

CONTENTS

INTRODUCTION	1
1 Traces	7
1.1 Definition and uniqueness of a trace	7
1.1.1 Traces and DGAs	12
1.1.2 Example: bounded operators	16
1.1.3 Matrix algebras	17
1.1.4 Example: commutative algebras	18
1.1.5 Example: non-commutative torus	19
1.1.6 Example: de Rham currents	19
1.1.7 Example: smoothing operators	23
1.1.8 Example: pseudodifferential operators	24
1.2 Characters and (algebraic) K -theory	28
1.2.1 Cyclic homology	29
1.2.2 $K_0(\mathcal{A})$	34
1.2.2.1 Chern character on $K_0(\mathcal{A})$	36
1.2.2.2 Example: topological even K -theory	37
1.2.2.3 Superconnections: Chern–Weil on $K_0(M)$	40
1.3 Trace ideals in $\mathbf{B}(H)$	41
1.3.1 $\mathbf{B}(H)$ has no trace if H is infinite-dimensional, and a unique trace if H is finite-dimensional	41
1.3.2 Traces on commutator ideals	45
1.3.3 The ideal $\mathbf{F}(H)$ of finite-rank operators has a unique trace	47
1.3.4 On the ideal $\mathbf{C}(H)$ of compact operators there is no trace	49
1.3.5 Schatten ideals \mathbf{C}_p and the classical trace	49
1.3.5.1 \mathbf{C}_1 : trace class operators	50
1.3.5.2 Tr on smoothing operators on \mathbb{R}^n	52
1.3.5.3 Tr on \mathbf{C}_1	54
1.3.6 Exotic traces	56
1.4 Holomorphic functional calculus on a Banach algebra	57
1.4.1 Functional integrals	57
1.4.1.1 Spectral cut and Agmon angle	57
1.4.1.2 Traces on functional integrals	58
1.4.1.3 Functional integrals over infinite contours	60
1.4.1.4 Functional integrals on $\mathbf{B}(H)$ and trace ideals	61
1.4.1.5 Products of functional integrals	62

1.4.2	Complex powers, logarithms, and heat operators	63
1.4.2.1	Integer powers	63
1.4.2.2	Logarithm and complex power functions	65
1.4.2.3	Logarithm and complex power operators	66
1.4.2.4	Dependence on the spectral cut	67
1.4.2.5	Spectral zeta functions on Banach algebras	70
1.4.2.6	Heat operators	70
1.4.3	Functional calculus on the de Rham algebra	73
1.4.3.1	Quillen–Chern–Weil theory	77
1.5	Traces on ψ dos	83
1.5.1	Symbol algebras	84
1.5.2	Classical symbols	86
1.5.3	Pseudodifferential operators (ψ dos)	87
1.5.3.1	Classical ψ dos	91
1.5.4	Unique trace on Ψ^z	92
1.5.4.1	Commutator structure of Ψ^∞	93
1.5.4.2	Commutator structure of $\Psi^{-\infty}$	95
1.5.5	Unique quasi-trace on $\Psi^{c \setminus \mathbb{Z}^n}$	95
1.5.6	Meromorphic extension of TR to $\Psi^{\mathbb{Z}^n}$	97
1.5.6.1	Gaugings of ψ dos and meromorphic continuation	99
1.5.6.2	Trace formulae for the Laurent coefficients	101
1.5.7	Example: zeta functions	104
1.5.7.1	Complex power gaugings	104
1.5.7.2	Spectral zeta functions	106
1.5.7.3	Higher Laurent expansion	108
1.5.7.4	Zeta quasi-traces	109
1.5.7.5	Dependence on the choice of principal angle θ	113
1.5.7.6	Resolvent trace expansion	114
1.5.7.7	Heat trace invariants	115
1.5.8	Traces on subalgebras of ψ dos	117
1.5.8.1	Classical L^2 trace	117
1.5.8.2	TR on even–even (odd-class) operators	118
1.5.8.3	Leading symbol traces	120
1.5.9	Suspended traces	121
1.5.10	Traces on families of smoothing operators	128
1.5.11	Traces on geometric families of ψ dos	131
1.6	Notes	134
2	Determinants	154
2.1	Logarithmic representations of semigroups	154
2.1.1	Global logarithms	154
2.1.1.1	Logs pull-back	156

