

Contents

Preface	vii
0 Univariate Distributions	1
0.1 Introduction	1
0.2 Notation and Definitions	2
0.2.1 Notation	2
0.2.2 Explanations	3
0.2.3 Characteristic Function	3
0.2.4 Cumulant Generating Function.....	4
0.3 Some Measures of Shape Characteristics.....	5
0.3.1 Location and Scale	5
0.3.2 Skewness and Kurtosis	5
0.3.3 Tail Behavior	6
0.3.4 Some Multiparameter Systems of Univariate Distributions.....	6
0.3.5 Reliability Classes	7
0.4 Normal Distribution and Its Transformations	7
0.4.1 Normal Distribution	7
0.4.2 Lognormal Distribution	8
0.4.3 Truncated Normal	8
0.4.4 Johnson's System	8
0.4.5 Box-Cox Power Transformations to Normality ...	9
0.4.6 g and h Families of Distributions	9
0.4.7 Efron's Transformation.....	10
0.4.8 Distribution of a Ratio	10
0.4.9 Compound Normal Distributions	10
0.5 Beta Distribution	11
0.5.1 The First Kind.....	11
0.5.2 Uniform Distribution	12
0.5.3 Symmetric Beta Distribution	12
0.5.4 Inverted Beta Distribution.....	12

0.6	Exponential, Gamma, Weibull, and Stacy Distributions . . .	13
0.6.1	Exponential Distribution	13
0.6.2	Gamma Distribution	14
0.6.3	Chi-Squared and Chi Distributions	14
0.6.4	Weibull Distribution	15
0.6.5	Stacy Distribution	15
0.6.6	Comments on Skew Distributions	16
0.6.7	Compound Exponential Distributions	16
0.7	Aging Distributions	17
0.7.1	Marshall and Olkin's Family of Distributions	17
0.7.2	Families of Generalized Weibull Distributions	18
0.8	Logistic, Laplace, and Cauchy Distributions	19
0.8.1	Logistic Distribution	19
0.8.2	Laplace Distribution	19
0.8.3	The Generalized Error Distribution	20
0.8.4	Cauchy Distribution	20
0.9	Extreme-Value Distributions	20
0.9.1	Type 1	20
0.9.2	Type 2	21
0.9.3	Type 3	21
0.10	Pareto Distribution	21
0.11	Pearson System	22
0.12	Burr System	23
0.13	t - and F -Distributions	23
0.13.1	t -Distribution	23
0.13.2	F -Distribution	24
0.14	The Wrapped t Family of Circular Distributions	24
0.15	Noncentral Distributions	25
0.16	Skew Distributions	25
0.16.1	Skew-Normal Distribution	25
0.16.2	Skew t -Distributions	26
0.16.3	Skew-Cauchy Distribution	27
0.17	Jones' Family of Distributions	28
0.18	Some Lesser-Known Distributions	28
0.18.1	Inverse Gaussian Distribution	28
0.18.2	Meixner Hypergeometric Distribution	29
0.18.3	Hyperbolic Distributions	29
0.18.4	Stable Distributions	29
	References	30
1	Bivariate Copulas	33
1.1	Introduction	33
1.2	Basic Properties	34
1.3	Further Properties of Copulas	35
1.4	Survival Copula	36

1.5	Archimedean Copula	37
1.6	Extreme-Value Copulas	38
1.7	Archimax Copulas	39
1.8	Gaussian, t -, and Other Copulas of the Elliptical Distributions	40
1.9	Order Statistics Copula	41
1.10	Polynomial Copulas	41
	1.10.1 Approximation of a Copula by a Polynomial Copula	43
1.11	Measures of Dependence Between Two Variables with a Given Copula	44
	1.11.1 Kendall’s Tau	44
	1.11.2 Spearman’s Rho	45
	1.11.3 Geometry of Correlation Under a Copula	45
	1.11.4 Measure Based on Gini’s Coefficient	46
	1.11.5 Tail Dependence Coefficients	46
	1.11.6 A Local Dependence Measure	48
	1.11.7 Tests of Dependence and Inferences	48
	1.11.8 “Concepts of Dependence” of Copulas	48
1.12	Distribution Function of $Z = C(U, V)$	48
1.13	Simulation of Copulas	49
	1.13.1 The General Case	50
	1.13.2 Archimedean Copulas	50
1.14	Construction of a Copula	50
	1.14.1 Rüschendorf’s Method	50
	1.14.2 Generation of Copulas by Mixture	52
	1.14.3 Convex Sums	53
	1.14.4 Univariate Function Method	53
	1.14.5 Some Other Methods	54
1.15	Applications of Copulas	55
	1.15.1 Insurance, Finance, Economics, and Risk Management	55
	1.15.2 Hydrology and Environment	56
	1.15.3 Management Science and Operations Research	57
	1.15.4 Reliability and Survival Analysis	57
	1.15.5 Engineering and Medical Sciences	57
	1.15.6 Miscellaneous	58
1.16	Criticisms about Copulas	58
1.17	Conclusions	59
	References	60
2	Distributions Expressed as Copulas	67
	2.1 Introduction	67
	2.2 Farlie–Gumbel–Morgenstern (F-G-M) Copula and Its Generalization	68

- 2.2.1 Applications 70
- 2.2.2 Univariate Transformations 70
- 2.2.3 A Switch-Source Model 71
- 2.2.4 Ordinal Contingency Tables 71
- 2.2.5 Iterated F-G-M Distributions 71
- 2.2.6 Extensions of the F-G-M Distribution 72
- 2.2.7 Other Related Distributions 75
- 2.3 Ali–Mikhail–Haq Distribution 76
 - 2.3.1 Bivariate Logistic Distributions 77
 - 2.3.2 Bivariate Exponential Distribution 78
- 2.4 Frank’s Distribution 78
- 2.5 Distribution of Cuadras and Augé and Its Generalization . . 79
 - 2.5.1 Generalized Cuadras and Augé Family (Marshall and Olkin’s Family) 79
- 2.6 Gumbel–Hougaard Copula 80
- 2.7 Plackett’s Distribution 82
- 2.8 Bivariate Lomax Distribution 84
 - 2.8.1 The Special Case of $c = 1$ 87
 - 2.8.2 Bivariate Pareto Distribution 88
- 2.9 Lomax Copula 89
 - 2.9.1 Pareto Copula (Clayton Copula) 90
 - 2.9.2 Summary of the Relationship Between Various Copulas 92
- 2.10 Gumbel’s Type I Bivariate Exponential Distribution 92
- 2.11 Gumbel–Barnett Copula 94
- 2.12 Kimeldorf and Sampson’s Distribution 95
- 2.13 Rodríguez-Lallena and Úbeda-Flores’ Family of Bivariate Copulas 96
- 2.14 Other Copulas 96
- 2.15 References to Illustrations 97
- References 98

