Chapter 1
Introduction

The aim of this chapter is to give the reader a better orientation. For convenience of the reader we summarize the contents of the following chapters first, then we continue with some remarks to the history and finally, we collect the definitions of various function spaces and their coincidence relations.

1.1 A Short Summary of the Book

Chapter 2. For all $s, \tau \in \mathbb{R}$, all $p \in (0,\infty]$, and all $q \in (0,\infty]$, we introduce the inhomogeneous Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ are defined for the same range of parameters except that p has to be less than infinity. Also corresponding sequence spaces, $b^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $f^{s,\tau}_{p,q}(\mathbb{R}^n)$ (see Definitions 2.1 and 2.2 below), are introduced. The spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ are the inhomogeneous counterparts of $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ introduced earlier in [164, 165]. Via the Calderón reproducing formulae we establish the φ-transform characterization of these spaces in the sense of Frazier and Jawerth for all admissible values of the parameters s, τ, p, and q (see Theorem 2.1 below). On the one side this generalizes the classical results for $B^{s}_{p,q}(\mathbb{R}^n)$ and $F^{s}_{p,q}(\mathbb{R}^n)$ in [64, 65] by taking $\tau = 0$, on the other hand it also implies that $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ are well-defined. This method has to be traced to Frazier and Jawerth ([62, 64]; see also [65]), and has been further developed by Bownik [23–25]. We continue by deriving some embedding properties for different metrics by using the φ-transform characterization; see Sect. 2.2 below. Finally, the Fatou property of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ is established.

Chapter 3. To begin with, in Definition 3.1, we introduce a class of ε-almost diagonal operators on $b^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $f^{s,\tau}_{p,q}(\mathbb{R}^n)$. Any ε-almost diagonal operator is an almost diagonal operator in the sense of Frazier and Jawerth [64]. The main result in the first part of this chapter is given in Theorem 3.1 and concerns the boundedness of these operators on $b^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $f^{s,\tau}_{p,q}(\mathbb{R}^n)$, respectively. As an application we establish characterizations by atomic and molecular decompositions (see Theorems 3.2 and 3.3). In case $\tau = 0$, Theorems 3.1, 3.2 and 3.3 reduce to the well-known characterizations of $B^{s}_{p,q}(\mathbb{R}^n)$ and $F^{s}_{p,q}(\mathbb{R}^n)$, for which we refer to [25, 64, 65].

D. Yang et al., *Morrey and Campanato Meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics 2005, DOI 10.1007/978-3-642-14606-0_1,

In the second section of this chapter we shall compare the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ with other approaches to introduce spaces of Besov-Triebel-Lizorkin type built on Morrey spaces. Let $\mathscr{N}^s_{pqu}(\mathbb{R}^n)$ denote the Besov-Morrey spaces; see (xxv) in Sect. 1.3. Then our main result consists in

$$B^{s,1/u-1/p}_{u,\infty}(\mathbb{R}^n) = \mathscr{N}^s_{p\infty u}(\mathbb{R}^n), \qquad 0 < u \le p \le \infty,$$

in the sense of equivalent quasi-norms and, if $0 < q < \infty$,

$$\mathscr{N}^s_{pqu}(\mathbb{R}^n) \subset B^{s,1/u-1/p}_{u,q}(\mathbb{R}^n), \quad \mathscr{N}^s_{pqu}(\mathbb{R}^n) \neq B^{s,1/u-1/p}_{u,q}(\mathbb{R}^n), \quad 0 < u \le p \le \infty.$$

Let $\mathscr{E}^s_{pqu}(\mathbb{R}^n)$ $(p \neq \infty)$ denote the Triebel-Lizorkin-Morrey spaces studied in [88, 126, 139]. Then we have

$$F^{s,1/u-1/p}_{u,q}(\mathbb{R}^n) = \mathscr{E}^s_{pqu}(\mathbb{R}^n), \qquad 0 < u \le p < \infty,$$

with equivalent quasi-norms. In particular, if $1 < u \le p < \infty$

$$F^{0,1/u-1/p}_{u,2}(\mathbb{R}^n) = \mathscr{E}^0_{p2u}(\mathbb{R}^n) = \mathscr{M}^p_u(\mathbb{R}^n),$$

also in the sense of with equivalent norms. Thus, these conclusions combined with Theorem 2.1 also give the φ-transform characterization of the spaces $\mathscr{N}^s_{p\infty u}(\mathbb{R}^n)$ and $\mathscr{E}^s_{pqu}(\mathbb{R}^n)$, which seems to be also new.

