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ON THE CONJUGACY OF REAL CARTAN SUBALGEBRAS. I

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Among the questions which have been raised concerning the structure of a connected semisimple Lie group are those relating to conjugacy of its Cartan subgroups.

In case the group is either compact or complex, it is a well-known fact (and indeed a fundamental one) that all Cartan subgroups are conjugate. It is also known that this is not true in general. The interest in the general case is heightened as a result of statements of Harish-Chandra[1] relating the "classes of conjugate Cartan subgroups and the various 'series' of unitary representations which occur in the Plancherel formula."

It is clear that one may reduce the problem to a consideration of the conjugate classes of Cartan subalgebras (C.S.'s) of a real, simple Lie algebra (conjugate, under the action of the adjoint group).[2] Moreover, in view of the above, one may restrict the Lie algebra to be a real noncompact form of A_l, B_l, C_l, D_l, G_2, F_4, E_6, E_7, and E_8. It is the purpose of this note to list a series of general theorems enabling us to "determine" the conjugate classes in every one of the real forms mentioned above. By "determine" is meant, among other things, giving (1) the number of conjugate classes, (2) the number of classes for which the "vector part" (or "toroidal part") of a C.S. in that class has a given dimension, (3) a *characterization* of each conjugacy class with respect to the full group of automorphisms. (This may be done in all but a few cases by giving the centralizers of the "vector and toroidal parts" of a C.S. in that class. A more convenient type of characterization may be given in the case of all but one classical algebra.)

In a succeeding note, tables will be given listing enough of this information so that every C.S. (up to conjugacy by the full group of automorphisms) in all the real forms mentioned above may be identified. A list of all W-subspaces of the simple complex algebras up to C-conjugacy (see below) will be included also. Proofs and further elaboration will appear elsewhere.

Let $\mathfrak{g}_0$ be a real, semisimple Lie algebra. Let G_0 be the adjoint group. Let $\mathfrak{k}_0$ be the subalgebra of $\mathfrak{g}_0$ corresponding to a maximal compact subgroup $K_0 \subseteq G_0$.

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Let $\mathfrak{p}_0$ be the orthogonal complement to $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Cartan-Killing bilinear form B. Then $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$.

Let $\mathfrak{m}_0$ be a maximal commutative subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{p}_0$. The group K_0 leaves $\mathfrak{p}_0$ invariant. Let W_s be the group of transformations of $\mathfrak{m}_0$ induced by the subgroup of K_0 which leaves $\mathfrak{m}_0$ invariant. W_s is a finite group.

Let $\mathfrak{h}_0$ be a Cartan subalgebra. Let $\mathfrak{h}_0^-$, the "vector part" ($\mathfrak{h}_0^+$, the "toroidal part"), be the subspace $\{X \in \mathfrak{h}_0 |$ the eigenvalues of adX are real (pure imaginary)$\}$. Then $\mathfrak{h}_0 = \mathfrak{h}_0^+ + \mathfrak{h}_0^-$.

We will call a Cartan subalgebra $\mathfrak{h}_0$ standard if $\mathfrak{h}_0^+ \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0^- \subseteq \mathfrak{m}_0 \subset \mathfrak{p}_0$.

Theorems, essentially equivalent to our first four theorems, have been proved independently (unpublished) by A. Borel.

THEOREM 1. *Every Cartan subalgebra is conjugate to a standard Cartan subalgebra. Moreover, two standard Cartan subalgebras are themselves conjugate if and only if their "vector parts" are conjugate with respect to W_s.*[2]

Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Let $\mathfrak{h}$ be the complexification of a Cartan subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{m}_0$. Let $\mathfrak{h}^\theta$ be the real subspace of $\mathfrak{h}$ spanned by the roots. Then $\mathfrak{h} = \mathfrak{h}^\theta + i\mathfrak{h}^\theta$ and $\mathfrak{g}_0 \cap \mathfrak{h}^\theta = \mathfrak{m}_0$. We may regard the Weyl group W or the Cartan group C associated with $\mathfrak{g}$ as operating in $\mathfrak{h}^\theta$.

We will call a subspace of $\mathfrak{h}^\theta$ a W-subspace if it is the eigenspace belonging to the eigenvalue -1 of an element of order 2 in W.