- 3 Concepts of Stochastic Dependence 105**
 - 3.1 Introduction 105
 - 3.2 Concept of Positive Dependence and Its Conditions 106
 - 3.3 Positive Dependence Concepts at a Glance 107
 - 3.4 Concepts of Positive Dependence Stronger than PQD 108
 - 3.4.1 Positive Quadrant Dependence 108
 - 3.4.2 Association of Random Variables 109
 - 3.4.3 Left-Tail Decreasing (LTD) and Right-Tail Increasing (RTI) 110
 - 3.4.4 Positive Regression Dependent (Stochastically Increasing) 112
 - 3.4.5 Left Corner Set Decreasing and Right Corner Set Increasing 114

3.4.6	Total Positivity of Order 2	115
3.4.7	$DTP_2(m, n)$ and Positive Dependence by Mixture	117
3.5	Concepts of Positive Dependence Weaker than PQD	117
3.5.1	Positive Quadrant Dependence in Expectation	117
3.5.2	Positively Correlated Distributions	118
3.5.3	Monotonic Quadrant Dependence Function	118
3.5.4	Summary of Interrelationships	120
3.6	Families of Bivariate PQD Distributions	121
3.6.1	Bivariate PQD Distributions with Simple Structures	122
3.6.2	Construction of Bivariate PQD Distributions	125
3.6.3	Tests of Independence Against Positive Dependence	126
3.6.4	Geometric Interpretations of PQD and Other Positive Dependence Concepts	127
3.7	Additional Concepts of Dependence	128
3.8	Negative Dependence	129
3.8.1	Neutrality	130
3.8.2	Examples of NQD	130
3.9	Positive Dependence Orderings	131
3.9.1	Some Other Positive Dependence Orderings	134
3.9.2	Positive Dependent Ordering with Different Marginals	135
3.9.3	Bayesian Concepts of Dependence	136
	References	136
4	Measures of Dependence	141
4.1	Introduction	141
4.2	Total Dependence	142
4.2.1	Functions	142
4.2.2	Mutual Complete Dependence	142
4.2.3	Monotone Dependence	143
4.2.4	Functional and Implicit Dependence	144
4.2.5	Overview	144
4.3	Global Measures of Dependence	144
4.4	Pearson's Product-Moment Correlation Coefficient	146
4.4.1	Robustness of Sample Correlation	147
4.4.2	Interpretation of Correlation	148
4.4.3	Correlation Ratio	151
4.4.4	Chebyshev's Inequality	151
4.4.5	ρ and Concepts of Dependence	151
4.5	Maximal Correlation (Sup Correlation)	152
4.6	Monotone Correlations	153
4.6.1	Definitions and Properties	153

4.6.2	Concordant and Discordant Monotone Correlations	154
4.7	Rank Correlations	155
4.7.1	Kendall's Tau	155
4.7.2	Spearman's Rho	156
4.7.3	The Relationship Between Kendall's Tau and Spearman's Rho	157
4.7.4	Other Concordance Measures	162
4.8	Measures of Schweizer and Wolff and Related Measures . . .	163
4.9	Matrix of Correlation	164
4.10	Tetrachoric and Polychoric Correlations	165
4.11	Compatibility with Perfect Rank Ordering	166
4.12	Conclusions on Measures of Dependence	167
4.13	Local Measures of Dependence	167
4.13.1	Definition of Local Dependence	168
4.13.2	Local Dependence Function of Holland and Wang	168
4.13.3	Local ρ_S and τ	169
4.13.4	Local Measure of LRD	169
4.13.5	Properties of $\gamma(x, y)$	170
4.13.6	Local Correlation Coefficient	170
4.13.7	Several Local Indices Applicable in Survival Analysis	171
4.14	Regional Dependence	171
4.14.1	Preliminaries	171
4.14.2	Quasi-Independence and Quasi-Independent Projection	172
4.14.3	A Measure of Regional Dependence	173
	References	173
5	Construction of Bivariate Distributions	179
5.1	Introduction	179
5.1.1	Fréchet Bounds	180
5.1.2	Transformations	181
5.2	The Marginal Transformation Method	181
5.2.1	General Description	181
5.2.2	Johnson's Translation Method	182
5.2.3	Uniform Representation: Copulas	183
5.2.4	Some Properties Unaffected by Transformation . . .	184
5.3	Methods of Constructing Copulas	185
5.3.1	The Inversion Method	185
5.3.2	Geometric Methods	185
5.3.3	Algebraic Methods	186
5.3.4	Rüschendorf's Method	186
5.3.5	Models Defined from a Distortion Function	187

5.3.6	Marshall and Olkin’s Mixture Method	187
5.3.7	Archimedean Copulas	188
5.3.8	Archimax Copulas	189
5.4	Mixing and Compounding	189
5.4.1	Mixing	189
5.4.2	Compounding	190
5.5	Variables in Common and Trivariate Reduction Techniques	193
5.5.1	Summary of the Method	193
5.5.2	Denominator-in-Common and Compounding	194
5.5.3	Mathai and Moschopoulos’ Methods	194
5.5.4	Modified Structure Mixture Model	195
5.5.5	Khinchine Mixture	195
5.6	Conditionally Specified Distributions	196
5.6.1	A Conditional Distribution with a Marginal Given	196
5.6.2	Specification of Both Sets of Conditional Distributions	196
5.6.3	Conditionals in Exponential Families	197
5.6.4	Conditions Implying Bivariate Normality	199
5.6.5	Summary of Conditionally Specified Distributions	199
5.7	Marginal Replacement	201
5.7.1	Example: Bivariate Non-normal Distribution	202
5.7.2	Marginal Replacement of a Spherically Symmetric Bivariate Distribution	202
5.8	Introducing Skewness	202
5.9	Density Generators	202
5.10	Geometric Approach	203
5.11	Some Other Simple Methods	204
5.12	Weighted Linear Combination	205
5.13	Data-Guided Methods	206
5.13.1	Conditional Distributions	206
5.13.2	Radii and Angles	207
5.13.3	The Dependence Function in the Extreme-Value Sense	208
5.14	Special Methods Used in Applied Fields	208
5.14.1	Shock Models	208
5.14.2	Queueing Theory	210
5.14.3	Compositional Data	211
5.14.4	Extreme-Value Models	211
5.14.5	Time Series: Autoregressive Models	213
5.15	Limits of Discrete Distributions	215
5.15.1	A Bivariate Exponential Distribution	215
5.15.2	A Bivariate Gamma Distribution	216
5.16	Potentially Useful Methods But Not in Vogue	216

- 5.16.1 Differential Equation Methods 217
- 5.16.2 Diagonal Expansion 219
- 5.16.3 Bivariate Edgeworth Expansion 220
- 5.16.4 An Application to Wind Velocity at the
Ocean Surface 221
- 5.16.5 Another Application to Statistical Spectroscopy .. 221
- 5.17 Concluding Remarks 222
- References 223