Chapter 4. Following a well-known but rather long and technical procedure (see, for example, [109] and [145]), we establish some equivalent characterizations of the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$. Step by step we establish the following chain of inequalities. First we shall show that Littlewood-Paley characterizations can be dominated by characterizations by differences. The second step consists in proving that characterizations by differences can be estimated from above either by characterizations by oscillations or in terms of wavelet coefficients. The third step consists in estimating oscillations by wavelet coefficients. Finally, as an application of our atomic characterizations we can close the circle and estimate these expressions in terms of wavelet coefficients by the Littlewood-Paley characterization. Here we obtain generalizations of the well-known corresponding results for $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ $(p < \infty)$. They seem to be new for the classes $F^s_{\infty,q}(\mathbb{R}^n)$. A few more interesting localization properties of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ will given as well. In fact, at least for small s, membership of a continuous function in $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ can be checked by investigating the local behavior of this function in the corresponding space with $\tau = 0$.

Chapter 5. Based on the smooth atomic and molecular decompositions, derived in Theorems 3.2 and 3.3, we shall prove here the boundedness of exotic pseudo-differential operators on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ (see Theorem 5.1) under some restrictions for τ. This has several useful consequences. As applications of Theorem 5.1, we can establish mapping properties of $f \to \partial f$ as well as the so-called lifting property. Furthermore, we study the boundedness of nonlinear composition operators $T_f:\ g \to f \circ g$ on spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Chapter 6. This chapter is devoted to so-called key theorems; see [146, Chap. 4]. Assertions on pointwise multipliers (see Theorem 6.1), on diffeomorphisms (see Theorem 6.7) and traces (see Theorem 6.8) belong to this group. These theorems are basic for the definitions of Besov-Triebel-Lizorkin-type spaces on domains. We finally introduce Besov-Triebel-Lizorkin-type spaces on $\mathbb{R}^n_+$ and on bounded C^∞ domains in $\mathbb{R}^n$ and discuss a few properties.

Chapter 7. The main aim of this chapter consists in defining and investigating a class of spaces which have as duals the classes $A^{s,\tau}_{p,q}(\mathbb{R}^n)$. These spaces are introduced by using the Hausdorff capacity. For this reason we call them Besov-Hausdorff spaces $BH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin-Hausdorff spaces $FH^{s,\tau}_{p,q}(\mathbb{R}^n)$, respectively. They are the predual spaces of $B^{-s,\tau}_{p',q'}(\mathbb{R}^n)$ and $F^{-s,\tau}_{p',q'}(\mathbb{R}^n)$ (see Theorem 7.3 below). If $\tau = 0$, these results reduce to the classical duality assertions for Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. These new scales $BH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $FH^{s,\tau}_{p,q}(\mathbb{R}^n)$ have many properties in common with the classes $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$. In particular, we establish the φ-transform characterization, characterizations by smooth atomic and molecular decompositions, boundedness of certain pseudo-differential operators, the lifting property, a pointwise multiplier and a diffeomorphism theorem and finally assertions on traces. However, the most important property is the following: let $s \in \mathbb{R}$, $p = q \in (0,\infty)$ and $\tau \in [0, \frac{1}{p}]$, then

$$({}_0B^{s,\tau}_{p,p}(\mathbb{R}^n))^* = BH^{-s,\tau}_{p',p'}(\mathbb{R}^n),$$

where ${}_0B^{s,\tau}_{p,p}(\mathbb{R}^n)$ denotes the closure of $C^\infty_c(\mathbb{R}^n) \cap B^{s,\tau}_{p,p}(\mathbb{R}^n)$ in $B^{s,\tau}_{p,p}(\mathbb{R}^n)$ (see Theorem 7.12 below). By taking $s = 0$, $p = 2$ and $\tau = 1/2$ we get back the well-known result

