

---

# Contents

<b>Introduction</b> .....	xi
<b>Acknowledgments</b> .....	xv
<b>1 General Facts About Groups</b> .....	1
1 Review of Definitions .....	1
2 Examples of Finite Groups .....	2
2.1 Cyclic Group of Order $n$ .....	2
2.2 Symmetric Group $\mathfrak{S}_n$ .....	3
2.3 Dihedral Group .....	3
2.4 Other Examples .....	3
3 Examples of Infinite Groups .....	3
4 Group Actions and Conjugacy Classes .....	5
References .....	6
Exercises .....	6
<b>2 Representations of Finite Groups</b> .....	9
1 Representations .....	9
1.1 General Facts .....	9
1.2 Irreducible Representations .....	11
1.3 Direct Sum of Representations .....	11
1.4 Intertwining Operators and Schur's Lemma .....	12
2 Characters and Orthogonality Relations .....	14
2.1 Functions on a Group, Matrix Coefficients .....	14
2.2 Characters of Representations and Orthogonality Relations .....	15
2.3 Character Table .....	18
2.4 Application to the Decomposition of Representations ..	19
3 The Regular Representation .....	20
3.1 Definition .....	20
3.2 Character of the Regular Representation .....	21
3.3 Isotypic Decomposition .....	21
3.4 Basis of the Vector Space of Class Functions .....	22

4	Projection Operators	24
5	Induced Representations	25
5.1	Definition	25
5.2	Geometric Interpretation	26
	References	26
	Exercises	27
<b>3</b>	<b>Representations of Compact Groups</b>	<b>33</b>
1	Compact Groups	33
2	Haar Measure	34
3	Representations of Topological Groups and Schur's Lemma	36
3.1	General Facts	36
3.2	Coefficients of a Representation	36
3.3	Intertwining Operators	37
3.4	Operations on Representations	38
3.5	Schur's Lemma	38
4	Representations of Compact Groups	39
4.1	Complete Reducibility	39
4.2	Orthogonality Relations	40
5	Summary of Chapter 3	42
	References	43
	Exercises	44
<b>4</b>	<b>Lie Groups and Lie Algebras</b>	<b>47</b>
1	Lie Algebras	47
1.1	Definition and Examples	47
1.2	Morphisms	49
1.3	Commutation Relations and Structure Constants	49
1.4	Real Forms	49
1.5	Representations of Lie Algebras	50
2	Review of the Exponential Map	51
3	One-Parameter Subgroups of $GL(n, \mathbb{K})$	54
4	Lie Groups	56
5	The Lie Algebra of a Lie Group	56
6	The Connected Component of the Identity	59
7	Morphisms of Lie Groups and of Lie Algebras	60
7.1	Differential of a Lie Group Morphism	60
7.2	Differential of a Lie Group Representation	62
7.3	The Adjoint Representation	64
	References	65
	Exercises	66
<b>5</b>	<b>Lie Groups <math>SU(2)</math> and <math>SO(3)</math></b>	<b>71</b>
1	The Lie Algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$	71
1.1	Bases of $\mathfrak{su}(2)$	71
1.2	Bases of $\mathfrak{so}(3)$	73
1.3	Bases of $\mathfrak{sl}(2, \mathbb{C})$	74

---

2	The Covering Morphism of $SU(2)$ onto $SO(3)$ .....	74
2.1	The Lie Group $SO(3)$ .....	74
2.2	The Lie Group $SU(2)$ .....	76
2.3	Projection of $SU(2)$ onto $SO(3)$ .....	77
	References .....	78
	Exercises .....	79
<b>6</b>	<b>Representations of <math>SU(2)</math> and <math>SO(3)</math></b> .....	<b>81</b>
1	Irreducible Representations of $\mathfrak{sl}(2, \mathbb{C})$ .....	81
1.1	The Representations $D^j$ .....	81
1.2	The Casimir Operator .....	84
1.3	Hermitian Nature of the Operators $J_3$ and $J^2$ .....	84
2	Representations of $SU(2)$ .....	86
2.1	The Representations $\mathcal{D}^j$ .....	86
2.2	Characters of the Representations $\mathcal{D}^j$ .....	89
3	Representations of $SO(3)$ .....	90
	References .....	90
	Exercises .....	91
<b>7</b>	<b>Spherical Harmonics</b> .....	<b>93</b>
1	Review of $L^2(S^2)$ .....	93
2	Harmonic Polynomials .....	94
2.1	Representations of Groups on Function Spaces .....	94
2.2	Spaces of Harmonic Polynomials .....	94
2.3	Representations of $SO(3)$ on Spaces of Harmonic Polynomials .....	95
3	Definition of Spherical Harmonics .....	97
3.1	Representations of $SO(3)$ on Spaces of Spherical Harmonics .....	97
3.2	The Casimir Operator .....	99
3.3	Eigenfunctions of the Casimir Operator .....	99
3.4	Bases of Spaces of Spherical Harmonics .....	100
3.5	Explicit Formulas .....	103
	References .....	104
	Exercises .....	104
<b>8</b>	<b>Representations of <math>SU(3)</math> and Quarks</b> .....	<b>107</b>
1	Review of $\mathfrak{sl}(n, \mathbb{C})$ , Representations of $\mathfrak{sl}(3, \mathbb{C})$ and $SU(3)$ .....	107
1.1	Review of $\mathfrak{sl}(n, \mathbb{C})$ .....	107
1.2	The Case of $\mathfrak{sl}(3, \mathbb{C})$ .....	107
1.3	The Bases $(I_3, Y)$ and $(I_3, T_8)$ of $\mathfrak{h}$ .....	109
1.4	Representations of $\mathfrak{sl}(3, \mathbb{C})$ and of $SU(3)$ .....	110
2	The Adjoint Representation and Roots .....	110
3	The Fundamental Representation and Its Dual .....	111
3.1	The Fundamental Representation .....	111
3.2	The Dual of the Fundamental Representation .....	112

---

4	Highest Weight of a Finite-Dimensional Representation . . . . .	113
4.1	Highest Weight . . . . .	113
4.2	Weights as Linear Combinations of the $\lambda_i$ . . . . .	114
4.3	Finite-Dimensional Representations and Weights . . . . .	115
4.4	Another Example: The Representation 6 . . . . .	116
4.5	One More Example: The Representation 10 . . . . .	117
5	Tensor Products of Representations . . . . .	118
6	The Eightfold Way . . . . .	121
6.1	Baryons ( $B = 1$ ) . . . . .	122
6.2	Mesons ( $B = 0$ ) . . . . .	122
6.3	Baryon Resonances . . . . .	123
7	Quarks and Antiquarks . . . . .	124
	References . . . . .	125
	Exercises . . . . .	125
<b>Problems and Solutions</b> . . . . .		129
1	Restriction of a Representation to a Finite Group . . . . .	129
2	The Group $O(2)$ . . . . .	131
3	Representations of the Dihedral and Quaternion Groups . . . . .	134
4	Representations of $SU(2)$ and of $\mathfrak{S}_3$ . . . . .	143
5	Pseudo-unitary and Pseudo-orthogonal Groups . . . . .	147
6	Irreducible Representations of $SU(2) \times SU(2)$ . . . . .	152
7	Projection Operators . . . . .	160
8	Symmetries of Fullerene Molecules . . . . .	168
9	Matrix Coefficients and Spherical Harmonics . . . . .	177
<b>Bibliography</b> . . . . .		185
<b>Index</b> . . . . .		191