6 Bivariate Distributions Constructed by the Conditional Approach 229

- 6.1 Introduction 229
 - 6.1.1 Contents 229
 - 6.1.2 Pertinent Univariate Distributions 230
 - 6.1.3 Compatibility and Uniqueness 231
 - 6.1.4 Early Work on Conditionally
Specified Distributions 232
 - 6.1.5 Approximating Distribution Functions Using the
Conditional Approach..... 232
- 6.2 Normal Conditionals 233
 - 6.2.1 Conditional Distributions 233
 - 6.2.2 Expression of the Joint Density 233
 - 6.2.3 Univariate Properties 234
 - 6.2.4 Further Properties 234
 - 6.2.5 Centered Normal Conditionals 234
- 6.3 Conditionals in Exponential Families 236
 - 6.3.1 Dependence in Conditional Exponential Families . 237
 - 6.3.2 Exponential Conditionals 237
 - 6.3.3 Normal Conditionals 240
 - 6.3.4 Gamma Conditionals 240
 - 6.3.5 Model II for Gamma Conditionals 241
 - 6.3.6 Gamma-Normal Conditionals 242
 - 6.3.7 Beta Conditionals 243
 - 6.3.8 Inverse Gaussian Conditionals..... 244
- 6.4 Other Conditionally Specified Families 245
 - 6.4.1 Pareto Conditionals 245
 - 6.4.2 Beta of the Second Kind (Pearson Type VI)
Conditionals 246
 - 6.4.3 Generalized Pareto Conditionals..... 248
 - 6.4.4 Cauchy Conditionals 249
 - 6.4.5 Student *t*-Conditionals 250
 - 6.4.6 Uniform Conditionals 251
 - 6.4.7 Translated Exponential Conditionals..... 252
 - 6.4.8 Scaled Beta Conditionals 253

6.5	Conditionally Specified Bivariate Skewed Distributions	254
6.5.1	Bivariate Distributions with Skewed Normal Conditionals	254
6.5.2	Linearly Skewed and Quadratically Skewed Normal Conditionals	256
6.6	Improper Bivariate Distributions from Conditionals	256
6.7	Conditionals in Location-Scale Families with Specified Moments	256
6.8	Conditional Distributions and the Regression Function	257
6.8.1	Assumptions and Specifications	257
6.8.2	Wesolowski’s Theorem	258
6.9	Estimation in Conditionally Specified Models	258
6.10	McKay’s Bivariate Gamma Distribution and Its Generalization	260
6.10.1	Conditional Properties	260
6.10.2	Expression of the Joint Density	260
6.10.3	Dussauchoy and Berland’s Bivariate Gamma Distribution	260
6.11	One Conditional and One Marginal Specified	261
6.11.1	Dubey’s Distribution	261
6.11.2	Blumen and Ypelaar’s Distribution	262
6.11.3	Exponential Dispersion Models	262
6.11.4	Four Densities of Barndorff-Nielsen and Blæsild	263
6.11.5	Continuous Bivariate Densities with a Discontinuous Marginal Density	263
6.11.6	Tiku and Kambo’s Bivariate Non-normal Distribution	264
6.12	Marginal and Conditional Distributions of the Same Variate	265
6.12.1	Example	266
6.12.2	Vardi and Lee’s Iteration Scheme	266
6.13	Conditional Survival Models	267
6.13.1	Exponential Conditional Survival Function	267
6.13.2	Weibull Conditional Survival Function	268
6.13.3	Generalized Pareto Conditional Survival Function	269
6.14	Conditional Approach in Modeling	269
6.14.1	Beta-Stacy Distribution	269
6.14.2	Sample Skewness and Kurtosis	270
6.14.3	Business Risk Analysis	271
6.14.4	Intercropping	271
6.14.5	Winds and Waves, Rain and Floods	272
	References	275

7 Variables-in-Common Method 279

7.1 Introduction 279

7.2 General Description 280

7.3 Additive Models 281

7.3.1 Background 281

7.3.2 Meixner Classes 282

7.3.3 Cherian’s Bivariate Gamma Distribution 283

7.3.4 Symmetric Stable Distribution 283

7.3.5 Bivariate Triangular Distribution 283

7.3.6 Summing Several I.I.D. Variables 284

7.4 Generalized Additive Models 285

7.4.1 Trivariate Reduction of Johnson and Tenenbein 285

7.4.2 Mathai and Moschopoulos’ Bivariate Gamma 286

7.4.3 Lai’s Structure Mixture Model 286

7.4.4 Latent Variables-in-Common Model 287

7.4.5 Bivariate Skew-Normal Distribution 288

7.4.6 Ordered Statistics 289

7.5 Weighted Linear Combination 290

7.5.1 Derivation 290

7.5.2 Expression of the Joint Density 290

7.5.3 Correlation Coefficients 290

7.5.4 Remarks 291

7.6 Bivariate Distributions Having a Common Denominator 291

7.6.1 Explanation 291

7.6.2 Applications 292

7.6.3 Correlation Between Ratios with a
Common Divisor 292

7.6.4 Compounding 293

7.6.5 Examples of Two Ratios with a Common Divisor 293

7.6.6 Bivariate *t*-Distribution with Marginals Having
Different Degrees of Freedom 295

7.6.7 Bivariate Distributions Having a Common
Numerator 295

7.7 Multiplicative Trivariate Reduction 295

7.7.1 Bryson and Johnson (1982) 296

7.7.2 Gokhale’s Model 296

7.7.3 Ulrich’s Model 297

7.8 Khintchine Mixture 297

7.8.1 Derivation 297

7.8.2 Exponential Marginals 297

7.8.3 Normal Marginals 298

7.8.4 References to Generation of Random Variates 298

7.9 Transformations Involving the Minimum 299

7.10 Other Forms of the Variables-in-Common Technique 299

7.10.1 Bivariate Chi-Squared Distribution 299

7.10.2	Bivariate Beta Distribution	300
7.10.3	Bivariate Z-Distribution	300
	References	301
8	Bivariate Gamma and Related Distributions	305
8.1	Introduction	305
8.2	Kibble's Bivariate Gamma Distribution	306
8.2.1	Formula of the Joint Density	306
8.2.2	Formula of the Cumulative Distribution Function	307
8.2.3	Univariate Properties	307
8.2.4	Correlation Coefficient	307
8.2.5	Moment Generating Function	307
8.2.6	Conditional Properties	308
8.2.7	Derivation	308
8.2.8	Relations to Other Distributions	309
8.2.9	Generalizations	309
8.2.10	Illustrations	309
8.2.11	Remarks	310
8.2.12	Fields of Applications	310
8.2.13	Tables and Algorithms	311
8.2.14	Transformations of the Marginals	311
8.3	Royen's Bivariate Gamma Distribution	311
8.3.1	Formula of the Cumulative Distribution Function	311
8.3.2	Univariate Properties	312
8.3.3	Derivation	312
8.3.4	Relation to Kibble's Bivariate Gamma Distribution	312
8.4	Izawa's Bivariate Gamma Distribution	312
8.4.1	Formula of the Joint Density	312
8.4.2	Correlation Coefficient	313
8.4.3	Relation to Kibble's Bivariate Gamma Distribution	313
8.4.4	Fields of Application	313
8.5	Jensen's Bivariate Gamma Distribution	313
8.5.1	Formula of the Joint Density	313
8.5.2	Univariate Properties	314
8.5.3	Correlation Coefficient	314
8.5.4	Characteristic Function	314
8.5.5	Derivation	315
8.5.6	Illustrations	315
8.5.7	Remarks	315
8.5.8	Fields of Application	316
8.5.9	Tables and Algorithms	316
8.6	Gunst and Webster's Model and Related Distributions ...	316
8.6.1	Case 3 of Gunst and Webster	317