$$(\mathrm{cmo}\,(\mathbb{R}^n))^* = h^1(\mathbb{R}^n),$$

where $\mathrm{cmo}\,(\mathbb{R}^n)$ is the local $\mathrm{CMO}(\mathbb{R}^n)$ space and $h^1(\mathbb{R}^n)$ is the local Hardy space; see Sect. 1.3. For suitable indices, the behavior of the scales $BH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $FH^{s,\tau}_{p,q}(\mathbb{R}^n)$ under real interpolation is investigated; see Theorem 7.14 below.

Chapter 8. In the last chapter we focus on the homogeneous case. The homogeneous spaces, including homogeneous Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, Triebel-Lizorkin-type spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and their preduals, homogeneous Besov-Hausdorff spaces $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin-Hausdorff spaces $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, were introduced and investigated in [127,164,165,168]. We gather some corresponding results for these spaces. In particular, we establish their wavelet characterizations (see Theorem 8.2 below).

1.2 A Piece of History

Here we will give a very rough overview about the history, mentioning some pioneering work, but without having the aim to reach completeness.

1.2.1 Besov-Triebel-Lizorkin Spaces

Nikol'skij [108] introduced in 1951 the Nikol'skij-Besov spaces, nowadays denoted by $B^s_{p,\infty}(\mathbb{R}^n)$. However, he was mentioning that this was based on earlier work of Bernstein [10] ($p = \infty$) and Zygmund [170] (periodic case, $n = 1, 1 < p < \infty$). Besov [11, 12] complemented the scale by introducing the third index q in 1959. We also refer to Taibleson [136–138] for the early investigations of Besov spaces. Around 1970 Lizorkin [91, 92] and Triebel [142] started to investigate the scale $F^s_{p,q}(\mathbb{R}^n)$, nowadays named after these two mathematicians. Further, we have to mention the contributions of Peetre [111, 113, 114], who extended around 1973–1975 the range of the admissible parameters p and q to values less than one.

Of particular importance for us has been the fundamental paper [64] of Frazier and Jawerth; see also [62, 63] and the monograph [65] of Frazier, Jawerth and Weiss in this connection. In these papers, the authors describe the Besov and Triebel-Lizorkin spaces in terms of a fixed countable family of functions with certain properties, namely, smooth atoms and molecules, which have been a second breakthrough in a certain sense (after the Fourier-analytic one in the seventieth), preparing the nowadays widely used wavelet decompositions. However, these decompositions were prepared by earlier contributions to the Calderón reproducing formula in [32, 38, 150, 155] and the studies in [41, 115]. We refer to the introduction in [64] for more details.

The theory is summarized in the monographs [14, 109, 114, 145–149]. A much more detailed history can be found in [146, 148]; see also [153].

1.2.2 Morrey-Campanato Spaces

In 1938 Morrey [102] introduced the classes $\mathscr{M}^p_u(\mathbb{R}^n)$ which are generalizations of the ordinary Lebesgue spaces. Next we would like to mention the work of John and Nirenberg, which introduced *BMO* in 1961 (see [79]). At the beginning of the sixties, in a series of papers, Campanato introduced and studied the spaces $\mathscr{L}^{p,\lambda}(\mathbb{R}^n)$, nowadays named after him; see also Meyers [101]. Peetre [110] gave a survey on the topic (to which we refer also for more detailed comments to the early history) and studied the interpolation properties of these classes. Section 2.4 in the monograph [88] of Kufner, John and Fučik is devoted to the study of Morrey and Campanato spaces and summarizes the state of the art at 1975.

Function spaces, defined by oscillations, i. e., local approximation by polynomials, were studied by Brudnij [26, 27], Il'in [13, 14], Christ [40], Bojarski [15], DeVore and Sharpley [46], Wallin [153], Seeger [130], and Triebel [146, Sect. 1.7], to mention only a few. Important for us has been also the general approach of Hedberg and Netrusov [70] to those function spaces.

