Preface

Bertram Kostant is to be counted as one of the remarkable mathematicians of the second half of the twentieth century through his fundamental and varied contributions to many aspects of Lie theory, a subject which itself pervades almost the whole of mathematics. His work is marked by a rare simplicity and characteristic elegance, making it eminently readable, wonderfully enjoyable, and easily understandable even to a true novice. It is often through his work that one may understand many sophisticated developments of modern mathematics.

The mathematical work of Bertram Kostant has spanned well over fifty years in which he has published at an almost constant rhythm over 100 papers of which more than a few have become a cornerstone of rich and fruitful theories. Here, in briefly listing some of his most important contributions and their eventual ramifications we can give only a small glimpse into the depth and nature of his legacy.

In one of his earliest works Kostant introduced a partition function describing weight multiplicities. Significantly, it provided a tool which allowed Jantzen to compute certain fundamental determinants concerning weight spaces in Verma modules in the much more difficult parabolic case.

Kostant studied extensively the principal three-dimensional subalgebra of a semisimple Lie algebra, extracting from its adjoint action the Betti numbers of the corresponding Lie group. In addition, he thereby obtained a remarkable linearization of the fundamental invariants. Geometrically this realizes an affine slice to the regular coadjoint orbits, a result subsequently broadened notably by Luna, Slowody, and Premet, and now of greatly renewed interest.

Kostant followed this work by a further fundamental paper describing harmonic polynomials in the symmetric algebra. This gave a separation of variables theorem and a rather precise description of the nilpotent cone. These results have been key building blocks in many important results in representation theory, including the theory of primitive ideals.

Kostant discovered a remarkable connection between the regular nilpotent element and the Coxeter element in the Weyl group. This was subsequently brought to a classification of nilpotent orbits by Carter and Elkington and then elevated to a map from nilpotent orbits to conjugacy classes in the Weyl group by Kazhdan and Lusztig.

Kostant himself has returned to the many beautiful themes which may be developed from these early papers. They include an extensive study of Whittaker modules, the mathematical underpinnings of the Toda lattice which he generalized well beyond the dreams of physicists, and quantum cohomology.

Building upon Bott's geometric results, Kostant computed the Lie algebra cohomology of certain nilpotent Lie algebras arising as nilradicals of parabolic subalgebras, studying at the same time their relationship to the cohomology of the flag variety and its generalizations. This recovered the Bernstein–Gelfand–Gelfand resolution of simple finite-dimensional modules admitting far-reaching generalizations to Kac–Moody and later to Borcherds algebras. Here he inadvertently rediscovered the Amitsur–Levitski identity, so important in algebras with polynomial identity, giving it a far simpler proof and placing the result in the more general context of semisimple Lie algebras.

Kostant was one of the first to realize that a construction of Kirillov to classify unitary representations of nilpotent Lie groups implied that any coadjoint orbit admitted the structure of a symplectic variety, that is, the phase space of classical mechanics. He quickly seized upon the passage to quantum mechanics, developed by physicists, as a process of "geometric quantization." The Kostant line bundle detects which symplectic manifolds are quantizable and this result is fundamental in Hamiltonian geometry. When applied to the coadjoint orbits, quantization is designed to produce most unitary representations of the corresponding real Lie group. With Auslander he classified the unitary representations for real class 1 simply connected solvable Lie groups, laying the groundwork for the more complete theory of Pukanszky. The general case, though even more resistant, has come under the intensive study of many mathematicians, including notably Duflo, Rossmann, Vergne and Vogan. In more algebraic terms geometric quantization dramatically laid open the path to the classification of primitive ideals initiated by Dixmier and Gabriel. Further deep manifestations of geometric quantization are exemplified by the Duflo isomorphism, the Kashiwara–Vergne conjecture, and the Drinfeld associator themselves having been unified and extended following notably the powerful techniques of Kontsevitch inspired by Feynman diagrams and knot theory.

Stepping outside the purely algebraic framework Kostant discovered a far-reaching generalization of the Golden–Thompson rule in a convexity theorem subsequently generalized by Atiyah–Bott and Guillemin–Sternberg. Going beyond zero characteristic, he introduced the "Kostant form" on the enveloping algebra which has played a major role in modular representation theory. Moreover its generalization for quantum groups and the subsequent development of the quantum Frobenius map due to Lusztig allowed Littelmann to complete the Lakshmibai–Seshadri programme of describing standard monomial bases.

In a veritable "tour de force" Kostant calculated certain fundamental determinants showing them to have linear factors from which their zeros could be determined. This led to a criterion for unitarity which has so far gone unsurpassed. Subsequently, many other such factorizations were achieved, particularly that of Shapovalov and Janzten noted above; but also of Gorelik–Lanzmann which determined exactly when a Verma module annihilator for a reductive super Lie algebra was generated by its intersection with the centre.

Kostant was one of the first to firmly lay the foundations of supermanifolds and super Lie groups. He also studied super Lie algebras providing a fundamental structure theorem for their enveloping algebras. He initiated the study of the tensor product of an infinite-dimensional representation with finite-dimensional ones, leading in the hands of Jantzen to the translation principle. The latter, combined with Kostant's description of the nilpotent cone, led to the Beilinson–Bernstein equivalence of categories which heralded a new enlightenment in Verma module and Harish-Chandra theory. Kostant described (in sl(4)) a non-trivial example of a unitary highest weight module, the theory of which was subsequently completed by Enright, Howe and Wallach (and independently by Jakobsen). In the theory of characteristics developed by Guillemin, Quillen and Sternberg, a result of Kostant proves in the finite-dimensional case the involutivity of characteristics ultimately resolved in full generality by Gabber.

Among some of his important collaborative works, we mention his paper with Hochschild and Rosenberg on the differential forms on regular affine algebras, his paper with Rallis on separation of variables for symmetric spaces, his work with Kumar on the cohomology and K-theory of flag varieties associated to Kac–Moody groups, his determination with R. Brylinski of invariant symplectic structures and a uniform construction of minimal representations, and his work with Wallach on Gelfand–Zeitlin theory from the point of view of the construction of maximal Poisson commutative subalgebras particularly by "shift of argument" coming from classical mechanics.

Even in the fifth and sixth decades of his career, Kostant has continued to produce results of astonishing beauty and significance. We cite here his work on the Toda lattice and the quantum cohomology of the flag variety, his re-examination of the Clifford algebra deformation of "wedge n" with its beautiful realization of the module whose highest weight is the sum of the fundamental weights, his introduction of the cubic Dirac operator, his generalization of the Borel–Bott–Weil theorem with its connection to Euler number multiplets of representations, his work on the set of abelian ideals in the nilradical of a Borel (wonderfully classified by certain elements in the affine Weyl group by Dale Peterson), his longstanding love affair with the icosahedron, his work with Wallach mentioned above and most recently an attempt to pin down the generators of the centralizer of a maximal compact subalgebra.

After receiving his Ph.D. from the University of Chicago in 1954, Bertram Kostant began his academic career as an Assistant Professor in 1956 at the University of California, Berkeley rose to full professor in 1962 and shortly after moved to MIT. He taught at MIT until retiring in 1993.

Kostant was elected to the National Academy of Sciences U.S.A. in 1978 and has received many other honours and prizes, including election as a Sackler Institute Fellow at Tel-Aviv University in 1982, a medal from the Collège de France in 1983, the Steele prize of the American Mathematical Society in 1990, and several honorary doctorates.

Kostant maintains his physical fitness after the strains of University life by a weight lifting regime well beyond the capabilities of many younger colleagues. This may be one secret to his scientific longevity.

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