8.6.2	Case 2 of Gunst and Webster	318
8.7	Smith, Aldelfang, and Tubbs' Bivariate Gamma Distribution	318
8.8	Sarmanov's Bivariate Gamma Distribution	319
8.8.1	Formula of the Joint Density	319
8.8.2	Univariate Properties	319
8.8.3	Correlation Coefficient	319
8.8.4	Derivation	320
8.8.5	Interrelationships	320
8.9	Bivariate Gamma of Loáiciga and Leipnik	320
8.9.1	Formula of the Joint Density	321
8.9.2	Univariate Properties	321
8.9.3	Joint Characteristic Function	321
8.9.4	Correlation Coefficient	321
8.9.5	Moments and Joint Moments	321
8.9.6	Application to Water-Quality Data	322
8.10	Cheriyán's Bivariate Gamma Distribution	322
8.10.1	Formula of the Joint Density	323
8.10.2	Univariate Properties	323
8.10.3	Correlation Coefficient	323
8.10.4	Moment Generating Function	323
8.10.5	Conditional Properties	323
8.10.6	Derivation	324
8.10.7	Generation of Random Variates	324
8.10.8	Remarks	324
8.11	Prékopa and Szántai's Bivariate Gamma Distribution	325
8.11.1	Formula of the Cumulative Distribution Function	325
8.11.2	Formula of the Joint Density	325
8.11.3	Univariate Properties	326
8.11.4	Relation to Other Distributions	326
8.12	Schmeiser and Lal's Bivariate Gamma Distribution	326
8.12.1	Method of Construction	326
8.12.2	Correlation Coefficient	327
8.12.3	Remarks	327
8.13	Farlie–Gumbel–Morgenstern Bivariate Gamma Distribution	327
8.13.1	Formula of the Joint Density	327
8.13.2	Univariate Properties	328
8.13.3	Moment Generating Function	328
8.13.4	Correlation Coefficient	328
8.13.5	Conditional Properties	328
8.13.6	Remarks	328
8.14	Moran's Bivariate Gamma Distribution	329
8.14.1	Derivation	329
8.14.2	Formula of the Joint Density	329

- 8.14.3 Computation of Bivariate Distribution Function . . 329
- 8.14.4 Remarks 329
- 8.14.5 Fields of Application 330
- 8.15 Crovelli’s Bivariate Gamma Distribution 330
 - 8.15.1 Fields of Application 330
- 8.16 Suitability of Bivariate Gammas for Hydrological Applications 330
- 8.17 McKay’s Bivariate Gamma Distribution 331
 - 8.17.1 Formula of the Joint Density 331
 - 8.17.2 Formula of the Cumulative Distribution Function 331
 - 8.17.3 Univariate Properties 331
 - 8.17.4 Conditional Properties 331
 - 8.17.5 Methods of Derivation 332
 - 8.17.6 Remarks 332
- 8.18 Dussauchoy and Berland’s Bivariate Gamma Distribution . 332
 - 8.18.1 Formula of the Joint Density 332
- 8.19 Mathai and Moschopoulos’ Bivariate Gamma Distributions 334
 - 8.19.1 Model 1 334
 - 8.19.2 Model 2 335
- 8.20 Becker and Roux’s Bivariate Gamma Distribution 336
 - 8.20.1 Formula of the Joint Density 336
 - 8.20.2 Derivation 336
 - 8.20.3 Remarks 337
- 8.21 Bivariate Chi-Squared Distribution 337
 - 8.21.1 Formula of the Cumulative Distribution Function 337
 - 8.21.2 Univariate Properties 337
 - 8.21.3 Correlation Coefficient 338
 - 8.21.4 Conditional Properties 338
 - 8.21.5 Derivation 338
 - 8.21.6 Remarks 338
- 8.22 Bivariate Noncentral Chi-Squared Distribution 339
- 8.23 Gaver’s Bivariate Gamma Distribution 339
 - 8.23.1 Moment Generating Function 339
 - 8.23.2 Derivation 340
 - 8.23.3 Correlation Coefficients 340
- 8.24 Bivariate Gamma of Nadarajah and Gupta 340
 - 8.24.1 Model 1 340
 - 8.24.2 Model 2 341
- 8.25 Arnold and Strauss’ Bivariate Gamma Distribution 342
 - 8.25.1 Remarks 343
- 8.26 Bivariate Gamma Mixture Distribution 343
 - 8.26.1 Model Specification 343
 - 8.26.2 Formula of the Joint Density 343

- 8.26.3 Formula of the Cumulative Distribution Function 344
- 8.26.4 Univariate Properties 344
- 8.26.5 Moments and Moment Generating Function 344
- 8.26.6 Correlation Coefficient 345
- 8.26.7 Fields of Application 345
- 8.26.8 Mixtures of Bivariate Gammas of Iwasaki and Tsubaki 345
- 8.27 Bivariate Bessel Distributions 345
- References 346

9 Simple Forms of the Bivariate Density Function 351

- 9.1 Introduction 351
- 9.2 Bivariate *t*-Distribution 352
 - 9.2.1 Formula of the Joint Density 352
 - 9.2.2 Univariate Properties 352
 - 9.2.3 Correlation Coefficients 353
 - 9.2.4 Moments 353
 - 9.2.5 Conditional Properties 353
 - 9.2.6 Derivation 354
 - 9.2.7 Illustrations 354
 - 9.2.8 Generation of Random Variates 354
 - 9.2.9 Remarks 354
 - 9.2.10 Fields of Application 355
 - 9.2.11 Tables and Algorithms 355
 - 9.2.12 Spherically Symmetric Bivariate *t*-Distribution 356
 - 9.2.13 Generalizations 356
- 9.3 Bivariate Noncentral *t*-Distributions 356
 - 9.3.1 Bivariate Noncentral *t*-Distribution with $\rho = 1$ 357
- 9.4 Bivariate *t*-Distribution Having Marginals with Different Degrees of Freedom 357
- 9.5 Jones' Bivariate Skew *t*-Distribution 359
 - 9.5.1 Univariate Skew *t*-Distribution 359
 - 9.5.2 Formula of the Joint Density 359
 - 9.5.3 Correlation and Local Dependence for the Symmetric Case 360
 - 9.5.4 Derivation 360
- 9.6 Bivariate Skew *t*-Distribution 361
 - 9.6.1 Formula of the Joint Density 361
 - 9.6.2 Moment Properties 361
 - 9.6.3 Derivation 361
 - 9.6.4 Possible Application due to Flexibility 362
 - 9.6.5 Ordered Statistics 362
- 9.7 Bivariate *t*-/Skew *t*-Distribution 362
 - 9.7.1 Formula of the Joint Density 362

