Chapter 2

Representations of Finite Groups

In mathematics and physics, the notion of a group representation is fundamental. The idea is to study the different ways that groups can act on vector spaces by *linear transformations.*

In this chapter, unless otherwise indicated, we shall consider only representations of finite *groups in* complex, finite-dimensional *vector spaces.*

1 Representations

1.1 General Facts

Let G be a finite group. If E is a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\mathrm{GL}(E)$ the group of $\mathbb{K}$-linear isomorphisms of E. (The group $\mathrm{GL}(E)$ is not finite unless $E = \{0\}$.)

Definition 1.1. *A* representation *of a group G is a finite-dimensional complex vector space E along with a group morphism of groups $\rho : G \to \mathrm{GL}(E)$.*

Thus, for every $g, g' \in G$,

$$\rho(gg') = \rho(g)\rho(g'), \quad \rho(g^{-1}) = (\rho(g))^{-1}, \quad \rho(e) = \mathrm{Id}_E.$$

The vector space E is called the *support* of the representation, and the dimension of E is called the *dimension of the representation*. We denote such a representation by (E, ρ) or simply ρ.

If in particular $E = \mathbb{C}^n$, we say that the representation is a *matrix representation* of dimension n.

The *fundamental representation* of a subgroup G of $\mathrm{GL}(E)$ is the representation of G on E defined by the canonical injection of G into $\mathrm{GL}(E)$.

Any representation such that $\rho(g) = \mathrm{Id}_E$ for each $g \in G$ is called a *trivial representation.*

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Example 1.2. Here is a first example of a representation of a nonabelian group. Let $t \in \mathfrak{S}_3$ be the transposition $123 \mapsto 132$ and c the cyclic permutation $123 \mapsto 231$ that generate $\mathfrak{S}_3$. We set $j = e^{2i\pi/3}$, so that $j^2 + j + 1 = 0$. We can represent $\mathfrak{S}_3$ on $\mathbb{C}^2$ by defining

$$\rho(e) = I, \quad \rho(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}.$$

Definition 1.3. *Let $(\ |\ )$ be a scalar product on E. We say that the representation ρ is* unitary *if $\rho(g)$ is unitary for every g, that is, if*

$$\forall g \in G,\ \forall x, y \in E, \quad (\rho(g)x \mid \rho(g)y) = (x \mid y).$$

A representation (E, ρ) is called *unitarizable* if there is a scalar product on E such that ρ is unitary.

In order to prove the following theorem, as well as many other propositions, we shall use a fundamental property:

Lemma 1.4. *Let G be a finite group. For every function φ on G taking values in a vector space,*

$$\forall g \in G, \quad \sum_{h \in G} \varphi(gh) = \sum_{h \in G} \varphi(hg) = \sum_{k \in G} \varphi(k). \tag{1.1}$$

Proof. In fact, once g is chosen, every element of G can be written uniquely in the form gh (or hg), where $h \in G$. □

Theorem 1.5. *Every representation of a finite group is unitarizable.*

Proof. Let (E, ρ) be a representation of a finite group G, and let $(\ \mid\ )$ be a scalar product on E. We consider

$$(x \mid y)' = \frac{1}{|G|} \sum_{g \in G} (\rho(g)x \mid \rho(g)y),$$

which is a scalar product on E. In fact, suppose that $(x \mid x)' = 0$, that is, $\sum_{g \in G} (\rho(g)x \mid \rho(g)x) = 0$. Then for each $g \in G$, $(\rho(g)x \mid \rho(g)x) = 0$, and in particular, $(x \mid x) = 0$, whence $x = 0$.

This scalar product on E is invariant under ρ. In fact,

$$\begin{aligned}(\rho(g)x \mid \rho(g)y)' &= \frac{1}{|G|} \sum_{h \in G} (\rho(h)\rho(g)x \mid \rho(h)\rho(g)y) \\ &= \frac{1}{|G|} \sum_{h \in G} (\rho(hg)x \mid \rho(hg)y) = (x \mid y)',\end{aligned}$$

where we have used the fundamental equation (1.1), which holds for any function φ on G. Thus ρ is a unitary representation of G on $(E, (\ \mid\ )')$. □

1.2 Irreducible Representations

Let (E, ρ) be a representation of G. A vector subspace $F \subset E$ is called *invariant* (or *stable*) under ρ (or under G, if the name of the representation is understood) if for every $g \in G$, $\rho(g)F \subset F$. (since F is finite-dimensional, the condition $\rho(g)F \subset F$ implies $\rho(g)F = F$.) We can then speak of the representation ρ restricted to F, which is a representation of G on F. We denote it by $\rho|_F$. Such a representation restricted to an invariant subspace is also called a *subrepresentation*.

Definition 1.6. *A representation (E, ρ) of G is called* irreducible *if $E \neq \{0\}$ and if the only vector subspaces of E invariant under ρ are $\{0\}$ and E itself.*

Example. The representation of dimension 2 of $\mathfrak{S}_3$ defined in Example 1.2 is irreducible, since the eigenspaces of $\rho(t)$ and $\rho(c)$ have trivial intersection.

Proposition 1.7. *Every irreducible representation of a finite group is finite-dimensional.*

Proof. Let (E, ρ) be an irreducible representation of a finite group G and let $x \in E$. Because the subset $\{\rho(g)x \mid g \in G\}$ is finite, it generates a finite-dimensional vector subspace of E. If $x \neq 0$, this vector subspace of E is not equal to $\{0\}$. Because this subspace is invariant under ρ, it coincides with E, which is thus finite-dimensional. □

1.3 Direct Sum of Representations

Definition 1.8. *Let (E_1, ρ_1) and (E_2, ρ_2) be representations of G. Then*

$$(E_1 \oplus E_2, \rho_1 \oplus \rho_2),$$

where $(\rho_1 \oplus \rho_2)(g)(x_1, x_2) = (\rho_1(g)(x_1), \rho_2(g)(x_2))$, for $g \in G, x_1 \in E_1, x_2 \in E_2$, is a representation of G called the direct sum *of the representations (E_1, ρ_1) and (E_2, ρ_2).*

Clearly a direct sum of representations of strictly positive dimensions cannot be irreducible, even if the summands are irreducible. For matrix representations ρ_1 and ρ_2, the matrices of the direct sum representation of ρ_1 and ρ_2 are block-diagonal matrices

$$\begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

More generally, if m is a strictly positive integer, we can use recursion to define the direct sum of m representations $\rho_1 \oplus \cdots \oplus \rho_m$. If (E, ρ) is a representation of G we denote by $m\rho$ the representation $\rho \oplus \cdots \oplus \rho$ (direct sum of m terms) on the vector space $E \oplus \cdots \oplus E$ (m terms).

A representation is called *completely reducible* if it is a direct sum of irreducible representations.

Lemma 1.9. *Let ρ be a unitary representation of G on $(E, (\ \mid\ ))$. If $F \subset E$ is invariant under ρ, then $F^\perp = \{y \in E \mid \forall x \in F, (x \mid y) = 0\}$ is also invariant under ρ.*

Proof. Let $y \in F^\perp$. Then, because F is invariant under ρ, for every $g \in G$ and $x \in F, (x \mid \rho(g)y) = (\rho(g^{-1})x \mid y) = 0$. Thus $\rho(g)y \in F^\perp$. □

Theorem 1.10 (Maschke's Theorem). *Every finite-dimensional representation of a finite group is completely reducible.*

Proof. Let (E, ρ) be be a representation of G. By Theorem 1.5, one may suppose this representation to be unitary. If ρ is not irreducible, let F be a vector subspace of E invariant under ρ such that $F \neq \{0\}$ and $F \neq E$. Then $E = F \oplus F^\perp$, where F (by hypothesis) and $F^\perp$ (by Lemma 1.9) are invariant under ρ, and $\dim F < \dim E$, $\dim F^\perp < \dim E$. By induction on the dimension of E, we obtain the desired result. □

In fact, this theorem is true under more general conditions. (See the study of compact groups in Chapter 3.)