1.2.3 Spaces Built on Morrey-Campanato Spaces

The Besov-Morrey spaces $\mathcal{N}^s_{pqu}(\mathbb{R}^n)$, $1 < u \le p < \infty$, $1 < q \le \infty$, were studied for the first time by Kozono and Yamazaki [88] in connection with applications to the Navier-Stokes equation. Also in connection with applications to pde the homogeneous version $\dot{\mathcal{N}}^s_{pqu}(\mathbb{R}^n)$, $1 < u \le p < \infty$, $1 < q \le \infty$, were studied by Mazzucato [97]. The next step has been done by Tang and Xu [139]. They introduced the scale $\mathcal{E}^s_{pqu}(\mathbb{R}^n)$ (the Triebel-Lizorkin counterpart of $\mathcal{N}^s_{pqu}(\mathbb{R}^n)$) and made first investigations for the extended range $0 < u \le p < \infty$, $0 < q \le \infty$, of parameters for both types of spaces. Later, Sawano and Tanaka [126] presented various decompositions including quarkonial, atomic and molecular characterizations of $\mathcal{A}^s_{pqu}(\mathbb{R}^n)$ and $\dot{\mathcal{A}}^s_{pqu}(\mathbb{R}^n)$, where $\mathcal{A} \in \{\mathcal{N}, \mathcal{E}\}$. Jia and Wang [78] investigated the Hardy-Morrey spaces, which are special cases of Triebel-Lizorkin-Morrey spaces. In [154], Wang obtained the atomic characterization and the trace theorem for Besov-Morrey and Triebel-Lizorkin-Morrey spaces independently of Sawano and Tanaka. Recently, Sawano [125] investigated the Sobolev embedding theorem for Besov-Morrey spaces. Recall that the Besov-Morrey and Triebel-Lizorkin-Morrey spaces cover many classic function spaces, such as Besov spaces, Triebel-Lizorkin spaces, Morrey spaces and Sobolev-Morrey spaces. For the Sobolev-Morrey spaces, we refer to Najafov [103–105].

The Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and its homogeneous version $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, restricted to the Banach space case, were first introduced by El Baraka in [49–51]. In these papers, El Baraka investigated embeddings as well as Littlewood-Paley characterizations of Campanato spaces. El Baraka showed that the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ cover certain Campanato spaces (see [51]).

Triebel-Lizorkin-Morrey spaces $\dot{\mathcal{E}}^s_{pqu}(\mathbb{R}^n)$ $(p \neq \infty)$ have been studied in [88, 126, 139]. The identity

$$\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{pqu}(\mathbb{R}^n)$$

has been proved in [127].

The Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the Triebel-Lizorkin-type spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ were introduced in [164, 165].

1.2.4 Q Spaces

The history of Q_α spaces (or simply Q spaces) started in 1995 with a paper by Aulaskari, Xiao and Zhao [7]. Originally they were defined as spaces of holomorphic functions on the unit disk, which are geometric in the sense that they transform naturally under conformal mappings (see [7, 160]). Following earlier contributions of Essén and Xiao [55] and Janson [76] on the boundary values of these functions on the unit circle, Essén, Janson, Peng and Xiao [54] extended these spaces to the n-dimensional Euclidean space $\mathbb{R}^n$. There is a rapidly increasing literature devoted to this subject, we refer to [7, 44, 45, 54, 55, 76, 157–162, 169].

Most recently, in [164, 165], two of the authors (W.Y and D.Y) have introduced the scales of homogeneous Besov-Triebel-Lizorkin-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ $(p \neq \infty)$, which generalize the homogeneous Besov-Triebel-Lizorkin spaces $(\dot{B}^s_{p,q}(\mathbb{R}^n)$, $\dot{F}^s_{p,q}(\mathbb{R}^n))$ and Q spaces simultaneously, and hence answered an open question posed by Dafni and Xiao in [44] concerning the relation of these spaces. In fact, it holds

$$\dot{F}^{\alpha,\frac{1}{2}-\frac{\alpha}{n}}_{2,2}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)$$

if $\alpha \in (0,1)$ $(n \geq 2)$.