2.1.2	Local logarithms	156
2.1.3	Determinant structures	157
	2.1.3.1 Counting log-determinant structures	159
2.1.4	Determinant structures and DGAs	160
2.1.5	Logarithms on categories	161
2.2	Example: Fredholm index	162
	2.2.1 Fredholm operators	162
	2.2.2 Index as a determinant structure	163
	2.2.3 The restricted general linear group $\text{Gl}_{\text{res}, \mathfrak{J}}(H)$	165
	2.2.4 Families index	166
	2.2.4.1 A logarithm on the category Cat_{Fred}	166
	2.2.4.2 Index bundle	167
2.3	Universal odd logarithm	170
	2.3.1 Universal odd log	171
	2.3.1.1 Example: $\text{Gl}_{\text{res}}(H)$	178
	2.3.2 Logs and $K_1(\mathcal{A})$	179
	2.3.2.1 Example: log-holonomy	179
	2.3.2.2 Logs on $\text{Gl}_{\infty}(\mathcal{A})$	180
	2.3.2.3 Odd Chern character on $K_1(\mathcal{A})$	181
	2.3.2.4 $K_{-1}(\mathcal{A})$ and topological odd K -theory	182
2.4	Logarithm structures on a Banach algebra	186
	2.4.1 Suspended log	188
	2.4.2 Example: classical logarithm	190
	2.4.3 The Fredholm-determinant structure	192
	2.4.3.1 $\det_{\mathfrak{F}}$ and the first odd Chern form	193
	2.4.3.2 Spectral zeta function and heat trace formulae	196
	2.4.3.3 $\det_{\mathfrak{F}}$ and $\text{Gl}_1(I + C_1(H))$	197
	2.4.4 Constructions of the Fredholm determinant	198
	2.4.4.1 Combinatorial spectral invariant	198
	2.4.4.2 Character of the fundamental representation	199
	2.4.4.3 Canonical trivialization of the determinant line	201
	2.4.4.4 Determinant line of a Fredholm operator	202
	2.4.4.5 Graded Fredholm determinant	205
	2.4.4.6 Gaussian and Fermionic integrals	206
	2.4.4.7 Determinants of self-adjoint operators	208
	2.4.5 Exotic determinants	209
2.5	de Rham determinants	210
	2.5.1 Classical local logarithm on $\Omega(M, \text{End}(E))$	210
	2.5.2 Odd log on $C^{\infty}(M, G)$	212
2.6	Trace structures on logarithmic ψ dos	213
	2.6.1 Log-classical ψ dos	213
	2.6.1.1 Log-polyhomogeneous symbols	213
	2.6.1.2 Log-classical ψ dos	216
	2.6.1.3 Logarithm operator	216

2.6.2	Trace structure of $\Psi^{z, \mathbb{N}}$	219
2.6.2.1	Higher residue traces on $\Psi^{z, \mathbb{N}}$	220
2.6.2.2	Commutator structure of $\Psi^{C, \mathbb{N}}$	222
2.6.2.3	TR on $\Psi^{C \setminus z, \mathbb{N}}(M, E)$	222
2.6.2.4	Meromorphic extension of TR to $\Psi^{z, \mathbb{N}}$	224
2.6.2.5	Zeta functions of log-classical ψ dos	226
2.6.2.6	Traces on proper subalgebras of $\Psi^{z, \mathbb{N}}$	228
2.6.2.7	Traces on families of log-classical ψ dos	229
2.7	Determinant structures on classical ψ dos	229
2.7.1	Residue determinant	230
2.7.1.1	Residue determinant and $\zeta(A, 0)$	233
2.7.1.2	Residue determinant and the index	234
2.7.1.3	A ‘better’ residue determinant structure	235
2.7.2	Leading symbol determinants	236
2.7.3	The ‘order’ determinant structure on Ψ^∞	238
2.7.4	A uniqueness result	238
2.7.5	Zeta quasi-determinants	239
2.7.5.1	Variation of the ζ -determinant	244
2.7.5.2	Multiplicative anomaly and Polyakov formula	245
2.7.6	Further log-determinant structures on ψ dos	247
2.7.6.1	ψ do determinant structures on trace ideals	247
2.7.6.2	TR determinants on even–even operators	247
2.7.6.3	Index determinant and odd Chern forms	248
2.7.6.4	Suspended determinants and the eta invariant	250
2.7.6.5	Pole structure of $\eta(D, z)$ and η_{sus}	255
2.7.6.6	Determinants on geometric families of ψ dos	258
2.8	Notes	258
3	Computations, transition formulae, and the local index formula	271
3.1	Computations using classical zeta functions	272
3.1.1	Dirac-type operators on the circle	272
3.1.1.1	Relative zeta determinants and formal quotients	277
3.1.1.2	Special values of the Riemann zeta function	278
3.1.2	Laplacians on higher dimensional spheres	280
3.1.3	Laplacian on the torus	282
3.2	Computations on compact quotients	287
3.2.1	Pushing down Schwartz kernels	287
3.2.1.1	Extension to flat Γ -bundles	289
3.2.2	Traces and regularized traces on compact quotients	290
3.2.2.1	Spectral-geometric trace formula	290
3.2.2.2	Selberg–Arthur trace formula	292
3.2.2.3	A regularized spectral-geometric trace formula	294