1.4 Intertwining Operators and Schur's Lemma

Definition 1.11. *Let (E_1, ρ_1) and (E_2, ρ_2) be representations of G. We say that a linear map $T : E_1 \to E_2$* intertwines *ρ_1 and ρ_2 if*

$$\forall g \in G,\ \ \rho_2(g) \circ T = T \circ \rho_1(g),$$

in which case T is called an intertwining operator *for ρ_1 and ρ_2.*

The definition can be expressed in the commutativity of the following diagram for each $g \in G$:

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ E_1 & \xrightarrow{T} & E_2 \end{array}$$

The following expressions are often used to express the same property:

- T is equivariant under ρ_1 and ρ_2,
- T is a morphism of G-vector spaces,
- T is a G-morphism,
- $T \in \mathrm{Hom}_G(E_1, E_2)$.

If $E_1 = E_2 = E$ and if $\rho_1 = \rho_2 = \rho$, an intertwining operator for ρ_1 and ρ_2 is just an operator that *commutes* with ρ.

Definition 1.12. *The representations ρ_1 and ρ_2 are called* equivalent *if there is a* bijective *intertwining operator for ρ_1 and ρ_2.*

If T is such a bijective intertwining operator, then

$$\forall g \in G,\ \rho_2(g) = T \circ \rho_1(g) \circ T^{-1}.$$

The existence of an intertwining operator is an equivalence relation on representations, which leads to the notion of an *equivalence class of representations.* We let $\sim$ denote this equivalence relation.

Two representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent if and only if there is a basis B_1 of E_1 and a basis B_2 of E_2 such that for every $g \in G$, the matrix of $\rho_1(g)$ in the basis B_1 is equal to the matrix of $\rho_2(g)$ in the basis B_2. In particular, if the representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent, then E_1 is isomorphic to E_2.

For equivalent matrix representations, we thus obtain *similar matrices*: if $E_1 = E_2 = \mathbb{C}^n$, and if ρ_1 and ρ_2 are equivalent, then the matrices $\rho_1(g)$ and $\rho_2(g)$ are similar via the same similarity matrix for every g.

If ρ_0 is an n-dimensional representation of G on E, the choice of a basis (e_i) of E determines a matrix representation $(\mathbb{C}^n, \rho)$; by changing to the basis (e_i') via a matrix T, one obtains the equivalent representation $(\mathbb{C}^n, \rho')$,

$$\rho'(g) = T \circ \rho(g) \circ T^{-1}.$$

Lemma 1.13. *If T intertwines ρ_1 and ρ_2, then the kernel of T,* Ker T*, is invariant under ρ_1, and the image of T,* ImT*, is invariant under ρ_2.*

Proof. If $x \in E_1$ and $Tx = 0$, then $T(\rho_1(g)x) = \rho_2(g)(Tx) = 0$. Thus Ker T is a subspace of E_1 invariant under ρ_1.

Let $y \in$ Im T. Then, there exists $x \in E_1$ such that $y = Tx$. Therefore $\rho_2(g)y = \rho_2(g)(Tx) = T(\rho_1(g)x)$, and hence Im T is a subspace of E_2 invariant under ρ_2. $\square$

Lemma 1.14. *If T commutes with ρ, each eigenspace of T is invariant under ρ.*

Proof. In fact, if $Tx = \lambda x, \lambda \in \mathbb{C}$, then $T(\rho(g)x) = \lambda\rho(g)x$. Thus the eigenspace of T corresponding to the eigenvalue λ is invariant under ρ. $\square$

Theorem 1.15 (Schur's Lemma). *Let T be an operator intertwining irreducible representations (E_1, ρ_1) and (E_2, ρ_2) of G.*

- *If ρ_1 and ρ_2 are not equivalent, then $T = 0$.*
- *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then T is a scalar multiple of the identity of E.*

Proof. If ρ_1 and ρ_2 are not equivalent, T is not bijective. Hence either Ker$T \neq \{0\}$, or Im $T \neq E_2$. By Lemma 1.13, Ker T is invariant under ρ_1. Because ρ_1 is irreducible, if Ker $T \neq \{0\}$, then Ker $T = E_1$; hence $T = 0$. By Lemma 1.13, Im T is invariant under ρ_2. Because ρ_2 is irreducible, if Im $T \neq E_2$, then Im $T = \{0\}$, and hence $T = 0$.

If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then for every $g \in G, \rho(g) \circ T = T \circ \rho(g)$, and T commutes with the representation ρ. Let λ be an eigenvalue of T, which must exist because T is an endomorphism of E, a vector space over $\mathbb{C}$, and let

E_λ be the eigenspace associated to λ. By Lemma 1.14, E_λ is invariant under ρ. By hypothesis $E_\lambda \neq \{0\}$, therefore, since ρ is irreducible, $E_\lambda = E$, which means that $T = \lambda\ \mathrm{Id}_E$. We remark that the proof of the second part of the theorem uses the hypothesis that the vector space of the representation is a complex vector space. □

Conversely, if each operator commuting with the representation ρ is a scalar multiple of the identity, then ρ is irreducible. In fact, if ρ were not irreducible, the projection onto a nontrivial invariant subspace would be a nonscalar operator commuting with ρ.

Remark. Lemma 1.14 has very important consequences in quantum mechanics. The symmetry operators of a system represented by a Hamiltonian $\widehat{H}$ (an operator acting on a Hilbert space) are precisely the operators that commute with $\widehat{H}$. For each energy level, that is, for each eigenvalue of the Hamiltonian, there is a corresponding eigenspace. By this lemma, each eigenspace is the support of a representation of the group of symmetries of the system. *Wigner's principle* then states that for each energy level, the corresponding representation is an irreducible representation of the full symmetry group of the system. The dimension of the representation corresponding to the given energy level is called the degree of *degeneracy* of the energy level.

2 Characters and Orthogonality Relations

2.1 Functions on a Group, Matrix Coefficients

We shall denote by $\mathcal{F}(G)$, or sometimes by $\mathbb{C}[G]$, the vector space of functions on G taking values in $\mathbb{C}$. When this vector space is equipped with the scalar product defined below, we call the resulting Hilbert space $L^2(G)$. (This definition will be extended to compact groups.)

We adopt the convention that a *scalar product* is antilinear in the first argument and linear in the second.

Definition 2.1. *On $L^2(G)$, the scalar product is defined by*

$$(f_1 \mid f_2) = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

We shall be interested in the matrix coefficients of representations.

Definition 2.2. *If ρ is a representation of G on $\mathbb{C}^n$, then for every ordered pair (i,j), $1 \leq i \leq n$, $1 \leq j \leq n$, the function $\rho_{ij} \in L^2(G)$ defined for each $g \in G$ to be the coefficient of the matrix $\rho(g)$ in the ith row and the jth column, $(\rho(g))_{ij} \in \mathbb{C}$, is called a* matrix coefficient *of ρ.*

For a representation ρ on a vector space E, we define the matrix coefficients ρ_{ij} relative to a basis (e_i) satisfying

$$\rho(g) e_j = \sum_i \rho_{ij}(g) e_i,$$

where i is the row index and j is the column index. If ρ is a unitary representation on a finite-dimensional Hilbert space, then

$$\rho(g^{-1}) = (\rho(g))^{-1} = {}^t\overline{(\rho(g))}.$$

Hence, in an orthonormal basis,

$$\rho_{ij}(g^{-1}) = \overline{\rho_{ji}(g)},$$

and in particular, the diagonal coefficients of $\rho(g)$ and $\rho(g^{-1})$ are complex conjugates.