LEMMA 1. *$\mathfrak{l} \subseteq \mathfrak{h}^\theta$ is a W-subspace if and only if it has a basis of orthogonal (with respect to B) roots.*

Where $\mathfrak{l}^\perp$ means the orthogonal complement of $\mathfrak{l}$ with respect to B in $\mathfrak{h}^\theta$, we have

THEOREM 2. *$\mathfrak{n} \subseteq \mathfrak{m}_0$ is the vector part of a standard C.S. of $\mathfrak{g}_0$ if and only if $\mathfrak{n}$ is of the form $\mathfrak{m}_0 \cap \mathfrak{l}^\perp$, where $\mathfrak{l} \subseteq \mathfrak{m}_0$ and $\mathfrak{l}$ is a W-subspace.*

THEOREM 3. *The group W_s is obtained by the transformations induced on $\mathfrak{m}_0$ by the subgroup of W leaving $\mathfrak{m}_0$ fixed. Moreover, two W-subspaces of $\mathfrak{m}_0$ are conjugate with respect to W if and only if they are conjugate with respect to W_s.*

We are thus reduced to the study of how W conjugates the W-subspaces. Assume that $\mathfrak{g}$ is simple. We begin with Theorem 4, which tells how W acts on one-dimensional W-subspaces.

THEOREM 4. *The orbits of W acting on the set of roots are the subsets of all roots having the same length.*

For a subspace $\mathfrak{l} \subseteq \mathfrak{h}^\theta$, let $g[\mathfrak{l}]$ denote the complex subalgebra of $\mathfrak{g}$ generated by $\mathfrak{l}$ and those root vectors whose roots are in $\mathfrak{l}$. Let $\mathfrak{f}_\varphi$ denote the hyperplane in $\mathfrak{h}^\theta$ orthogonal to the root φ.

For any root α we will say that a root β is related to α if (*a*) $B(\alpha, \beta) = 0$; (*b*) $B(\alpha, \alpha) = B(\beta, \beta)$; (*c*) for any root φ, $B(\alpha, \varphi) = 0$, $\varphi \neq \pm\beta$ implies $B(\beta, \varphi) = 0$.

If we now say that a root is related to itself and its negative, we have

THEOREM 5. *The relation above is an equivalence relation partitioning the roots into equivalence classes.*

Clearly C preserves these classes. If we order the roots in any one of the usual ways and define n_φ to be the number of positive roots in the equivalence class of φ, we have

THEOREM 6. *$\mathfrak{g}[\mathfrak{f}_\varphi] = (n_\varphi - 1)A_1 \oplus \mathfrak{g}_\varphi$, where $\mathfrak{g}_\varphi$ is simple or simple $\oplus D_1$ (the latter case occurs only in A_l).*[3] *Also, the $n_\varphi - 1$ copies of A_1 are in fact $\mathfrak{g}[(\varphi_i)]$, where φ_i runs through positive roots other than φ which are related to φ.*

The key factor in our method of determining the class of W-conjugate W-subspaces[4] is the simplicity of $[\mathfrak{g}_\varphi, \mathfrak{g}_\varphi]$ in the above theorem; it enables us to repeat the use of Theorem 4, that is, in view of Lemma 1 and

LEMMA 2. *For any* $X \in \mathfrak{h}^\theta$, *let* Δ_X *be the set of roots* φ *such that* $B(\varphi, X) = 0$; *then, if* $\sigma \in W$ *leaves* X *invariant,* σ *is generated by the reflections through* $\mathfrak{f}_\varphi$ *for* $\varphi \in \Delta_X$.

We need

LEMMA 3. *Let* W_α *be the (commutative) subgroup of* W *generated by reflections through* $\mathfrak{f}_\beta$ *for all* β *related to* α. *If* φ *is not related to* α *or orthogonal to* α, *then the set* $\pm W_\alpha \varphi$ *is a union of equivalence classes.*

LEMMA 4. *If* $\mathfrak{l}$ *is a* W-*subspace and* $\alpha \in \mathfrak{l}$, *then, if* β *is related to* α, *either* $\beta \in \mathfrak{l}$ *or* $\beta \in \mathfrak{l}^\perp$.

For a W-subspace $\mathfrak{l}$, let $\mathfrak{n} \subseteq \mathfrak{l}$ be the subspace generated by all $\alpha \in \mathfrak{l}$ such that $\mathfrak{l}$ contains all β related to α. Both $\mathfrak{n}$ and $\mathfrak{n}^\perp \cap \mathfrak{l}$ are W-subspaces. We will call $\mathfrak{l}$ complete if $\mathfrak{n} = \mathfrak{l}$, incomplete if $\mathfrak{n} \neq \mathfrak{l}$, and totally incomplete if $\mathfrak{n} = 0$.[5]

Because $B_2 = C_2$, our definitions make B_2 a degenerate case (its W-subspaces up to W-conjugacy are obviously 0, (φ), (ψ), and $\mathfrak{h}^\theta$, where φ and ψ are roots and $B(\psi, \psi) = 2B(\varphi, \varphi)$). The following theorems are our main results:

THEOREM 7. *Assume that* $\mathfrak{g}$ *is classical but not* B_2; *then two* W-*subspaces* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are conjugate with respect to the Cartan group if and only if*

$$dim\ \mathfrak{l}_1 = dim\ \mathfrak{l}_2, \qquad dim\ \mathfrak{n}_1 = dim\ \mathfrak{n}_2.$$

They are conjugate with respect to W *if and only if these conditions hold, with the following exceptions:* $\mathfrak{g} = D_4$, $dim\ \mathfrak{l} = 2$, $dim\ \mathfrak{n} = 0$ *(three conjugate classes), and* $\mathfrak{g} = D_{2k}$, $k \geq 3$, $dim\ \mathfrak{l} = k$, $dim\ \mathfrak{n} = 0$ *(two conjugate classes).*

For the exceptional Lie algebras two W-subspaces are W-conjugate if and only if they are C-conjugate. For any subspace $\mathfrak{l} \subseteq \mathfrak{h}^\theta$, let $\Delta[\mathfrak{l}]$ be the set of roots in $\mathfrak{l}$. We now have

THEOREM 8. *If* $\mathfrak{g}$ *is any simple complex Lie algebra and* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are two* W-*subspaces, then* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are* C-*conjugate if and only if there exist (one to one, onto) maps* i *and* j $(i\colon \Delta[\mathfrak{l}_1] \rightarrow \Delta[\mathfrak{l}_2], j\colon \Delta[\mathfrak{l}_1^\perp] \rightarrow \Delta[\mathfrak{l}_2^\perp])$ *which preserve the structure (length, addition, negatives), with the sole exception* $\mathfrak{g} = D_{2k}$, $k \geq 3$, $dim\ \mathfrak{l}_1 = k$, $dim\ \mathfrak{n}_1 = 0$, $dim\ \mathfrak{l}_2 = k$, $dim\ \mathfrak{n}_2 = 2$. *In this case maps* i *and* j *exist but* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are not* C-*conjugate.*

Returning to a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and a C.S. $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$, we first observe that $i\mathfrak{h}_0^+$ and $\mathfrak{h}_0^-$ are complementary orthogonal subspaces of some $\mathfrak{h}^\theta \subseteq \mathfrak{g}$. Consequently, we may speak of $\Delta[i\mathfrak{h}_0^+]$ and $\Delta[\mathfrak{h}_0^-]$. We have, finally,

THEOREM 9. *For a real, simple Lie algebra* $\mathfrak{g}_0$, *the structure of* $\Delta[i\mathfrak{h}_0^+]$, *together with that of* $\Delta[\mathfrak{h}_0^-]$, *uniquely determines the conjugacy class of any C.S.* $\mathfrak{h}_0$ *under the full group of automorphisms of* $\mathfrak{g}_0$, *with the sole exception* $\mathfrak{g} = D_{2k}$, $k \geq 3$; $\mathfrak{g}_0$ *is the unique real form for which dim* $\mathfrak{m}_0 = 2k$. *For this case, corresponding to the pair of non-conjugate* W-*subspaces* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *of Theorem* 8, *there exist two distinct conjugate classes of C.S.'s both determining, for a representative C.S.* $\mathfrak{h}_0$ *of either class, identical structures for* $\Delta[i\mathfrak{h}_0^+]$ *and for* $\Delta[\mathfrak{h}_0^-]$.

We may substitute the adjoint group for the full automorphism group in the above statement, altering only the exceptional case by letting $k \geq 2$. *Corresponding now to the exceptional* W-*subspaces of Theorems* 7 *and* 8 *there are three distinct conjugacy classes of C.S.'s all determining, for a representative C.S.* $\mathfrak{h}_0$, *identical structures for* $\Delta[i\mathfrak{h}_0^+]$ *and for* $\Delta[\mathfrak{h}_0^-]$.

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[1] Harish-Chandra, "Plancherel Formula for the 2 × 2 Real Unimodular Group," these PROCEEDINGS, **38,** 337–342, 1952.

[2] In general, the word "conjugate" will be used to mean transformable with respect to the action of the group under consideration. When speaking of conjugacy of C.S.'s, this group will be understood to be the adjoint group unless the full group of automorphisms is specified.

[3] D_1, the one-dimensional complex Lie algebra, is not semisimple, and $D_2 = A_1 \oplus A_1$ is not simple.

[4] We say that $l_1 \subseteq \mathfrak{h}_\theta$, $l_2 \subseteq \mathfrak{h}^\theta$ are W- (resp. C-) conjugate if that can be carried into one another by an element W (resp. C).

[5] A maximal W-subspace contained in the subspace $\mathfrak{m}_o \subseteq \mathfrak{h}^\theta$ associated with a real form is always complete or totally incomplete.