examples</i>                        |     |
| 2.4 | The Fundamental Theorem of finite automata . . . . .   | 409 |
|     | <i>Proper automata · proper families of series · Statement and proof – Notes and corollaries</i>   |     |
| 2.5 | $\mathbb{K}$ -coverings – $\mathbb{K}$ -quotients . . . . .  | 416 |
|     | <i>From coverings to <math>\mathbb{K}</math>-coverings · Matrix description · Co-<math>\mathbb{K}</math>-covering · Minimal <math>\mathbb{K}</math>-quotient</i>                     |     |
| 3   | $\mathbb{K}$ -recognisable series . . . . .  | 424 |
| 3.1 | $\mathbb{K}$ -representations . . . . .  | 424 |
| 3.2 | Products . . . . .   | 426 |
|     | <i>Tensor product of <math>\mathbb{K}</math>-representations – Hadamard product · Tensor product of series – Shuffle product</i>   |     |
| 3.3 | The Kleene–Schützenberger Theorem . . . . .  | 433 |
| 4   | Series on a free monoid . . . . .  | 438 |
| 4.1 | A characterisation of recognisable series . . . . .  | 438 |
|     | <i>Quotients of series – Stable modules – The Kleene–Schützenberger Theorem revisited</i>  |     |
| 4.2 | Derivation of rational $\mathbb{K}$ -expressions . . . . .   | 443 |
|     | <i>Polynomials of <math>\mathbb{K}</math>-expressions – <math>\mathbb{K}</math>-derivatives of a <math>\mathbb{K}</math>-expression – Derived terms · The derived term automaton</i> |     |
| 4.3 | Series on a field . . . . .  | 451 |
|     | <i>Rank of a series – Reduced representation · Linear recurrence · Effective computations</i>  |     |
| 4.4 | Rational series and their supports . . . . .   | 463 |
|     | <i>Rationality of supports · The Rational Skimming Theorem, I – Undecidable questions</i>  |     |
| 5   | Series on an arbitrary monoid . . . . .  | 470 |
| 5.1 | Complete semirings, continuous semirings . . . . .   | 470 |
| 5.2 | Star of a series . . . . .   | 472 |
| 5.3 | $\mathbb{K}$ -rational series . . . . .  | 474 |
| 6   | Rational subsets in free products . . . . .  | 476 |
| 6.1 | Free product of monoids . . . . .  | 476 |
| 6.2 | Bipartite automaton over a free product . . . . .  | 478 |
| 6.3 | Bipartite deterministic automaton . . . . .  | 482 |
| 6.4 | Minimal deterministic bipartite automaton . . . . .  | 484 |
| 7   | A non-commutative linear algebra primer . . . . .  | 488 |
|     | Solutions to the exercises . . . . .   | 498 |

|                              |     |
|------------------------------|-----|
| Notes & references . . . . . | 519 |
|------------------------------|-----|

## Rationality in relations

|  |     |
|--|-----|
| IV THE RICHNESS OF TRANSDUCERS . . . . .   | 523 |
| 1 Rational relations: an introduction . . . . .  | 525 |
| 1.1 Rational relations . . . . .   | 525 |
| <i>Rational relations between free monoids -- Rational relations between arbitrary monoids</i>   |     |
| 1.2 Realisation by automata . . . . .  | 529 |
| 1.3 Realisation by morphisms . . . . .   | 531 |
| <i>Realisation -- Evaluation Theorem -- Composition Theorem -- Star Lemma</i>  |     |
| 1.4 Recognisable relations . . . . .   | 539 |
| 1.5 Realisation by representation . . . . .  | 540 |
| <i>Real-time transducers -- From real-time transducers to representations -- Theorem of evaluation and composition of representations</i>    |     |
| 1.6 The Rabin–Scott model . . . . .  | 545 |
| 2 $\mathbb{K}$ -relations . . . . .  | 546 |
| 2.1 Definitions . . . . .  | 548 |
| <i>The canonical isomorphism -- <math>\mathbb{K}</math>-relations -- Support of relations -- characteristic relations -- Continuity</i>      |     |
| 2.2 Composition . . . . .  | 553 |
| 2.3 Multiplicative $\mathbb{K}$ -relations . . . . .   | 555 |
| 3 Rational $\mathbb{K}$ -relations . . . . .   | 557 |
| 3.1 Reasonable semirings . . . . .   | 558 |
| <i>Image of series under continuous morphisms -- Image of series under projections -- <math>\mathbb{K}</math>-intersections</i>              |     |
| 3.2 Realisation of rational $\mathbb{K}$ -relations . . . . .  | 561 |
| <i>Realisation by <math>\mathbb{K}</math>-automaton -- Realisation by <math>\mathbb{K}</math>-representation -- Realisation by morphisms</i> |     |
| 3.3 Evaluation and Composition Theorems . . . . .  | 564 |
| <i>Using recognition by morphisms -- Using recognition by representation</i>   |     |
| 4 Equivalence of finite $\mathbb{K}$ -transducers . . . . .  | 568 |
| 4.1 Equivalence of $\mathbb{B}$ -transducers, general case . . . . .   | 569 |
| 4.2 Equivalence of $\mathbb{B}$ -transducers, case of small alphabets . . . . .  | 571 |
| 4.3 Equivalence of $\mathbb{N}$ -transducers . . . . .   | 574 |
| 5 Deterministic rational relations . . . . .   | 577 |
| 5.1 Transducers with an endmarker . . . . .  | 577 |
| 5.2 Deterministic transducers . . . . .  | 578 |
| <i>Definition -- Uniqueness of computations -- Almost an action</i>  |     |
| 5.3 Deterministic relations . . . . .  | 584 |
| <i>Definitions -- Complement -- Iteration Lemma</i>  |     |
| 5.4 Geography of $\text{Rat } A^* \times B^* \text{ I}$ . . . . .  | 588 |