2.2 Characters of Representations and Orthogonality Relations

We denote by Tr the trace of an endomorphism.

Definition 2.3. *Let (E, ρ) be a representation of G. The character of ρ is the function χ_ρ on G taking complex values defined by*

$$\forall g \in G,\ \ \chi_\rho(g) = \mathrm{Tr}\,(\rho(g)).$$

Equivalent representations have the same character.

For a matrix representation of dimension n,

$$\chi_\rho(g) = \sum_{i=1}^{n} (\rho(g))_{ii}. \tag{2.1}$$

On each conjugacy class of G, the function χ_ρ is constant.

Definition 2.4. *A* class function *on G is a function constant on each conjugacy class.*

Thus characters of representations are class functions on the group.

Proposition 2.5. *The following are elementary properties of characters:*

- $\chi_\rho(e) = \dim \rho$.
- $\forall g \in G,\ \ \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.
- *The character of a direct sum of representations is the sum of the characters,* $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.

Proof. The first property is a consequence of formula (2.1). To prove the second formula, we may assume that ρ is unitary in a certain scalar product and choose an orthonormal basis. The direct sum property is obvious. □

If (E_1, ρ_1) and (E_2, ρ_2) are representations of the same group G, we define their *tensor product* to be $(E_1 \otimes E_2, \rho_1 \otimes \rho_2)$, where

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g),$$

for each $g \in G$. (See Exercise 2.5 for a review of the relevant definitions.) The following is an important property of characters.

Proposition 2.6. *The character of a tensor product of representations is the product of the characters,*

$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\ \chi_{\rho_2}. \tag{2.2}$$

Proof. The equality follows from the fact that the trace of a tensor product of matrices is the product of the traces. $\square$

By Proposition 2.5, for representations ρ_1 and ρ_2 of G,

$$(\chi_{\rho_1} \mid \chi_{\rho_2}) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(g). \tag{2.3}$$

We shall show that the characters of inequivalent irreducible representations are orthogonal and that the character of an irreducible representation is of norm 1.

Proposition 2.7. *Let (E_1, ρ_1) and (E_2, ρ_2) be representations of G and let $u : E_1 \to E_2$ be a linear map. Then the linear map $T_u : E_1 \to E_2$ defined by*

$$T_u = \frac{1}{|G|} \sum_{g \in G} \rho_2(g)\ u\ \rho_1(g)^{-1} \tag{2.4}$$

intertwines ρ_1 and ρ_2.

Proof. We calculate

$$\begin{aligned} \rho_2(g)\ T_u &= \frac{1}{|G|} \sum_{h \in G} \rho_2(gh)\ u\ \rho_1(h^{-1}) \\ &= \frac{1}{|G|} \sum_{k \in G} \rho_2(k)\ u\ \rho_1(k^{-1}g), \end{aligned}$$

by the fundamental equation (1.1). Hence,

$$\rho_2(g)\ T_u = T_u\ \rho_1(g).$$

The operator T_u is thus an intertwining operator for ρ_1 and ρ_2. $\square$

Proposition 2.8. *Let (E_1, ρ_1) and (E_2, ρ_2) be irreducible representations of G, let $u : E_1 \to E_2$ be a linear map, and define T_u by equation* (2.4)*.*

(*i*) *If ρ_1 and ρ_2 are inequivalent, then $T_u = 0$.*
(*ii*) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then*

$$T_u = \frac{\operatorname{Tr}\ u}{\dim E}\ \operatorname{Id}_E.$$

Proof. The first assertion is clear by Schur's lemma (Theorem 1.15). For the second, we need only calculate λ given that $T_u = \lambda \operatorname{Id}_E$. So we obtain $\operatorname{Tr} T_u = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} u = \operatorname{Tr} u$, and thus $\lambda = \frac{\operatorname{Tr} u}{\dim E}$. $\square$

Proposition 2.9. *Let (E_1, ρ_1) and (E_2, ρ_2) be irreducible representations of G. We choose bases in E_1 and E_2.*

(*i*) *If ρ_1 and ρ_2 are inequivalent, then*

$$\forall i, j, k, \ell, \sum_{g \in G} (\rho_2(g))_{k\ell} (\rho_1(g^{-1}))_{ji} = 0.$$

(*ii*) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then*

$$\frac{1}{|G|} \sum_{g \in G} (\rho(g))_{k\ell} (\rho(g^{-1}))_{ji} = \frac{1}{\dim E} \delta_{ki} \delta_{\ell j}.$$

Proof. We use a basis (e_j) of E_1, $1 \leq j \leq \dim E_1$, and a basis (f_ℓ) of E_2, $1 \leq \ell \leq \dim E_2$. For $u : E_1 \to E_2$, T_u is defined by (2.4). We have, for $1 \leq i \leq \dim E_1, 1 \leq k \leq \dim E_2$,

$$(T_u)_{ki} = \frac{1}{|G|} \sum_{g \in G} \sum_{m=1}^{\dim E_1} \sum_{p=1}^{\dim E_2} (\rho_2(g))_{kp} \; u_{pm} \; (\rho_1(g^{-1}))_{mi}.$$

Let us choose our linear map u to be the map $u_{(\ell j)} : E_1 \to E_2$ defined by $u_{(\ell j)}(e_k) = \delta_{jk} f_\ell$. Then

$$(u_{(\ell j)})_{pm} = \delta_{\ell p} \delta_{jm},$$

and consequently,

$$(T_{u_{(\ell j)}})_{ki} = \frac{1}{|G|} \sum_{g \in G} (\rho_2(g))_{k\ell} (\rho_1(g^{-1}))_{ji}.$$

Next we apply Proposition 2.8. If ρ_1 and ρ_2 are inequivalent, then $T_{u_{(\ell j)}}$ is always zero, whence (i). If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then

$$\frac{1}{|G|} \sum_{g \in G} (\rho(g))_{k\ell} (\rho(g^{-1}))_{ji} = (T_{u_{(\ell j)}})_{ki} = \frac{\operatorname{Tr} u_{(\ell j)}}{\dim E} \delta_{ki} = \frac{\delta_{ki} \delta_{\ell j}}{\dim E},$$

which proves (ii). □

Corollary 2.10. *Let (E_1, ρ_1) and (E_2, ρ_2) be unitary irreducible representations of G. We choose orthonormal bases in E_1 and E_2.*

(*i*) *If ρ_1 and ρ_2 are inequivalent, then for every i, j, k, l,*

$$((\rho_1)_{ij} \mid (\rho_2)_{k\ell}) = 0.$$

(*ii*) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then for every i, j, k, l,*

$$(\rho_{ij} \mid \rho_{k\ell}) = \frac{1}{\dim E} \delta_{ik} \delta_{j\ell}.$$

Proof. In fact, if ρ_1 is unitary for a scalar product on E_1 and if the chosen basis in E_1 is orthonormal, then

$$\frac{1}{|G|}\sum_{g\in G}(\rho_2(g))_{k\ell}(\rho_1(g^{-1}))_{ji} = \frac{1}{|G|}\sum_{g\in G}(\rho_2(g))_{k\ell}\overline{(\rho_1(g))_{ij}} = ((\rho_1)_{ij} \mid (\rho_2)_{k\ell}).$$

Proposition 2.9 thus implies (i) and (ii). □

Theorem 2.11 (Orthogonality Relations). *Let G be a finite group.*

(*i*) *If ρ_1 and ρ_2 are inequivalent irreducible representations of G, then*

$$(\chi_{\rho_1} \mid \chi_{\rho_2}) = 0.$$

(*ii*) *If ρ is an irreducible representation of G, then*

$$(\chi_\rho \mid \chi_\rho) = 1.$$

Proof. By the equality (2.3) and the preceding proposition, if ρ_1 and ρ_2 are inequivalent irreducible representations, then $(\chi_{\rho_1} \mid \chi_{\rho_2}) = 0$. If $\rho_1 = \rho_2 = \rho$, then $\frac{1}{|G|}\sum_{g\in G}\rho(g)_{ii}\rho(g^{-1})_{jj} = \frac{\delta_{ij}}{\dim E}$, whence $(\chi_\rho \mid \chi_\rho) = 1$. □

We define the *irreducible characters* of G to be the set of characters of inequivalent irreducible representations of G. We write χ_{ρ_i} or even χ_i to denote the character of an irreducible representation ρ_i. The preceding results can be formulated as follows.