9.7.2	Univariate Properties	363
9.7.3	Conditional Properties	363
9.7.4	Other Properties	363
9.7.5	Derivation	363
9.8	Bivariate Heavy-Tailed Distributions	364
9.8.1	Formula of the Joint Density	364
9.8.2	Univariate Properties	364
9.8.3	Remarks	364
9.8.4	Fields of Application	364
9.9	Bivariate Cauchy Distribution	365
9.9.1	Formula of the Joint Density	365
9.9.2	Formula of the Cumulative Distribution Function	365
9.9.3	Univariate Properties	365
9.9.4	Conditional Properties	365
9.9.5	Illustrations	366
9.9.6	Remarks	366
9.9.7	Generation of Random Variates	366
9.9.8	Generalization	366
9.9.9	Bivariate Skew-Cauchy Distribution	367
9.10	Bivariate F -Distribution	367
9.10.1	Formula of the Joint Density	368
9.10.2	Formula of the Cumulative Distribution Function	368
9.10.3	Univariate Properties	368
9.10.4	Correlation Coefficients	368
9.10.5	Product Moments	368
9.10.6	Conditional Properties	369
9.10.7	Methods of Derivation	369
9.10.8	Relationships to Other Distributions	369
9.10.9	Fields of Application	370
9.10.10	Tables and Algorithms	370
9.11	Bivariate Pearson Type II Distribution	371
9.11.1	Formula of the Joint Density	371
9.11.2	Univariate Properties	371
9.11.3	Correlation Coefficient	371
9.11.4	Conditional Properties	371
9.11.5	Relationships to Other Distributions	371
9.11.6	Illustrations	372
9.11.7	Generation of Random Variates	372
9.11.8	Remarks	372
9.11.9	Tables and Algorithms	372
9.11.10	Jones' Bivariate Beta/Skew Beta Distribution	372
9.12	Bivariate Finite Range Distribution	373
9.12.1	Formula of the Survival Function	373
9.12.2	Characterizations	374

- 9.12.3 Remarks 374
- 9.13 Bivariate Beta Distribution 374
 - 9.13.1 Formula of the Joint Density 374
 - 9.13.2 Univariate Properties 375
 - 9.13.3 Correlation Coefficient 375
 - 9.13.4 Product Moments 375
 - 9.13.5 Conditional Properties 375
 - 9.13.6 Methods of Derivation 375
 - 9.13.7 Relationships to Other Distributions 376
 - 9.13.8 Illustrations 376
 - 9.13.9 Generation of Random Variates 376
 - 9.13.10 Remarks 376
 - 9.13.11 Fields of Application 377
 - 9.13.12 Tables and Algorithms 378
 - 9.13.13 Generalizations 378
- 9.14 Jones' Bivariate Beta Distribution 379
 - 9.14.1 Formula of the Joint Density 379
 - 9.14.2 Univariate Properties 380
 - 9.14.3 Product Moments 380
 - 9.14.4 Correlation and Local Dependence 380
 - 9.14.5 Other Dependence Properties 380
 - 9.14.6 Illustrations 381
- 9.15 Bivariate Inverted Beta Distribution 381
 - 9.15.1 Formula of the Joint Density 381
 - 9.15.2 Formula of the Cumulative Distribution Function 381
 - 9.15.3 Derivation 381
 - 9.15.4 Tables and Algorithms 382
 - 9.15.5 Application 382
 - 9.15.6 Generalization 382
 - 9.15.7 Remarks 382
- 9.16 Bivariate Liouville Distribution 382
 - 9.16.1 Definitions 383
 - 9.16.2 Moments and Correlation Coefficient 384
 - 9.16.3 Remarks 385
 - 9.16.4 Generation of Random Variates 385
 - 9.16.5 Generalizations 386
 - 9.16.6 Bivariate p th-Order Liouville Distribution 386
 - 9.16.7 Remarks 386
- 9.17 Bivariate Logistic Distributions 387
 - 9.17.1 Standard Bivariate Logistic Distribution 387
 - 9.17.2 Archimedean Copula 389
 - 9.17.3 F-G-M Distribution with Logistic Marginals 389
 - 9.17.4 Generalizations 389
 - 9.17.5 Remarks 389

9.18	Bivariate Burr Distribution	390
9.19	Rhodes' Distribution	390
9.19.1	Support	390
9.19.2	Formula of the Joint Density	390
9.19.3	Derivation	391
9.19.4	Remarks	391
9.20	Bivariate Distributions with Support Above the Diagonal	391
9.20.1	Formula of the Joint Density	391
9.20.2	Formula of the Cumulative Distribution Function	392
9.20.3	Univariate Properties	392
9.20.4	Other Properties	392
9.20.5	Rotated Bivariate Distribution	392
9.20.6	Some Special Cases	393
9.20.7	Applications	394
	References	394
10	Bivariate Exponential and Related Distributions	401
10.1	Introduction	401
10.2	Gumbel's Bivariate Exponential Distributions	402
10.2.1	Gumbel's Type I Bivariate Exponential Distribution	403
10.2.2	Characterizations	403
10.2.3	Estimation Method	403
10.2.4	Other Properties	403
10.2.5	Gumbel's Type II Bivariate Exponential Distribution	404
10.2.6	Gumbel's Type III Bivariate Exponential Distribution	405
10.3	Freund's Bivariate Distribution	406
10.3.1	Formula of the Joint Density	406
10.3.2	Formula of the Cumulative Distribution Function	406
10.3.3	Univariate Properties	406
10.3.4	Correlation Coefficient	407
10.3.5	Conditional Properties	407
10.3.6	Joint Moment Generating Function	407
10.3.7	Derivation	407
10.3.8	Illustrations	408
10.3.9	Other Properties	408
10.3.10	Remarks	408
10.3.11	Fields of Application	409
10.3.12	Transformation of the Marginals	409
10.3.13	Compounding	409
10.3.14	Bhattacharya and Holla's Generalizations	410