|     |  |     |
|-----|--|-----|
| 5.5 | Matrix representations . . . . .   | 590 |
|     | <i>Representation of a deterministic transducer -- Representation of a deterministic relation</i>  |     |
| 5.6 | An example: the map equivalence of a morphism . . . . .  | 592 |
| 6   | Synchronisation of transducers . . . . .   | 595 |
| 6.1 | Rational relations of bounded length discrepancy . . . . .   | 596 |
|     | <i>Definitions, notation and conventions -- Characterisation of rational bld-relations -- Translation into automata theoretic terms, and corollaries</i>   |     |
| 6.2 | Transducers of bounded lag . . . . .   | 602 |
|     | <i>Lag in a computation or transducer -- Resynchronisation algorithm for transducers -- Composition of letter-to-letter transducers</i>  |     |
| 6.3 | Synchronous relations . . . . .  | 609 |
|     | <i>Another family of rational relations -- Determinisation and minimisation -- Geography of <math>\text{Rat } A^* \times B^*</math> II</i>   |     |
| 7   | Malcev Neumann series . . . . .  | 616 |
| 7.1 | Order on the free group . . . . .  | 617 |
|     | <i>On ordered groups -- Representation of the free group -- A detour via ordered rings -- Order on the free group</i>  |     |
| 7.2 | Series on an ordered group . . . . .   | 622 |
|     | <i>The semiring <math>\mathbb{K}_{\text{no}}\langle\langle G \rangle\rangle</math> -- Ordered semigroups -- The field <math>\mathbb{K}_{\text{no}}\langle\langle G \rangle\rangle</math> -- A last inclusion</i> |     |
|     | Solutions to the exercises . . . . .   | 627 |
|     | Notes & references . . . . .   | 641 |
| V   | THE SIMPLICITY OF FUNCTIONAL TRANSDUCERS . . . . .   | 643 |
| 1   | Functionary . . . . .  | 645 |
| 1.1 | Deciding functionality . . . . .   | 645 |
|     | <i>An effective characterisation of functionality -- Equivalence of rational functions</i>   |     |
| 1.2 | Sequential functions . . . . .   | 651 |
|     | <i>Some unconventional terminology -- Dual definitions -- Composition</i>  |     |
| 1.3 | Pure sequential functions . . . . .  | 658 |
| 1.4 | Local functions . . . . .  | 661 |
| 2   | Uniformisation of rational relations . . . . .   | 664 |
| 2.1 | Proof of Theorem 2.1 (transducer version) . . . . .  | 666 |
| 2.2 | Proof of Theorem 2.1 (representation version) . . . . .  | 667 |
|     | <i>Represent of <math>S</math>-immersions of an automaton -- Semi-monomial matrices -- Representation of <math>S</math>-uniformisations</i>  |     |
| 2.3 | Decomposition of rational functions . . . . .  | 673 |
|     | <i>The Weak Decomposition Theorem -- The Strong Decomposition Theorem</i>  |     |
| 2.4 | The Rational Skimming Theorem II . . . . .   | 677 |
| 3   | Cross-section of rational functions . . . . .  | 679 |

|     |  |     |
|-----|--|-----|
| 3.1 | The rational cross-section property . . . . .  | 680 |
|     | <i>The Rational Cross-Section Theorem · The rational cross-section property for a monoid · Return to simplification monoids</i>  |     |
| 3.2 | Choosing the uniformisation (or the cross-section) . . . . .   | 685 |
|     | <i>Uniformisation of synchronous relations · Uniformisation of deterministic relations · Th. 3.3 back on the loom</i>  |     |
| 4   | Sequential functions . . . . .   | 692 |
| 4.1 | Two characterisations . . . . .  | 692 |
|     | <i>Translations of a function · A functional characterisation · A quasi-topological point of view</i>  |     |
| 4.2 | Deciding sequentiality . . . . .   | 699 |
| 4.3 | Minimisation . . . . .   | 704 |
|     | <i>Conjugation · Blockage of a sequential transducer · Reduction · Effective computation</i>   |     |
| 4.4 | The (Great) Sequentiality Theorem . . . . .  | 711 |
|     | <i>Differential of a function – Proof of Theorem 4.5 iii) <math>\Rightarrow</math> i) · Proof of Theorem 4.5 ii) <math>\Rightarrow</math> iii) · Return to the Sequentiality Theorem</i> |     |
| 4.5 | Pure sequential functions and local functions . . . . .  | 717 |
|     | Solutions to the exercises . . . . .   | 719 |
|     | Notes & references . . . . .   | 737 |
|     | BIBLIOGRAPHY . . . . .   | 739 |
|     | INDEX . . . . .  | 749 |