Theorem 2.12. *The irreducible characters of G form an orthonormal set in $L^2(G)$.*

Corollary 2.13. *The inequivalent irreducible representations of a finite group G are finite in number.*

We shall denote by $\widehat{G}$ the set of equivalence classes of irreducible representations of G.

2.3 Character Table

"Character table" is the name given to the table whose columns correspond to conjugacy classes of a group and whose rows correspond to inequivalent irreducible representations of the group. At the intersection of the row and the column one writes the value of the character of the representation, evaluated on an element (any element) of the conjugacy class. Let N be the number of conjugacy classes of the group G. (In other words, N is the number of columns; we shall show that it is also the number of rows.) Let g_i be an element of G in the conjugacy class C_{g_i}, $1 \le i \le N$, which consists of $|C_{g_i}|$ elements. Let ρ_k and ρ_ℓ be irreducible representations of G. Then

$$(\chi_{\rho_k} \mid \chi_{\rho_\ell}) = \frac{1}{|G|}\sum_{i=1}^{N}|C_{g_i}|\;\overline{\chi_{\rho_k}(g_i)}\;\chi_{\rho_\ell}(g_i) = \delta_{k\ell}.$$

This formula can be restated as the following result.

Proposition 2.14. *If the ith column is given weight $|C_{g_i}|$, the rows of the character table are orthogonal and of norm $\sqrt{|G|}$.*

We write character tables in the following form:

	$\lvert C_{g_1}\rvert$	$\ldots\ldots$	$\lvert C_{g_N}\rvert$
	g_1	$\ldots\ldots$	g_N
$\ldots$	$\ldots$	$\ldots\ldots$	$\ldots$
χ_{ρ_k}	$\chi_{\rho_k}(g_1)$	$\ldots\ldots$	$\chi_{\rho_k}(g_N)$
$\ldots$	$\ldots$	$\ldots\ldots$	$\ldots$
χ_{ρ_ℓ}	$\chi_{\rho_\ell}(g_1)$	$\ldots\ldots$	$\chi_{\rho_\ell}(g_N)$
$\ldots$	$\ldots$	$\ldots\ldots$	$\ldots$

2.4 Application to the Decomposition of Representations

We denote by $\rho_1, \ldots, \rho_N$ the inequivalent irreducible representations of G. (We shall see in Corollary 3.7 that this number N equals the number of conjugacy classes of G.) More precisely, we choose from each equivalence class of representations of G a representative that we denote by ρ_i.

In the equalities below, the equal sign denotes membership in the same equivalence class.

Theorem 2.15. *Let ρ be any representation of G and let χ_ρ be its character. Then*

$$\rho = \bigoplus_{i=1}^{N} m_i \rho_i,$$

where

$$m_i = (\chi_{\rho_i} \mid \chi_\rho).$$

Proof. We know by Theorem 1.10 that ρ is direct sum of irreducible representations. We can group the terms corresponding to the same equivalence class of irreducible representations ρ_i, and we obtain $\rho = \oplus_{i=1}^{N} m_i \rho_i$, for some nonnegative integers m_i. One sees then that $\chi_\rho = \sum_{i=1}^{N} m_i \, \chi_{\rho_i}$, and hence by orthogonality $(\chi_{\rho_i} \mid \chi_\rho) = m_i \, (\chi_{\rho_i} \mid \chi_{\rho_i}) = m_i$. □

Definition 2.16. *If ρ admits the decomposition*

$$\rho = m_1 \rho_1 \oplus m_2 \rho_2 \oplus \cdots \oplus m_N \rho_N,$$

then the nonnegative integer m_i is the multiplicity *of ρ_i in ρ, and $m_i \rho_i$ is the* isotypic component *of type ρ_i of ρ.*

Corollary 2.17. *The decomposition into isotypic components is unique up to order.*

Corollary 2.18. *Two representations with the same character are equivalent.*

By the previous theorem,

$$(\chi_\rho \mid \chi_\rho) = \sum_{i=1}^{N} m_i^2.$$

Hence we have the following result.

Theorem 2.19 (Irreducibility Criterion). *A representation ρ is irreducible if and only if $(\chi_\rho \mid \chi_\rho) = 1$.*

3 The Regular Representation

3.1 Definition

In general, if a group G acts on a set M, then G acts linearly on the space $\mathcal{F}(M)$ of functions on M taking values in $\mathbb{C}$ by $(g, f) \in G \times \mathcal{F}(M) \mapsto g \cdot f \in \mathcal{F}(M)$, where

$$\forall x \in M,\ (g \cdot f)(x) = f(g^{-1}x).$$

We can see immediately that this gives us a representation of G on $\mathcal{F}(M)$.

Take $M = G$, the group acting on itself by left multiplication. One obtains a representation R of G on $\mathcal{F}(G)$ called the *left regular representation* (or simply *regular representation*) of G. Thus, by definition,

$$\forall g, h \in G,\ (R(g)f)(h) = f(g^{-1}h).$$

In the same way one can define the *right regular representation* R', associated to the right action of G on itself, by $(R'(g)f)(h) = f(hg)$. The right and left regular representations are equivalent. For a finite group G the vector space $\mathcal{F}(G)$ of maps of G into $\mathbb{C}$ is finite-dimensional, of dimension $|G|$. The regular representation is thus of dimension $|G|$.

We use the basis $(\epsilon_g)_{g\in G}$ of $\mathcal{F}(G)$ defined by

$$\epsilon_g : G \to \mathbb{C} \begin{cases} \epsilon_g(g) = 1, \\ \epsilon_g(h) = 0, \ \text{if } h \neq g. \end{cases}$$

The regular representation of G satisfies

$$\forall g,\ h \in G,\ R(g)(\epsilon_h) = \epsilon_{gh}.$$

In fact, for every $k \in G$, $(R(g)\epsilon_h)(k) = \epsilon_h(g^{-1}k)$, and $\epsilon_h(g^{-1}k) = 1$ if $k = gh$, while $\epsilon_h(g^{-1}k) = 0$ otherwise. (In the right regular representation, $\epsilon_h \mapsto \epsilon_{hg^{-1}}$.)

Proposition 3.1. *On $L^2(G) = \mathcal{F}(G)$ with scalar product $(\ \mid\ )$, the regular representation is unitary.*

Proof. For f_1 and $f_2 \in L^2(G)$ we have, for every $g \in G$,

$$\begin{aligned}(R(g)f_1 \mid R(g)f_2) &= \frac{1}{|G|}\sum_{h\in G}\overline{(R(g)f_1)(h)}(R(g)f_2)(h)\\ &= \frac{1}{|G|}\sum_{h\in G}\overline{f_1(g^{-1}h)}f_2(g^{-1}h)\\ &= \frac{1}{|G|}\sum_{k\in G}\overline{f_1(k)}f_2(k) = (f_1 \mid f_2).\end{aligned}$$

The operator $R(g)$ is thus unitary for every $g \in G$. □

3.2 Character of the Regular Representation

On the one hand,

$$\chi_R(e) = \mathrm{Tr}(R(e)) = \dim \mathcal{F}(G) = |G|.$$

On the other hand, if $g \neq e$, then

$$\chi_R(g) = \mathrm{Tr}(R(g)) = 0$$

because in this case, for every $h \in G$, $R(g)\epsilon_h \neq \epsilon_h$.

