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# Rules

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| 0  | ALGORITHMIC DIFFERENTIATION DOES NOT INCUR TRUNCATION ERRORS.  | 2   |
| 1  | DIFFERENCE QUOTIENTS MAY SOMETIMES BE USEFUL TOO.  | 4   |
| 2  | WHAT'S GOOD FOR FUNCTION VALUES IS GOOD FOR THEIR DERIVATIVES.   | 20  |
| 3  | THE JACOBIAN-VECTOR AND JACOBIAN TRANSPOSED VECTOR PRODUCTS CALCULATED IN THE FORWARD AND REVERSE MODE, RESPECTIVELY, CORRESPOND TO THE EXACT VALUES FOR AN EVALUATION PROCEDURE WHOSE ELEMENTALS ARE PERTURBED AT THE LEVEL OF THE MACHINE PRECISION. | 51  |
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| 9  | THE SEQUENTIAL ACCESS MEMORY REQUIREMENT OF THE BASIC REVERSE MODE IS PROPORTIONAL TO THE TEMPORAL COMPLEXITY OF THE FUNCTION.   | 88  |
| 10 | NEVER GO BACK MORE THAN ONCE (AFTER ALL WEIGHT VECTORS ARE KNOWN).   | 92  |
| 11 | INCREMENTAL ADJOINTS OF LINEAR PROCEDURES ARE REFLEXIVE IRRESPECTIVE OF ALLOCATION.  | 98  |
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