- 10.3.15 Proschan and Sullo’s Extension of Freund’s Model 410
- 10.3.16 Becker and Roux’s Generalization 411
- 10.4 Hashino and Sugi’s Distribution 411
 - 10.4.1 Formula of the Joint Density 411
 - 10.4.2 Remarks 411
 - 10.4.3 An Application 412
- 10.5 Marshall and Olkin’s Bivariate Exponential Distribution . . 412
 - 10.5.1 Formula of the Cumulative Distribution Function 412
 - 10.5.2 Formula of the Joint Density Function 413
 - 10.5.3 Univariate Properties 413
 - 10.5.4 Conditional Distribution 413
 - 10.5.5 Correlation Coefficients 413
 - 10.5.6 Derivations 414
 - 10.5.7 Fisher Information 414
 - 10.5.8 Estimation of Parameters 414
 - 10.5.9 Characterizations 415
 - 10.5.10 Other Properties 415
 - 10.5.11 Remarks 416
 - 10.5.12 Fields of Application 418
 - 10.5.13 Transformation to Uniform Marginals 418
 - 10.5.14 Transformation to Weibull Marginals 419
 - 10.5.15 Transformation to Extreme-Value Marginals 419
 - 10.5.16 Transformation of Marginals: Approach of Muliere and Scarsini 419
 - 10.5.17 Generalization 420
- 10.6 ACBVE of Block and Basu 421
 - 10.6.1 Formula of the Joint Density 421
 - 10.6.2 Formula of the Cumulative Distribution Function 421
 - 10.6.3 Univariate Properties 421
 - 10.6.4 Correlation Coefficient 421
 - 10.6.5 Moment Generating Function 422
 - 10.6.6 Derivation 422
 - 10.6.7 Remarks 422
 - 10.6.8 Applications 423
- 10.7 Sarkar’s Distribution 423
 - 10.7.1 Formula of the Joint Density 423
 - 10.7.2 Formula of the Cumulative Distribution Function 424
 - 10.7.3 Univariate Properties 424
 - 10.7.4 Correlation Coefficient 424
 - 10.7.5 Derivation 424
 - 10.7.6 Relation to Marshall and Olkin’s Distribution 424

10.8	Comparison of Four Distributions	425
10.9	Friday and Patil's Generalization	425
10.10	Tosch and Holmes' Distribution	426
10.11	A Bivariate Exponential Model of Wang	427
	10.11.1 Formula of the Joint Density	427
	10.11.2 Univariate Properties	427
	10.11.3 Remarks	427
10.12	Lawrance and Lewis' System of Exponential Mixture Distributions	428
	10.12.1 General Form	428
	10.12.2 Model EP1	428
	10.12.3 Model EP3	429
	10.12.4 Model EP5	429
	10.12.5 Models with Negative Correlation	430
	10.12.6 Models with Uniform Marginals	430
	10.12.7 The Distribution of Sums, Products, and Ratios	430
	10.12.8 Mixture Models	430
	10.12.9 Models with Line Singularities	430
10.13	Raftery's Scheme	431
	10.13.1 First Special Case	431
	10.13.2 Second Special Case	431
	10.13.3 Formula of the Joint Density	432
	10.13.4 Formula of the Cumulative Distribution Function	432
	10.13.5 Derivation	432
	10.13.6 Illustrations	432
	10.13.7 Remarks	433
	10.13.8 Applications	433
10.14	Linear Structures of Iyer et al.	433
	10.14.1 Positive Cross Correlation	434
	10.14.2 Negative Cross Correlation	434
	10.14.3 Fields of Application	435
10.15	Moran-Downton Bivariate Exponential Distribution	436
	10.15.1 Formula of the Joint Density	436
	10.15.2 Formula of the Cumulative Distribution Function	436
	10.15.3 Univariate Properties	436
	10.15.4 Correlation Coefficients	436
	10.15.5 Conditional Properties	437
	10.15.6 Moment Generating Function	437
	10.15.7 Regression	437
	10.15.8 Derivation	438
	10.15.9 Fisher Information	438
	10.15.10 Estimation of Parameters	439

10.15.11	Illustrations	439
10.15.12	Random Variate Generation	439
10.15.13	Remarks	440
10.15.14	Fields of Application	441
10.15.15	Tables or Algorithms	442
10.15.16	Weibull Marginals	442
10.15.17	A Bivariate Laplace Distribution	443
10.16	Sarmanov's Bivariate Exponential Distribution	443
10.16.1	Formula of the Joint Density	443
10.16.2	Other Properties	444
10.17	Cowan's Bivariate Exponential Distribution	444
10.17.1	Formula of the Cumulative Distribution Function	444
10.17.2	Formula of the Joint Density	445
10.17.3	Univariate Properties	445
10.17.4	Correlation Coefficients	445
10.17.5	Conditional Properties	445
10.17.6	Derivation	446
10.17.7	Illustrations	446
10.17.8	Remarks	446
10.17.9	Transformation of the Marginals	446
10.18	Singpurwalla and Youngren's Bivariate Exponential Distribution	446
10.18.1	Formula of the Cumulative Distribution Function	447
10.18.2	Formula of the Joint Density	447
10.18.3	Univariate Properties	447
10.18.4	Derivation	447
10.18.5	Remarks	447
10.19	Arnold and Strauss' Bivariate Exponential Distribution	448
10.19.1	Formula of the Joint Density	448
10.19.2	Formula of the Cumulative Distribution Function	448
10.19.3	Univariate Properties	448
10.19.4	Conditional Distribution	448
10.19.5	Correlation Coefficient	449
10.19.6	Derivation	449
10.19.7	Other Properties	449
10.20	Mixtures of Bivariate Exponential Distributions	449
10.20.1	Lindley and Singpurwalla's Bivariate Exponential Mixture	449
10.20.2	Sankaran and Nair's Mixture	450
10.20.3	Al-Mutairi's Inverse Gaussian Mixture of Bivariate Exponential Distribution	450
10.20.4	Hayakawa's Mixtures	451

10.21	Bivariate Exponentials and Geometric Compounding Schemes	451
10.21.1	Background	451
10.21.2	Probability Generating Function	451
10.21.3	Bivariate Geometric Distribution	452
10.21.4	Bivariate Geometric Distribution Arising from a Shock Model	452
10.21.5	Bivariate Exponential Distribution Compounding Scheme	453
10.21.6	Wu’s Characterization of Marshall and Olkin’s Distribution via a Bivariate Random Summation Scheme	455
10.22	Lack of Memory Properties of Bivariate Exponential Distributions	455
10.22.1	Extended Bivariate Lack of Memory Distributions	457
10.23	Effect of Parallel Redundancy with Dependent Exponential Components	457
10.23.1	Mean Lifetime under Gumbel’s Type I Bivariate Exponential Distribution	458
10.24	Stress–Strength Model and Bivariate Exponential Distributions	459
10.24.1	Basic Idea	459
10.24.2	Marshall and Olkin’s Model	460
10.24.3	Downton’s Model	460
10.24.4	Two Dependent Components Subjected to a Common Stress	460
10.24.5	A Component Subjected to Two Stresses	461
10.25	Bivariate Weibull Distributions	461
10.25.1	Marshall and Olkin (1967)	462
10.25.2	Lee (1979)	462
10.25.3	Lu and Bhattacharyya (1990): I	463
10.25.4	Farlie–Gumbel–Morgenstern System	463
10.25.5	Lu and Bhattacharyya (1990): II	463
10.25.6	Lee (1979): II	464
10.25.7	Comments	464
10.25.8	Applications	464
10.25.9	Gamma Frailty Bivariate Weibull Models	465
10.25.10	Bivariate Mixture of Weibull Distributions	465
10.25.11	Bivariate Generalized Exponential Distribution	466
	References	466
11	Bivariate Normal Distribution	477
11.1	Introduction	477
11.2	Basic Formulas and Properties	479