The regular representation R is reducible because $\sum_{g\in G}\epsilon_g$ generates a vector subspace W of $\mathcal{F}(G)$ of dimension 1 that is invariant under R. In fact, for every $g \in G$, $R(g)(\sum_{h\in G}\epsilon_h) = \sum_{h\in G}\epsilon_{gh} = \sum_{k\in G}\epsilon_k$. Furthermore, $R\mid_W$ is equivalent to the trivial representation, since for every $x \in W$, $R(g)(x) = x$. We shall show that, in fact, the regular representation contains each irreducible representation of G with multiplicity equal to its dimension.

Example 3.2. The regular representation of $\mathfrak{S}_3$ on $\mathbb{C}[\mathfrak{S}_3]$ is of dimension 6. It decomposes into the direct sum of the one-dimensional trivial representation, the one-dimensional sign representation, and two copies of the two-dimensional irreducible representation studied in Example 1.2.

3.3 Isotypic Decomposition

We now use the notation introduced in Section 2.4.

Proposition 3.3. *The decomposition of the regular representation of G into isotypic components is $R = \oplus_{i=1}^{N} n_i\rho_i$, where $\rho_i, i = 1,\ldots,N$, are the irreducible representations of G, and $n_i = \dim\rho_i$.*

Proof. We know that

$$\chi_R(g) = \begin{cases}|G| & \text{if } g = e,\\ 0 & \text{if } g \neq e,\end{cases}$$

and hence $(\chi_{\rho_i} \mid \chi_R) = \chi_{\rho_i}(e) = \dim\rho_i$. □

Theorem 3.4. *We have*

$$\sum_{i=1}^{N}(n_i)^2 = |G|,$$

where $n_i = \dim \rho_i$.

Proof. We have $|G| = \chi_R(e) = \sum_{i=1}^{N} n_i \chi_{\rho_i}(e) = \sum_{i=1}^{N}(n_i)^2$. □

The equality $\sum_{i=1}^{N}(n_i)^2 = |G|$ is often used, for example, in order to determine the dimension of a "missing" irreducible representation when one already knows $N-1$ representations.

3.4 Basis of the Vector Space of Class Functions

The vector space of class functions on G taking values in $\mathbb{C}$ has for dimension the number of conjugacy classes of G. We shall show that this is also the number of equivalence classes of irreducible representations.

Let (E, ρ) be be a representation of G, and let f be a function on G. We consider the endomorphism ρ_f of E defined by

$$\rho_f = \sum_{g \in G} f(g)\rho(g). \tag{3.1}$$

Thus, by definition, for every $x \in E$, $\rho_f(x) = \sum_{g \in G} f(g)\rho(g)(x)$.

Lemma 3.5. *The endomorphism* ρ_f *has the following properties:*

(*i*) *If* f *is a class function,* ρ_f *commutes with* ρ.
(*ii*) *If* f *is a class function and if* ρ *is irreducible, then*

$$\rho_f = \frac{|G|(\overline{f} \mid \chi_\rho)}{\dim \rho} Id_E.$$

Proof. For every function f, we have

$$\begin{aligned}\rho_f \circ \rho(g) &= \sum_{h \in G} f(h)\rho(h)\rho(g) = \sum_{h \in G} f(h)\rho(hg) \\ &= \sum_{k \in G} f(kg^{-1})\rho(k) = \sum_{h \in G} f(ghg^{-1})\rho(gh).\end{aligned}$$

If f is assumed to be a class function, we obtain

$$\rho_f \circ \rho(g) = \rho(g) \sum_{h \in G} f(h)\rho(h) = \rho(g) \circ \rho_f.$$

Let us prove (ii). By (i) and Schur's lemma (Theorem 1.15), there is a $\lambda \in \mathbb{C}$ such that $\rho_f = \lambda Id_E$. On the other hand, $\mathrm{Tr}\rho_f = \sum_{g \in G} f(g)\mathrm{Tr}\rho(g) = \sum_{g \in G} f(g)\chi_\rho(g) = |G|(\overline{f} \mid \chi_\rho)$, from which the result follows. □

Theorem 3.6. *The irreducible characters form an orthonormal basis of the vector space of class functions.*

Proof. We know that the characters $\rho_1, \ldots, \rho_N$ of inequivalent irreducible representations of G form an orthonormal set in $L^2(G)$ (Theorem 2.12). Let us show that this set spans the vector subspace of class functions. Let f be a class function such that for $1 \leq i \leq N$, $(f|\chi_{\rho_i}) = 0$. We consider $(\rho_i)_{\overline{f}} = \sum_{g \in G} \overline{f}(g)\rho_i(g)$. By the previous lemma, $(\rho_i)_{\overline{f}} = 0$, and we deduce, by decomposition, that for any representation ρ we have $\rho_{\overline{f}} = 0$. In particular, $R_{\overline{f}} = 0$, where R is the regular representation. Thus,

$$0 = R_{\overline{f}}(\epsilon_g) = \sum_{h \in G} \overline{f}(h)R(h)(\epsilon_g) = \sum_{h \in G} \overline{f}(h)\epsilon_{hg},$$

for $g \in G$, and, in particular,

$$0 = R_{\overline{f}}(\epsilon_e) = \sum_{h \in G} \overline{f}(h)\epsilon_h = \overline{f},$$

so $f = 0$. $\square$

Corollary 3.7. *The number of equivalence classes of irreducible representations of a finite group is equal to the number of conjugacy classes of that group.*

In other words, *the character table is square.*

Proposition 3.8. *The columns of the character table of a finite group G are orthogonal and of norm $\sqrt{|G|/|C_g|}$, where $|C_g|$ denotes the number of elements of the conjugacy class of g. Explicitly,*

$$\sum_{i=1}^{N} \overline{\chi_{\rho_i}(g)}\chi_{\rho_i}(g') = 0, \ \textit{if } g \textit{ and } g' \textit{ are not conjugate,}$$

$$\frac{1}{|G|}\sum_{i=1}^{N} \overline{\chi_{\rho_i}(g)}\chi_{\rho_i}(g) = \frac{1}{|C_g|}.$$

In particular, when $g = e$, we recover the equation $\sum_{i=1}^{N}(\dim \rho_i)^2 = |G|$.

Proof. By Theorem 3.6, if f is a class function, then

$$f = \sum_{i=1}^{N} (\chi_{\rho_i} \mid f)\chi_{\rho_i}.$$

For $g \in G$, consider the class function f_g that takes the value 1 on g and the value 0 on every other conjugacy class of G. We have

$$\begin{aligned}(\chi_{\rho_i} \mid f_g) &= \frac{1}{|G|}\sum_{h \in G} \overline{\chi_{\rho_i}(h)} f_g(h) \\ &= \frac{|C_g|}{|G|}\overline{\chi_{\rho_i}(g)}\ ,\end{aligned}$$

and thus $f_g = \frac{|C_g|}{|G|} \sum_{i=1}^N \overline{\chi_{\rho_i}(g)} \chi_{\rho_i}$. In particular, if $g' \notin C_g$, then

$$0 = f_g(g') = \frac{|C_g|}{|G|} \sum_{i=1}^N \overline{\chi_{\rho_i}(g)} \chi_{\rho_i}(g'),$$

which proves the first formula and hence the orthognality of the columns of the character table. On the other hand, $1 = f_g(g) = \frac{|C_g|}{|G|} \sum_{i=1}^N \overline{\chi_{\rho_i}(g)} \chi_{\rho_i}(g)$, which proves the second formula. $\square$

4 Projection Operators

We introduce the projection operators onto the isotypic components of the decomposition of the vector space of any representation. Let (E, ρ) be a representation of G and let $\rho = \oplus_{i=1}^N m_i \rho_i$ be the decomposition of ρ into isotypic components. The support of the isotypic component $m_i \rho_i$, is $m_i E_i = E_i \oplus \cdots \oplus E_i$ (m_i terms). We denote this vector subspace of E by V_i. We shall write

$$V_i = m_i E_i = \overset{m_i}{\underset{j=1}{\oplus}} E_{i,j},$$

where each $E_{i,j}$, $1 \leq j \leq m_i$, is equal to E_i. We thus have $E = \oplus_{i=1}^N V_i$.