3.2.3	Spectral-geometric trace formulae on tori: Poisson summation	295
3.2.3.1	Spectral-geometric trace formulae on S^1	295
3.2.3.2	Variational computation of $\det_{\zeta} D$ on S^1	301
3.2.3.3	Spectral-geometric heat trace on tori	303
3.2.3.4	Laplacian determinant from the heat trace on S^1	304
3.2.3.5	Determinant of the Laplacian coupled to a flat bundle on a two-torus	307
3.2.3.6	Regularized $\det \bar{\partial}_{\Sigma}$ on an elliptic curve	311
3.2.4	Computations on higher genus Riemann surfaces	314
3.2.4.1	A Selberg heat trace formula	314
3.2.4.2	Relative Laplacian ζ -determinant on a surface	315
3.2.4.3	Conformal variation of the Laplacian determinant	316
3.2.5	Relative holomorphic torsion on negatively curved manifolds	319
3.3	Computations using the complex powers	321
3.3.1	Complex powers and transition formulae	321
3.3.1.1	Basic properties	323
3.3.1.2	Functional integral computation on S^1	325
3.3.2	Resolvent trace and zeta trace	329
3.3.2.1	Mellin transform formulae	330
3.3.2.2	Resolvent trace and zeta trace transition formulae	336
3.3.3	Heat trace and zeta trace	338
3.3.3.1	Resolvent trace to heat trace to zeta trace	339
3.3.3.2	General heat–zeta trace transition formulae	341
3.3.4	Relative determinant formulae	343
3.3.4.1	Relative heat kernel regularization	347
3.3.4.2	Multiplicativity property	348
3.3.4.3	Contour integral ζ -determinants on S^1	350
3.4	Residue determinant computations	353
3.4.1	Relative formulae for $\zeta(A, 0)$	354
3.4.2	Determinant property on $I + \Psi < 0$	356
3.4.3	Relative Laplacians	357
3.5	Local Atiyah–Singer index formula	360
3.5.1	Residue determinant and the index	361
3.5.2	Riemann–Roch–Hirzebruch formula	363
3.5.2.1	Local Riemann–Roch formula on a surface	365
3.5.3	An elementary proof of the local Atiyah–Singer index formula	372
3.5.3.1	Statement of the formula	373
3.5.3.2	Two traces	375

	3.5.3.3	Resolvent symbols	378
	3.5.3.4	Computing the index density	387
	3.5.3.5	Reformulation for heat coefficients	394
3.6		Notes	399
4		Pseudodifferential operator trace formulae	407
4.1		Homogeneous distributions	407
	4.1.1	Local distributions	407
	4.1.2	Fourier transform	409
	4.1.3	Extendibility of homogeneous distributions	412
		4.1.3.1 Example: dimension 1	415
	4.1.4	Gauged distributions	420
		4.1.4.1 Extensions via a gauging	422
		4.1.4.2 Regularized limit method	426
	4.1.5	Pull-back property of extensions	428
	4.1.6	Log-homogeneous distributions	430
		4.1.6.1 Log-homogeneous functions	430
		4.1.6.2 Extending log-homogeneous functions	433
4.2		Distributions on manifolds	443
	4.2.1	Densities and distributions	443
	4.2.2	Push-forward map	445
	4.2.3	Localizations	447
4.3		Schwartz kernel theorem	450
	4.3.1	SKT push-forward	453
	4.3.2	The unique trace on smoothing operators	454
4.4		Pseudodifferential operators	458
	4.4.1	Oscillatory integrals	458
		4.4.1.1 Regularization of oscillatory integrals	460
	4.4.2	ψ dos on manifolds	462
	4.4.3	Kernel singularity structure	470
		4.4.3.1 Stationary phase approximation expansion	473
	4.4.4	Gaugings of ψ dos: holomorphic parameters	476
		4.4.4.1 Holomorphic families of symbols	476
		4.4.4.2 Holomorphic families of ψ dos	479
4.5		TR on $\Psi^{C\setminus Z_n, N}$	482
	4.5.1	TR on $\Psi^{C\setminus Z_n}$	482
	4.5.2	Traciality properties of TR	487
	4.5.3	TR and finite-part integrals	488
	4.5.4	TR on $\Psi^{C\setminus Z_n, N}$	494
4.6		TR ^{mer} on Ψ^{Z_n}	498
	4.6.1	Laurent expansion of the gauged trace density	498
		4.6.1.1 Global densities defined on M by the expansion	504
		4.6.1.2 Full Laurent expansion	506
	4.6.2	Trace and quasi-trace densities	513