11.2.1	Notation	479
11.2.2	Support	479
11.2.3	Formula of the Joint Density	479
11.2.4	Formula of the Cumulative Distribution Function	480
11.2.5	Univariate Properties	481
11.2.6	Correlation Coefficients	481
11.2.7	Conditional Properties	481
11.2.8	Moments and Absolute Moments	481
11.3	Methods of Derivation	482
11.3.1	Differential Equation Method	482
11.3.2	Compounding Method	483
11.3.3	Trivariate Reduction Method	483
11.3.4	Bivariate Central Limit Theorem	483
11.3.5	Transformations of Diffuse Probability Distributions	483
11.4	Characterizations	484
11.5	Order Statistics	486
11.5.1	Linear Combination of the Minimum and the Maximum	487
11.5.2	Concomitants of Order Statistics	487
11.6	Illustrations	489
11.7	Relationships to Other Distributions	489
11.8	Parameter Estimation	490
11.8.1	Estimate and Inference of ρ	491
11.8.2	Estimation Under Censoring	492
11.9	Other Interesting Properties	492
11.10	Notes on Some More Specialized Fields	494
11.11	Applications	494
11.12	Computation of Bivariate Normal Integrals	495
11.12.1	The Short Answer	495
11.12.2	Algorithms—Rectangles	495
11.12.3	Algorithms: Owen's T Function	499
11.12.4	Algorithms: Triangles	502
11.12.5	Algorithms: Wedge-Shaped Domain	503
11.12.6	Algorithms: Arbitrary Polygons	504
11.12.7	Tables	504
11.12.8	Computer Programs	504
11.12.9	Literature Reviews	505
11.13	Testing for Bivariate Normality	505
11.13.1	How Might Bivariate Normality Fail?	506
11.13.2	Outliers	506
11.13.3	Graphical Checks	507
11.13.4	Formal Tests: Univariate Normality	511
11.13.5	Formal Tests: Bivariate Normality	514

- 11.13.6 Tests of Bivariate Normality
 - After Transformation 521
- 11.13.7 Some Comments and Suggestions 522
- 11.14 Distributions with Normal Conditionals 524
- 11.15 Bivariate Skew-Normal Distribution 524
 - 11.15.1 Bivariate Skew-Normal Distribution of Azzalini
and Dalla Valle 524
 - 11.15.2 Bivariate Skew-Normal Distribution of
Sahu et al. 524
 - 11.15.3 Fundamental Bivariate Skew-Normal Distributions 526
 - 11.15.4 Review of Bivariate Skew-Normal Distributions . . 526
- 11.16 Univariate Transformations 526
 - 11.16.1 The Bivariate Lognormal Distribution 526
 - 11.16.2 Johnson's System 528
 - 11.16.3 The Uniform Representation 530
 - 11.16.4 The g and h Transformations 530
 - 11.16.5 Effect of Transformations on Correlation 530
- 11.17 Truncated Bivariate Normal Distributions 532
 - 11.17.1 Properties 532
 - 11.17.2 Application to Selection Procedures 533
 - 11.17.3 Truncation Scheme of Arnold et al. (1993) 535
 - 11.17.4 A Random Right-Truncation Model of Gürler 535
- 11.18 Bivariate Normal Mixtures 536
 - 11.18.1 Construction 536
 - 11.18.2 References to Illustrations 536
 - 11.18.3 Generalization and Compounding 537
 - 11.18.4 Properties of a Special Case 537
 - 11.18.5 Estimation of Parameters 537
 - 11.18.6 Estimation of Correlation Coefficient for Bivariate
Normal Mixtures 538
 - 11.18.7 Tests of Homogeneity in Normal
Mixture Models 539
 - 11.18.8 Sharpening a Scatterplot 539
 - 11.18.9 Digression Analysis 540
 - 11.18.10 Applications 540
 - 11.18.11 Bivariate Normal Mixing with
Bivariate Lognormal 541
- 11.19 Nonbivariate Normal Distributions with Normal Marginals 541
 - 11.19.1 Simple Examples with Normal Marginals 541
 - 11.19.2 Normal Marginals with Linear Regressions 542
 - 11.19.3 Linear Combinations of Normal Marginals 542
 - 11.19.4 Uncorrelated Nonbivariate Normal Distributions
with Normal Marginals 542
- 11.20 Bivariate Edgeworth Series Distribution 543
- 11.21 Bivariate Inverse Gaussian Distribution 543

11.21.1	Formula of the Joint Density	543
11.21.2	Univariate Properties	544
11.21.3	Correlation Coefficients	544
11.21.4	Conditional Properties	544
11.21.5	Derivations	544
11.21.6	References to Illustrations	545
11.21.7	Remarks	545
	References	546
12	Bivariate Extreme-Value Distributions	563
12.1	Preliminaries	563
12.2	Introduction to Bivariate Extreme-Value Distribution	564
12.2.1	Definition	564
12.2.2	General Properties	564
12.3	Bivariate Extreme-Value Distributions in General Forms	565
12.4	Classical Bivariate Extreme-Value Distributions with Gumbel Marginals	566
12.4.1	Type A Distributions	566
12.4.2	Type B Distributions	568
12.4.3	Type C Distributions	570
12.4.4	Representations of Bivariate Extreme-Value Distributions with Gumbel Marginals	571
12.5	Bivariate Extreme-Value Distributions with Exponential Marginals	572
12.5.1	Pickands' Dependence Function	572
12.5.2	Properties of Dependence Function A	573
12.5.3	Differentiable Models	573
12.5.4	Nondifferentiable Models	574
12.5.5	Tawn's Extension of Differentiable Models	574
12.5.6	Negative Logistic Model of Joe	575
12.5.7	Normal-Like Bivariate Extreme-Value Distributions	576
12.5.8	Correlations	576
12.6	Bivariate Extreme-Value Distributions with Fréchet Marginals	577
12.6.1	Bilogistic Distribution	577
12.6.2	Negative Bilogistic Distributions	578
12.6.3	Beta-Like Extreme-Value Distribution	578
12.7	Bivariate Extreme-Value Distributions with Weibull Marginals	579
12.7.1	Formula of the Cumulative Distribution Function	579
12.7.2	Univariate Properties	579
12.7.3	Formula of the Joint Density	579
12.7.4	Fisher Information Matrix	580
12.7.5	Remarks	580