Theorem 4.1. *For each i, $1 \leq i \leq N$, we set*

$$P_i = \frac{\dim \rho_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(g).$$

Then

(i) P_i is the projection of E onto V_i under the decomposition $E = \oplus_{i=1}^N V_i$.
(ii) $P_i P_j = \delta_{ij} P_i$, for $1 \leq i \leq N$, $1 \leq j \leq N$.
(iii) If ρ is unitary, then P_i is Hermitian, that is, ${}^t\overline{P_i} = P_i$.

Proof. (i) Let us choose i_0, $1 \leq i_0 \leq N$, and show that $P_{i_0}\big|_{V_{i_0}} = Id_{V_{i_0}}$, while if $i \neq i_0$, then $P_{i_0}\big|_{V_i} = 0$. Let $x = \sum_{i=1}^N x_i$, where $x_i \in V_i$, and let $x_i = \sum_{j=1}^{m_i} x_{i,j}$, where $x_{i,j} \in E_{i,j}$, whence $x = \sum_{i=1}^N \sum_{j=1}^{m_i} x_{i,j}$. Then

$$\begin{aligned} P_{i_0}(x) &= \frac{\dim \rho_{i_0}}{|G|} \sum_{i=1}^N \sum_{j=1}^{m_i} \sum_{g \in G} \overline{\chi_{i_0}(g)} \rho(g) x_{i,j} \\ &= \frac{\dim \rho_{i_0}}{|G|} \sum_{i=1}^N \sum_{j=1}^{m_i} \left(\sum_{g \in G} \overline{\chi_{i_0}(g)} \rho_i(g) \right) x_{i,j}. \end{aligned}$$

Because χ_{i_0} is a class function and ρ_i is irreducible, we may apply Lemma 3.5, and we obtain

$$\sum_{g \in G} \overline{\chi_{i_0}(g)} \rho_i(g) = \rho_{i, \overline{\chi_{i_0}}} = \frac{|G|}{\dim \rho_i} (\chi_{i_0} \mid \chi_i) \mathrm{Id}_{E_i} = \frac{|G|}{\dim \rho_{i_0}} \delta_{i i_0} \mathrm{Id}_{E_{i_0}},$$

which finally leads to

$$P_{i_0}(x) = \sum_{i=1}^{N} \sum_{j=1}^{m_i} \delta_{i_0 i} x_{i,j} = \sum_{j=1}^{m_{i_0}} x_{i_0,j} = x_{i_0}.$$

(ii) The equations $P_i P_j = 0$ if $i \neq j$ and $P_i^2 = P_i$ follow from (i).
(iii) If ρ is unitary, then

$$\begin{aligned}\frac{|G|}{\dim \rho_i} {}^t\overline{P_i} &= \sum_{g\in G} \chi_i(g) {}^t\overline{\rho(g)} = \sum_{g\in G} \chi_i(g)\rho(g^{-1}) \\ &= \sum_{g\in G} \chi_i(g^{-1})\rho(g) = \sum_{g\in G} \overline{\chi_i(g)}\rho_i(g),\end{aligned}$$

which is equal to $\frac{|G|}{\dim \rho_i} P_i$, which proves (iii). □

The decomposition $E = \oplus_{i=1}^N V_i$ is unique up to order. On the other hand, the decomposition $V_i = \oplus_{j=1}^{m_i} E_{i,j}$ is not always unique. For example, if $\rho = \mathrm{Id}_E$, then ρ can be written in an infinite number of ways as a direct sum of one-dimensional representations.

5 Induced Representations

Induction is an operation that associates to a representation of a subgroup H of a group G a representation of the group G itself.

5.1 Definition

Let G be a finite group and H a subgroup. Let (F, π) be a representation of H. We define the vector space

$$E = \{\varphi : G \to F \mid \forall h \in H, \varphi(gh) = \pi(h^{-1})\varphi(g)\}, \tag{5.1}$$

and a representation $\rho = \pi^{\uparrow G}$ of G on E by

$$\forall \varphi \in E, \quad (\rho(g_0)\varphi)(g) = \varphi(g_0^{-1}g), \tag{5.2}$$

for every $g_0 \in G$ and for every $g \in G$. We can see that $\rho(g_0)\varphi$ lies in E because

$$(\rho(g_0)\varphi)(gh) = \varphi(g_0^{-1}gh) = \pi(h^{-1})\varphi(g_0^{-1}g) = \pi(h^{-1})\left((\rho(g_0)\varphi)(g)\right),$$

and on the other hand, we see that $g \mapsto \rho(g)$ is a group morphism of G into $\mathrm{GL}(E)$.

Definition 5.1. *The representation $\rho = \pi^{\uparrow G}$ of G on E is called the representation of G* induced *from the representation π of the subgroup H of G.*

For example, if $H = \{e\}$ and if π is the trivial representation of H on $\mathbb{C}$, then the vector space E is equal to $\mathbb{C}[G]$ and the representation of G induced from π is the regular representation of G.

5.2 Geometric Interpretation

We can interpret the vector space E as the space of sections of a "vector bundle." We consider the Cartesian product $G \times F$ and we introduce the equivalence relation

$$(g, x) \sim (gh, \pi(h^{-1})x), \ \forall h \in H. \tag{5.3}$$

Let $G \times_\pi F$ be the quotient of $G \times F$ by this equivalence relation, and let

$$q : G \times_\pi F \to G/H$$

be the projection that sends the equivalence class of (g, x) to gH. Note that this is well defined, because if $(g', x') \sim (g, x)$, then $g' = gh$, for some $h \in H$. The inverse image under the projection q of any point in G/H is isomorphic to the vector space F. We call $G \times_\pi F$ a *vector bundle* over G/H with *fiber* F.

A *section* of the projection $q : G \times_\pi F \to G/H$ (or of the vector bundle $G \times_\pi F$) is, by definition, a map ψ from G/H to $G \times_\pi F$ such that $q \circ \psi = \mathrm{Id}_{G/H}$.

Proposition 5.2. *The support E of the induced representation $\pi^{\uparrow G}$ is the vector space of sections of the projection $q : G \times_\pi F \to G/H$.*

Proof. To $\varphi \in E$ and $g \in G$ we associate the equivalence class of $(g, \varphi(g))$. The result depends only on the class of g modulo H. In fact, if $g' = gh$, with $h \in H$, we obtain the equivalence class of $(gh, \varphi(gh))$, which is equal to the equivalence class of $(g, \pi(h)\varphi(gh)) = (g, \varphi(g))$, since $\varphi \in E$. Thus one defines a section of $q : G \times_\pi F \to G/H$. On the other hand, to any given section of q we may associate an element of E by considering the second component of the equivalence class associated to an element of G/H. Since this construction is the inverse of the previous one, we have thus obtained an isomorphism of the space E of the induced representation onto the vector space of sections of the vector bundle $G \times_\pi F$. □

The notion of an induced representation can be defined more generally than just for finite groups, and has many applications in mathematics and physics.

References

The representations of finite groups are the subject of Serre's book (1997), of which Part I is an exposition of fundamental results. Finite groups are studied in Sternberg (1994), Simon (1996), Artin (1991) and Ledermann–Weir (1996). Also see the first chapters of Fulton–Harris (1991), which are followed by chapters on Lie algebra representations, or the textbook by James and Liebeck (1993, 2001), which stresses arithmetic. All of these works discuss induced representations. For applications to physics, see Ludwig–Falter (1996), Tung (1985), or Blaizot–Tolédano (1997).