4.6.3	The pole (the residue trace)	515
4.6.3.1	Formulae for sphere integrals	516
4.6.3.2	Existence and traciality of res	521
4.6.3.3	Uniqueness of res	523
4.6.4	The constant term (a quasi-trace)	524
4.6.4.1	Global existence of the density	525
4.6.4.2	Trace defect formulae	529
4.6.4.3	Traciality on even–even operators	533
4.7	TR^{mer} on $\Psi^{z_n, N}$	537
4.7.1	Construction of the quasi-trace density	538
4.7.2	Densities and (quasi-)traces	544
4.7.2.1	A higher residue trace and various quasi-traces	545
4.7.2.2	TR_{quasi} on $\Psi^{z_n, N}$	546
4.7.2.3	Primitives in $\Psi^{z_n, N}$	548
4.8	Complex power gauging	550
4.8.1	Weakly parameter-dependent ψ dos	550
4.8.2	Trace formulae for the basic gauging on $\Psi^{c, z_n, N}$	553
4.8.3	Formulae for the zeta determinant	560
4.8.3.1	Trace formulae for zeta-regularized logs	561
4.8.3.2	Determinant defect/anomaly formulae	565
4.8.4	Residue determinant	569
4.8.4.1	Commutators of logarithmic ψ dos are classical	569
4.8.4.2	Residue density on logarithmic ψ dos	570
4.8.4.3	$\text{res}([S, T]) = 0$ for $S, T \in \Psi_{\log}^{0,1} + \Psi^\infty$	571
4.8.4.4	$\text{res}(\log_\theta A)$ is independent of θ	572
4.8.4.5	Residue determinant and $\zeta(A, 0)$	573
4.8.4.6	$\text{res} \log AB = \text{res} \log A + \text{res} \log B$	578
4.8.4.7	\log_θ is a logarithm map	581
4.9	Notes	581

5 Geometric families of ψ dos operators and determinant line bundles

5.1	Families of ψ dos associated to fibrations	599
5.1.1	Vertical ψ dos	599
5.1.1.1	Classical vertical ψ dos of non-constant order	602
5.1.1.2	Vertical ψ dos on fibred manifolds	603
5.1.2	Traces and quasi-traces on vertical ψ dos	603
5.1.3	Form-valued vertical ψ dos	605
5.1.3.1	Vertical ellipticity and traces	607
5.1.4	Asymptotic expansion of the resolvent trace form	608
5.1.5	Zeta forms and zeta-determinant forms	613
5.2	Determinant structures on vertical ψ dos	616
5.2.1	Vertical pseudodifferential logarithms	617
5.2.2	Review of facts about connections on $\pi_*(E)$	618

5.2.3	A ζ -form local family index theorem	623
5.2.4	A ζ -Chern form local family index theorem	627
	5.2.4.1 Example: counting holomorphic sections	636
5.3	Determinant line bundles for families of ψ dos	642
5.3.1	The determinant line bundle	643
	5.3.1.1 Line bundle structure	644
	5.3.1.2 Determinant bundle functor	645
	5.3.1.3 Regularized determinant functions	646
5.3.2	ζ metric and connection	649
	5.3.2.1 ζ metric	649
	5.3.2.2 ζ connection	650
	5.3.2.3 Example: conformal anomaly and the Polyakov formula	652
	5.3.2.4 Example: Quillen's computation of determinants of Cauchy–Riemann operators over a Riemann surface	653
	5.3.2.5 Parametrix formulae 5.3.2.26 and 5.3.2.27	658
5.4	Notes	662
	REFERENCES	664
	INDEX	673