- 12.8 Methods of Derivation 580
- 12.9 Estimation of Parameters 581
- 12.10 References to Illustrations 581
- 12.11 Generation of Random Variates 581
 - 12.11.1 Shi et al.'s (1993) Method 581
 - 12.11.2 Ghoudi et al.'s (1998) Method 582
 - 12.11.3 Nadarajah's (1999) Method 582
- 12.12 Applications 582
 - 12.12.1 Applications to Natural Environments 582
 - 12.12.2 Financial Applications 584
 - 12.12.3 Other Applications 584
- 12.13 Conditionally Specified Gumbel Distributions 584
 - 12.13.1 Bivariate Model Without Having
Gumbel Marginals 585
 - 12.13.2 Nonbivariate Extreme-Value Distributions with
Gumbel Marginals 586
 - 12.13.3 Positive or Negative Correlation 587
 - 12.13.4 Fields of Applications 587
- References 588

- 13 Elliptically Symmetric Bivariate and Other Symmetric
Distributions 591**
 - 13.1 Introduction 591
 - 13.2 Elliptically Contoured Bivariate Distributions: Formulations 592
 - 13.2.1 Formula of the Joint Density 592
 - 13.2.2 Alternative Definition 593
 - 13.2.3 Another Stochastic Representation 593
 - 13.2.4 Formula of the Cumulative Distribution 594
 - 13.2.5 Characteristic Function 595
 - 13.2.6 Moments 595
 - 13.2.7 Conditional Properties 596
 - 13.2.8 Copulas of Bivariate Elliptical Distributions 596
 - 13.2.9 Correlation Coefficients 596
 - 13.2.10 Fisher Information 596
 - 13.2.11 Local Dependence Functions 597
 - 13.3 Other Properties 597
 - 13.4 Elliptical Compound Bivariate
Normal Distributions 598
 - 13.5 Examples of Elliptically and Spherically Symmetric
Bivariate Distributions 599
 - 13.5.1 Bivariate Normal Distribution 599
 - 13.5.2 Bivariate *t*-Distribution 599
 - 13.5.3 Kotz-Type Distribution 599
 - 13.5.4 Bivariate Cauchy Distribution 599
 - 13.5.5 Bivariate Pearson Type II Distribution 600

13.5.6	Symmetric Logistic Distribution	600
13.5.7	Bivariate Laplace Distribution	600
13.5.8	Bivariate Power Exponential Distributions	600
13.6	Extremal Type Elliptical Distributions	601
13.6.1	Kotz-Type Elliptical Distribution	602
13.6.2	Fréchet-Type Elliptical Distribution	604
13.6.3	Gumbel-Type Elliptical Distribution	605
13.7	Tests of Spherical and Elliptical Symmetry	607
13.8	Extreme Behavior of Bivariate Elliptical Distributions	607
13.9	Fields of Application	608
13.10	Bivariate Symmetric Stable Distributions	608
13.10.1	Explanations	608
13.10.2	Characteristic Function	608
13.10.3	Probability Densities	609
13.10.4	Association Parameter	609
13.10.5	Correlation Coefficients	609
13.10.6	Remarks	610
13.10.7	Application	610
13.11	Generalized Bivariate Symmetric Stable Distributions	611
13.11.1	Characteristic Functions	611
13.11.2	de Silva and Griffith's Class	611
13.11.3	A Subclass of de Silva's Stable Distribution	612
13.12	α -Symmetric Distribution	612
13.13	Other Symmetric Distributions	613
13.13.1	l_p -Norm Symmetric Distributions	613
13.13.2	Bivariate Liouville Family	613
13.13.3	Bivariate Linnik Distribution	613
13.14	Bivariate Hyperbolic Distribution	614
13.14.1	Formula of the Joint Density	614
13.14.2	Univariate Properties	614
13.14.3	Derivation	615
13.14.4	References to Illustrations	615
13.14.5	Remarks	615
13.14.6	Fields of Application	616
13.15	Skew-Elliptical Distributions	616
13.15.1	Bivariate Skew-Normal Distributions	617
13.15.2	Bivariate Skew t -Distributions	617
13.15.3	Bivariate Skew-Cauchy Distribution	618
13.15.4	Asymmetric Bivariate Laplace Distribution	618
13.15.5	Applications	618
	References	619

14	Simulation of Bivariate Observations	623
14.1	Introduction	623
14.2	Common Approaches in the Univariate Case	624
14.2.1	Introduction	624
14.2.2	Inverse Probability Integral Transform	625
14.2.3	Composition	625
14.2.4	Acceptance/Rejection	626
14.2.5	Ratio of Uniform Variates	626
14.2.6	Transformations	627
14.2.7	Markov Chain Monte Carlo—MCMC	627
14.3	Simulation from Some Specific Univariate Distributions	628
14.3.1	Normal Distribution	628
14.3.2	Gamma Distribution	629
14.3.3	Beta Distribution	630
14.3.4	<i>t</i> -Distribution	630
14.3.5	Weibull Distribution	631
14.3.6	Some Other Distributions	631
14.4	Software for Random Number Generation	631
14.4.1	Random Number Generation in IMSL Libraries	632
14.4.2	Random Number Generation in S-Plus and R	632
14.5	General Approaches in the Bivariate Case	632
14.5.1	Setting	633
14.5.2	Conditional Distribution Method	633
14.5.3	Transformation Method	634
14.5.4	Gibbs' Method	634
14.5.5	Methods Reflecting the Distribution's Construction	635
14.6	Bivariate Normal Distribution	635
14.7	Simulation of Copulas	637
14.8	Simulating Bivariate Distributions with Simple Forms	638
14.8.1	Bivariate Beta Distribution	638
14.9	Bivariate Exponential Distributions	639
14.9.1	Marshall and Olkin's Bivariate Exponential Distribution	639
14.9.2	Gumbel's Type I Bivariate Exponential Distribution	639
14.10	Bivariate Gamma Distributions and Their Extensions	639
14.10.1	Cherian's Bivariate Gamma Distribution	639
14.10.2	Kibble's Bivariate Gamma Distribution	640
14.10.3	Becker and Roux's Bivariate Gamma	640
14.10.4	Bivariate Gamma Mixture of Jones et al.	640
14.11	Simulation from Conditionally Specified Distributions	640

14.12	Simulation from Elliptically Contoured Bivariate Distributions	641
14.13	Simulation of Bivariate Extreme-Value Distributions	642
14.13.1	Method of Shi et al.	642
14.13.2	Method of Ghoudi et al.	642
14.13.3	Method of Nadarajah	643
14.14	Generation of Bivariate and Multivariate Skewed Distributions	643
14.15	Generation of Bivariate Distributions with Given Marginals	643
14.15.1	Background	643
14.15.2	Weighted Linear Combination and Trivariate Reduction	644
14.15.3	Schmeiser and Lal's Methods	645
14.15.4	Cubic Transformation of Normals	646
14.15.5	Parrish's Method	646
14.16	Simulating Bivariate Distributions with Specified Correlations	646
14.16.1	Li and Hammond's Method for Distributions with Specified Correlations	646
14.16.2	Generating Bivariate Uniform Distributions with Prescribed Correlation Coefficients	647
14.16.3	The Mixture Approach for Simulating Bivariate Distributions with Specified Correlations	647
	References	648
	Author Index	655
	Subject Index	667