Tensor products of vector spaces are introduced in Exercise 2.5 below. (Also see Exercise 2.7.) For supplementary material on the tensor, exterior, and symmetric algebras of a vector space, see Greub (1967), Warner (1983), Sternberg (1994), or Knapp (2002).

The theory of characters was created by Frobenius in a series of articles published, starting in 1896, in the *Sitzungsberichte* of the Berlin Academy. These articles, reprinted in Frobenius (1968), contain beautiful character tables, p. 345, for a subgroup of $\mathfrak{S}_{12}$ of order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$ with 15 irreducible representations, and, on the folding page between p. 346 and p. 347, for a subgroup of $\mathfrak{S}_{24}$ of order $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$ with 26 irreducible representations. One can find a historical and mathematical analysis of this theory in the book by Curtis (1999). Also see Hawkins (2000) and Rossmann (2002).

Heinrich Maschke (1858–1908) published the theorem that bears his name in 1899, as a preliminary result in an attempt to prove a property of finite groups of matrices with complex coefficients.

For and in-depth study of induced representations, including those of Lie groups, see, e.g., Gurarie (1992), which includes applications to physics.

Exercises

Exercise 2.1 *The symmetric group* $\mathfrak{S}_3$.

We write c for the cyclic permutation (123) and t for the transposition (23). Show that $\{c, t\}$ generates $\mathfrak{S}_3$, and that $tc = c^2t, ct = tc^2$. Find the conjugacy classes of the group $\mathfrak{S}_3$.

Exercise 2.2 *Representations of* $\mathfrak{S}_3$.

(a) Find the one-dimensional representations of the group $\mathfrak{S}_3$.
(b) Let e_1, e_2, e_3 be the canonical basis of $\mathbb{C}^3$. For $g \in \mathfrak{S}_3$, set $\sigma(g)e_i = e_{g(i)}$. Show that this defines a three-dimensional representation σ of $\mathfrak{S}_3$ and that $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\}$ is invariant under σ. This representation is called the *permutation representation* of the symmetric group.

   We denote by ρ the restriction to V of the representation σ.

(c) Show that there is a basis (u_1, u_2) of V such that $\rho(t)u_1 = u_2$, $\rho(t)u_2 = u_1$, $\rho(c)u_1 = ju_1$, $\rho(c)u_2 = j^2u_2$, where $j^2 + j + 1 = 0$. Is the representation ρ irreducible?
(d) Find the character table of $\mathfrak{S}_3$.
(e) What is the geometric interpretation of $\mathfrak{S}_3$ as a group of symmetries? What is the geometric interpretation of the representation ρ?

Exercise 2.3 *The symmetric group* $\mathfrak{S}_4$.

Find the conjugacy classes and character table of the symmetric group $\mathfrak{S}_4$.

Exercise 2.4 *The alternate group* $\mathfrak{A}_4$.

Find the character table of $\mathfrak{A}_4$. Which representations of $\mathfrak{A}_4$ are the restriction of a representation of $\mathfrak{S}_4$? Which representations of $\mathfrak{S}_4$ have an irreducible restriction to $\mathfrak{A}_4$? Which have a reducible restriction?

Exercise 2.5 *Tensor products of vector spaces and of representations.*

We denote the dual of a vector space E by E^*, and the duality pairing by $\langle\ ,\ \rangle$.

If E and F are finite-dimensional vector spaces over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), one can define the tensor product $E \otimes F$ as the vector space of bilinear maps of $E^* \times F^*$ into the scalar field $\mathbb{K}$. For $x \in E$, $y \in F$, we define the element $x \otimes y \in E \otimes F$ by

$$(x \otimes y)(\xi, \eta) = \langle \xi, x\rangle \langle \eta, y\rangle,$$

for every $\xi \in E^*$, $\eta \in F^*$.

(a) Let $(e_1, \dots, e_n)$ be a basis of E and let $(f_1, \dots, f_p)$ be a basis of F. Show that $(e_i \otimes f_j)_{1\le i\le n, 1\le j\le p}$ is a basis of $E \otimes F$.

(b) An element of $E \otimes E$ is called a *contravariant tensor* (or simply a *tensor*) of order 2 on E. Every contravariant tensor of order 2 on E can be written $T = \sum_{i,j=1}^{n} T^{ij} e_i \otimes e_j$, where the T^{ij} are scalars, called the *components* of T in the basis (e_i). What are the components of T after a change of basis?

(c) We can associate to $\xi \otimes y \in E^* \otimes F$ the linear map u of E into F defined by $u(x) = \langle \xi, x\rangle y$, for $x \in E$. Show that this defines an isomorphism of $E^* \otimes F$ onto the vector space of linear maps of E into F, $\mathcal{L}(E, F)$.

(d) Show that if $u : E \to E$ and $v : F \to F$ are linear maps, then there is a unique endomorphism $u \otimes v$ of $E \otimes F$ satisfying $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$ for each $x \in E$, $y \in F$. In $E \otimes F$, we choose the basis

$$(e_1 \otimes f_1, e_1 \otimes f_2, \dots, e_1 \otimes f_p, e_2 \otimes f_1, e_2 \otimes f_2, \dots, e_2 \otimes f_p, \dots, e_n \otimes f_1, \dots, e_n \otimes f_p).$$

Write the matrix of $u \otimes v$, where u (respectively, v) is an endomorphism of E (respectively, F) with matrix $A = (a_{ij})$ (respectively, $B = (b_{ij})$) in the chosen bases.

(e) If (E_1, ρ_1) and (E_2, ρ_2) are representations of a group G, we set, for $g \in G$,

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g).$$

Show that this defines a representation $\rho_1 \otimes \rho_2$ of G on $E_1 \otimes E_2$. What can one say about the character of $\rho_1 \otimes \rho_2$? If ρ_1 and ρ_2 are irreducible, is the representation $\rho_1 \otimes \rho_2$ irreducible?

Exercise 2.6 *The dual representation.*

Let (E, π) be a representation of a group G. For $g \in G$, $\xi \in E^*$, $x \in E$, we set $\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle$. (As in Excercise 2.5, E^* is the dual of E, and $\langle\ ,\ \rangle$ is the duality pairing.)

(a) Show that this defines a representation π^* of G on E^*. The representation π^* is called the *dual* (or *contragredient*) of π.

(b) Show that if (E, π) and (F, ρ) are representations of a group G, then $g \cdot u = \rho(g) \circ u \circ \pi(g^{-1})$, for $u \in \mathcal{L}(E, F)$ and $g \in G$, defines a representation of G on $\mathcal{L}(E, F)$, equivalent to $\pi^* \otimes \rho$.

Exercise 2.7 *Exterior and symmetric powers.*

Let E be a finite-dimensional vector space, with basis $(e_1, \ldots, e_n)$. We denote by $\bigwedge^2 E$ (respectively, S^2E) the vector subspace of $E \otimes E$ generated by $e_i \otimes e_j - e_j \otimes e_i$, $1 \leq i < j \leq n$ (respectively, $e_i \otimes e_j + e_j \otimes e_i$, $1 \leq i \leq j \leq n$). These definitions are independent of the choice of basis and $E \otimes E = \bigwedge^2 E \oplus S^2 E$. The space $\bigwedge^2 E$ is the *exterior* (or *antisymmetric*) *power* of degree 2 of E, and the space S^2E is the *symmetric power* of degree 2 of E.