Recently, Xiao [161], Li and Zhai [90] applied certain special cases of $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, including the Q spaces, to study the Navier-Stokes equation.

1.3 A Collection of the Function Spaces Appearing in the Book

As a service for the reader and also for having convenient references inside the book we give a list of definitions of the spaces of functions (distributions) showing up in this book. Sometimes a few comments will be added. We picked up this idea from [145, Sect. 2.2.2] and [153] and a part of our list is just a copy of the list given in [145].

As a general rule within this book we state that all spaces consist of complex-valued functions. We shall divide our collection into three groups:

- Function spaces defined by derivatives and differences.
- Function spaces defined by mean values and oscillations (local polynomial approximations).
- Function spaces defined by Fourier analytic tools.

The first item contains the classical approaches to define smoothness. In the second item we recall the definitions of spaces related to Morrey-Campanato spaces. Finally, in the third item we define spaces by Fourier analytic tools, in most of the cases by using a smooth dyadic resolution of unity.

1.3.1 Function Spaces Defined by Derivatives and Differences

(i) **Lebesgue spaces**. Let $p \in (0,\infty)$. By $L^p(\mathbb{R}^n)$ we denote the space of all measurable functions f such that

$$\|f\|_{L^p(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{1/p} < \infty.$$

In case $p = \infty$ the space $L^\infty(\mathbb{R}^n)$ is the collection of all measurable functions f such that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \equiv \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| < \infty.$$

Of a certain importance for the book are the following modified Lebesgue-type spaces. Let $\tau \in [0,\infty)$ and $p \in (0,\infty]$. Let $L^p_\tau(\mathbb{R}^n)$ be the collection of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L^p_\tau(\mathbb{R}^n)} \equiv \sup \frac{1}{|P|^\tau} \left(\int_P |f(x)|^p \, dx \right)^{1/p},$$

where the supremum is taken over all dyadic cubes P with side length $l(P) \geq 1$.

(ii) The space $C(\mathbb{R}^n)$ consists of all uniformly continuous functions f such that

$$\|f\|_{C(\mathbb{R}^n)} \equiv \sup_{x\in\mathbb{R}^n} |f(x)| < \infty.$$

(iii) Let $m \in \mathbb{N}$. The space $C^m(\mathbb{R}^n)$ consists of all functions $f \in C(\mathbb{R}^n)$, having all classical derivatives $\partial^\alpha f \in C(\mathbb{R}^n)$ up to order $|\alpha| \leq m$ and such that

$$\|f\|_{C^m(\mathbb{R}^n)} \equiv \sum_{|\alpha|\leq m} \|\partial^\alpha f\|_{C(\mathbb{R}^n)} < \infty.$$

We put $C^0(\mathbb{R}^n) \equiv C(\mathbb{R}^n)$.

(iv) **Hölder spaces**. Let $m \in \mathbb{Z}_+$ and $s \in (m,m+1)$. Then $C^s(\mathbb{R}^n)$ denotes the collection of all functions $f \in C^m(\mathbb{R}^n)$ such that

$$\|f\|_{C^s(\mathbb{R}^n)} \equiv \|f\|_{C^m(\mathbb{R}^n)} + \sum_{|\alpha|=m} \sup_{x\neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^{s-m}} < \infty.$$

(v) **Lipschitz spaces**. Let $s \in (0,1]$. The Lipschitz space $\mathrm{Lip}\, s(\mathbb{R}^n)$ consists of all functions $f \in C(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{Lip}\, s(\mathbb{R}^n)} \equiv \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty.$$

(vi) **Zygmund spaces**. Let $m \in \mathbb{N}$. The Zygmund space $\mathscr{Z}^m(\mathbb{R}^n)$ consists of all functions $f \in C^{m-1}(\mathbb{R}^n)$ such that

$$\begin{aligned}\|f\|_{\mathscr{Z}^m(\mathbb{R}^n)} &\equiv \|f\|_{C^{m-1}(\mathbb{R}^n)} \\ &\quad + \max_{|\alpha|=m} \sup_{h\neq 0} \sup_{x\in\mathbb{R}^n} \frac{|\partial^\alpha f(x+2h) - 2\,\partial^\alpha f(x+h) + \partial^\alpha f(x)|}{|h|} < \infty.\end{aligned}$$

In case of $s > 0$, $s \notin \mathbb{N}$, we use the convention $\mathscr{Z}^s(\mathbb{R}^n) = C^s(\mathbb{R}^n)$.

(vii) **Sobolev spaces**. Let $p \in (1,\infty)$ and $m \in \mathbb{N}$. Then $W^m_p(\mathbb{R}^n)$ is the collection of all functions $f \in L^p(\mathbb{R}^n)$ such that the distributional derivatives $\partial^\alpha f$ are functions belonging to $L^p(\mathbb{R}^n)$ for all α, $|\alpha| \leq m$. We equip this set with the norm

$$\|f\|_{W^m_p(\mathbb{R}^n)} \equiv \sum_{|\alpha|\leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

As usual, we define $W^0_p(\mathbb{R}^n) \equiv L^p(\mathbb{R}^n)$.

(viii) **Slobodeckij spaces**. Let $p \in [1,\infty)$ and let $s \in (0,\infty)$ be not an integer. Let $m \in \mathbb{Z}_+$ such that $s \in (m, m+1)$. Then $W_p^s(\mathbb{R}^n)$ consists of all functions $f \in W_p^m(\mathbb{R}^n)$ such that

$$\|f\|_{W_p^s(\mathbb{R}^n)} \equiv \|f\|_{W_p^m(\mathbb{R}^n)} + \sum_{|\alpha|=m} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+(m+1-s)p}} \, dx\,dy \right)^{1/p} < \infty.$$

(ix) **Besov spaces** (classical variant). Let $s \in (0,\infty)$ and $p, q \in [1,\infty]$. Let $M \in \mathbb{N}$. Then, if $s \in [M-1, M)$, the space $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all functions $f \in L^p(\mathbb{R}^n)$ satisfying

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \equiv \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} |h|^{-sq} \, \|\Delta_h^M f(\,\cdot\,)\|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^n} \right)^{1/q} < \infty.$$

Besov spaces can be defined in various ways; see in particular item (xx) below. In Chaps. 2–4 we shall prove the equivalence of some of these approaches in a much more general context.

1.3.2 Function Spaces Defined by Mean Values and Oscillations

Now we turn to a group of spaces which are related to Morrey-Campanato spaces.

(x) Functions of **bounded mean oscillations**. The space $\mathrm{BMO}\,(\mathbb{R}^n)$ is the set of locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathrm{BMO}\,(\mathbb{R}^n)} \equiv \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken on all cubes Q with sides parallel to the coordinate axes and where

$$f_Q \equiv \frac{1}{|Q|} \int_Q f(x) \, dx$$

denotes the mean value of the function f on Q.

(xi) According to Sarason [122], a function f of $\mathrm{BMO}\,(\mathbb{R}^n)$ which satisfies the limiting condition

$$\lim_{a \to 0} \left(\sup_{|Q| \le a} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \right) = 0$$

is said to be of **vanishing mean oscillation**. The subspace of $\mathrm{BMO}\,(\mathbb{R}^n)$ consisting of the functions of vanishing mean oscillation is denoted by $\mathrm{VMO}\,(\mathbb{R}^n)$. We note that the space $\mathrm{VMO}\,(\mathbb{R}^n)$ considered by Coifman and Weiss [42] is different from that considered by Sarason, and it coincides with our $\mathrm{CMO}\,(\mathbb{R}^n)$; see the next item.

(xii) We denote by $\mathrm{CMO}(\mathbb{R}^n)$ the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$, and we endow $\mathrm{CMO}(\mathbb{R}^n)$ with the norm of $\mathrm{BMO}(\mathbb{R}^n)$.