(a) If (E, ρ) is a representation of a group G, then $\bigwedge^2 E$ and S^2E are invariant under $\rho \otimes \rho$. We denote the restriction of $\rho \otimes \rho$ to $\bigwedge^2 E$ (respectively, S^2E) by $\bigwedge^2 \rho$ (respectively, $S^2\rho$). Suppose that G is finite. Show that the characters of these representations satisfy, for each $g \in G$,

$$\chi_{\wedge^2\rho}(g) = \frac{1}{2}\Big((\chi_\rho(g))^2 - \chi_\rho(g^2)\Big), \quad \chi_{S^2\rho}(g) = \frac{1}{2}\Big((\chi_\rho(g))^2 + \chi_\rho(g^2)\Big).$$

(b) If ρ is the two-dimensional irreducible representation of $\mathfrak{S}_3$, find $\chi_{\wedge^2\rho}$ and $\chi_{S^2\rho}$. Decompose $\rho \otimes \rho$ into a direct sum of irreducible representations.

Exercise 2.8 *Equivalence of the left and right regular representations.*

Show that the left and right regular representations of a finite group are equivalent.

Exercise 2.9 *Representations of abelian and cyclic groups.*

(a) Show that every irreducible representation of a finite group is one-dimensional if and only if the group is abelian.
(b) Find all the inequivalent irreducible representations of the cyclic group of order n.

Exercise 2.10 *An application of the orthogonality relations.*

Let ρ_i and ρ_j be irreducible representations of a finite group G. Let $\chi_i = \chi_{\rho_i}$ and $\chi_j = \chi_{\rho_j}$. Show that for every $h \in G$,

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}h) = \frac{1}{\dim\, \rho_i} \chi_i(h)\delta_{ij}.$$

Exercise 2.11 *Regular representation of $\mathfrak{S}_3$.*

Decompose the regular representation of $\mathfrak{S}_3$ into a direct sum of irreducible representations.

Find a basis of each one-dimensional invariant subspace and a projection onto the support of the representation 2ρ, where ρ is the irreducible representation of dimension 2.

Exercise 2.12 *Real and complexified representations.*

Let E be a vector space over $\mathbb{R}$, of dimension n. A morphism of a finite group G into $\mathrm{GL}(E)$ is called a *real representation* of G, of (real) dimension n.

We consider $E^{\mathbb{C}} = E \oplus iE = E \otimes \mathbb{C}$, a vector space over $\mathbb{C}$, of complex dimension n, called the *complexification* of E.

(a) Show that every real representation of G on E can be extended uniquely to a (complex) representation of G on $E^{\mathbb{C}}$. This representation is called the *complexification* of the real representation.
(b) Let the symmetric group $\mathfrak{S}_3$ act on $\mathbb{R}^2$ by rotation through angles of $2k\pi/3$ and reflection. Show that the complexification of this representation is equivalent to the irreducible representation of $\mathfrak{S}_3$ on $\mathbb{C}^2$.
(c) Let the cyclic group of order 3 act on $\mathbb{R}^2$ by rotations through angles of $2k\pi/3$. Is this real representation irreducible?
(d) Are all irreducible real representations of abelian groups one-dimensional?

Exercise 2.13 *Representations of the dihedral group.*

(a) Show that if H is an abelian subgroup of order p of a finite group G of order n, then every irreducible representation of G is of dimension $\leq n/p$.
(b) Conclude that for every $n \geq 3$, every irreducible representation of the dihedral group $D_{(n)}$ is one- or two-dimensional.

Exercise 2.14 *Peter–Weyl theorem for finite groups.*

Let $\rho^1, \rho^2, \ldots, \rho^N$ be unitary representations of a finite group G, chosen from each equivalence class of irreducible representations.

Show that the matrix coefficients of the representations ρ^k, $k = 1, \ldots, N$, in orthonormal bases form an orthogonal basis of $L^2(G)$. Conclude that every function $f \in L^2(G)$ has a "Fourier series"

$$f = \sum_{k=1}^{N} \sum_{i,j=1}^{\dim \rho_k} d_k (\rho_{ij}^k | f) \rho_{ij}^k,$$

where the d_k are integers.

Exercise 2.15 *Representation of* $\mathrm{GL}(2,\mathbb{C})$ *on the polynomials of degree* 2.

Let G be be a group and let ρ be a representation of G on $V = \mathbb{C}^n$. Let $P^{(k)}(V)$ be the vector space of complex polynomials on V that are homogeneous of degree k.

(a) For $f \in P^{(k)}(V)$, we set $\rho^{(k)}(g)(f) = f \circ \rho(g^{-1})$. Show that this defines a representation $\rho^{(k)}$ of G on $P^{(k)}(V)$.
(b) Compare $\rho^{(1)}$ and the dual representation of ρ.
(c) Suppose that $G = \mathrm{GL}(2,\mathbb{C})$, $V = \mathbb{C}^2$, and ρ is the fundamental representation. Let $k = 2$. To the polynomial $f \in P^{(2)}(\mathbb{C}^2)$ defined by $f(x,y) = ax^2 + 2bxy + cy^2$ we associate the vector $v_f = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{C}^3$. Let $\tilde{\rho}$ denote the representation of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^3$ defined by $\rho^{(2)}$ and the isomorphism above. Find the dual of $\tilde{\rho}$.

Exercise 2.16 *Convolution.*

Let G be be a finite group and let $\mathbb{C}[G]$ be the *group algebra*, of G that is, the vector space $\mathcal{F}(G)$ with multiplication defined by $\epsilon_g \epsilon_{g'} = \epsilon_{gg'}$, for g and $g' \in G$, and extended by linearity.

(a) Show that the product of two functions $f_1, f_2 \in \mathbb{C}[G]$ is the *convolution product* $(f_1 * f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1}g)$.
(b) Let ρ be a representation of G and suppose $f \in \mathbb{C}[G]$. Set $\rho_f = \sum_{g \in G} f(g)\rho(g)$. Show that $\rho_{f_1 * f_2} = \rho_{f_1} \circ \rho_{f_2}$.
(c) Show that $f \in \mathbb{C}[G]$ is a class function if and only if f is in the center of the algebra $\mathbb{C}[G]$ equipped with convolution (that is, f commutes in the sense of convolution with every function on G).

Exercise 2.17 *On the map $f \mapsto \rho_f$.*

For every representation (E, ρ) of G and each function f on G, consider the endomorphism ρ_f of E defined by

$$\rho_f = \sum_{g \in G} f(g)\rho(g).$$

(a) Let R be the regular representation of G. Consider $R_f(\epsilon_g)$, for $g \in G$. Show that $R_f(\epsilon_e) = f$. Is the map $f \in \mathbb{C}[G] \mapsto R_f \in \mathrm{End}(\mathbb{C}[G])$ injective?
(b) Let ρ_i and ρ_j be irreducible representations of G and let χ_i (respectively, χ_j) be the character of ρ_i (respectively, ρ_j). Find ρ_f for $\rho = \rho_j$ and $f = \overline{\chi_i}$.

Exercise 2.18 *Tensor products of representations.*

Let ρ be the irreducible representation of dimension 2 of the symmetric group $\mathfrak{S}_3$. We set $\rho = \rho^{\otimes 1}$, and by induction we define for every integer $k \geq 2$,

$$\rho^{\otimes k} = \rho^{\otimes (k-1)} \otimes \rho.$$

(a) For each positive integer k, decompose $\rho^{\otimes k}$ into a direct sum of irreducible representations.
(b) Let $\mathfrak{A}_3 \subset \mathfrak{S}_3$ denote the alternate group. For each positive integer k, decompose the restriction of $\rho^{\otimes k}$ to $\mathfrak{A}_3$ into a direct sum of irreducible representations.

Issai Schur, born in 1875 in Mohilev, Belorussia, professor in Bonn and then in Berlin, member of the Prussian Academy of Sciences, lost his university position in 1935. Forced to emigrate to Palestine, he died in Tel Aviv in 1941. Schur is, with Frobenius, a founder of representation theory.

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