(xiii) Functions of **local bounded mean oscillations**. The space $\mathrm{bmo}(\mathbb{R}^n)$ consists of all functions $f \in \mathrm{BMO}(\mathbb{R}^n)$ which satisfy also the following condition

$$\sup_{|Q|\geq 1} \frac{1}{|Q|} \int_Q |f(x)|\,dx < \infty.$$

We equip this space with the norm

$$\|f\|_{\mathrm{bmo}(\mathbb{R}^n)} \equiv \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + \sup_{|Q|=1} \int_Q |f(x)|\,dx.$$

(xiv) Functions of **local vanishing mean oscillations**. We set

$$\mathrm{vmo}(\mathbb{R}^n) \equiv \mathrm{VMO}(\mathbb{R}^n) \cap \mathrm{bmo}(\mathbb{R}^n),$$

and we endow the space $\mathrm{vmo}(\mathbb{R}^n)$ with the norm of $\mathrm{bmo}(\mathbb{R}^n)$.

(xv) We denote by $\mathrm{cmo}(\mathbb{R}^n)$ the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{bmo}(\mathbb{R}^n)$, and we endow $\mathrm{cmo}(\mathbb{R}^n)$ with the norm of $\mathrm{bmo}(\mathbb{R}^n)$.

(xvi) **Morrey spaces**. Let $0 < u \leq p \leq \infty$. The space $\mathscr{M}_u^p(\mathbb{R}^n)$ is defined to be the set of all u-locally Lebesgue-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathscr{M}_u^p(\mathbb{R}^n)} \equiv \sup_B |B|^{1/p-1/u} \left(\int_B |f(x)|^u\,dx \right)^{1/u} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$; see [89, Sect. 2.4].

(xvii) **Campanato spaces**. Let $\lambda \in [0,\infty)$ and $p \in [1,\infty)$. The collection of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathscr{L}^{p,\lambda}(\mathbb{R}^n)} \equiv \sup_B \frac{1}{|B|^{\lambda/n}} \left(\int_B |f(x) - f_B|^p\,dx \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

This set becomes a Banach space if functions are considered modulo constants. Furthermore, $\mathscr{L}^{p,\lambda}(\mathbb{R}^n)$ consists of the constant functions only if $\lambda > n+p$; see [33–36], [110] and [89, Sect. 2.4].

(xviii) **Local approximation spaces I.** Let $p \in [1,\infty)$ and $s \in [-n/p,\infty)$. Let $B(x,t)$ be the ball with center x and radius t. Let $M \in \mathbb{Z}_+$. Denote by $\mathscr{P}_M(\mathbb{R}^n)$ the set of all polynomials of total degree less than or equal to M. For $u \in (0,\infty]$ we define the local oscillation of $f \in L^u_{\ell oc}(\mathbb{R}^n)$ by setting, for all $x \in \mathbb{R}^n$ and all $t \in (0,\infty)$,

$$\mathrm{osc}_u^M f(x,t) \equiv \inf \left(t^{-n} \int_{B(x,t)} |f(y) - P(y)|^u\,dy \right)^{1/u},$$

where the infimum is taken over all polynomials $P \in \mathscr{P}_M(\mathbb{R}^n)$ with the usual modification if $u = \infty$, i. e.,

$$\mathrm{osc}_\infty^M f(x,t) \equiv \inf \sup_{y \in B(x,t)} |f(y) - P(y)| \,.$$

Now we define the associated sharp maximal function

$$f_u^{M,s}(x) \equiv \sup_{0<t<1} t^{-s} \mathrm{osc}_u^M f(x,t) \,.$$

Let $M \equiv \max\{-1, \lfloor s \rfloor\}$. Then $T_p^s(\mathbb{R}^n)$ is the collection of all functions in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{T_p^s(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,1)} |f(x)|^p \, dx \right)^{1/p} + \sup_{x \in \mathbb{R}^n} f_u^{M,s}(x) < \infty .$$

We followed [146, Sect. 1.7.2] (but change the notation because of item (i)); see also [153].

(xix) **Local approximation spaces II.** Let $p \in (0,\infty]$, $s \in (0,\infty)$ and $M \equiv \lfloor s \rfloor$. The local approximation space $C_s^p(\mathbb{R}^n)$ is the collection of all functions $f \in L^{\max\{p,1\}}(\mathbb{R}^n)$ such that

$$\|f\|_{C_s^p(\mathbb{R}^n)} \equiv \|f\|_{L^p(\mathbb{R}^n)} + \|f_p^{M,s}\|_{L^p(\mathbb{R}^n)} < \infty .$$

We refer to [15, 40, 46, 153] and [146, Sect. 1.7.2].

(xx) Let $\alpha \in \mathbb{R}$. The *space* $Q_\alpha(\mathbb{R}^n)$ is defined to be the collection of all $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{Q_\alpha(\mathbb{R}^n)} \equiv \sup_I \left\{ \frac{1}{|I|^{1-\frac{2\alpha}{n}}} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \right\}^{1/2} < \infty,$$

where I ranges over all cubes in $\mathbb{R}^n$; see, for example, [7, 44, 54].

1.3.3 Function Spaces Defined by Fourier Analytic Tools

Except the first two all spaces here will be defined by using a decomposition in the Fourier image induced by a smooth dyadic decomposition of unity. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a radial function such that $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 3/2$. Then by means of

$$\psi^0(x) \equiv \psi(x) \,, \qquad \psi^j(x) \equiv \psi(2^{-j}x) - \psi(2^{-j+1}x) \,, \quad j \in \mathbb{N}, \quad x \in \mathbb{R}^n, \quad (1.1)$$

we obtain a smooth dyadic decomposition of unity, namely,

$$\sum_{j=0}^{\infty} \psi^j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

We put

$$\varphi_0 \equiv \Phi \equiv \mathscr{F}^{-1}\psi,\ \varphi(x) \equiv \mathscr{F}^{-1}[\psi(2\xi)](x) \quad \text{and} \quad \varphi_j \equiv \mathscr{F}^{-1}\psi^j, \quad j \in \mathbb{Z}_+. \tag{1.2}$$

Then $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, $j \in \mathbb{Z}_+$, follows.

(xxi) **Local Hardy spaces**. Let $p \in (0,\infty)$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\widehat{\varphi}(0) = 1$. Then $h^p(\mathbb{R}^n)$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{h^p(\mathbb{R}^n)} \equiv \left\| \sup_{0<t<1} \mathscr{F}^{-1}[\varphi(t\xi)\mathscr{F}f(\xi)](\cdot) \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

(xxii) **Bessel-potential spaces** (sometimes also called Lebesgue or Liouville spaces). Let $s \in \mathbb{R}$ and $p \in (1,\infty)$. Then $H^s_p(\mathbb{R}^n)$ is the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\mathscr{F}^{-1}[(1+|\xi|^2)^{s/2}\mathscr{F}f(\xi)](\cdot)$ is a regular distribution and

$$\|f\|_{H^s_p(\mathbb{R}^n)} \equiv \left\| \mathscr{F}^{-1}[(1+|\xi|^2)^{s/2}\mathscr{F}f(\xi)](\cdot) \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

(xxiii) **Besov spaces** (general case). Let $p, q \in (0,\infty]$ and $s \in \mathbb{R}$. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in (1.2). Then $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \equiv \left\{ \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j * f\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

(xxiv) **Triebel-Lizorkin spaces**. Let $p \in (0,\infty)$, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in (1.2). Then $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \equiv \left\| \left\{ \sum_{j=0}^{\infty} (2^{js}|\varphi_j * f|)^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

We refer to [64, 145]. The *Triebel-Lizorkin space* $F^s_{\infty,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{\infty,q}(\mathbb{R}^n)} \equiv \sup_{\substack{P \text{dyadic} \\ l(P)\le 1}} \left\{ \frac{1}{|P|} \int_P \sum_{j=j_P}^{\infty} [2^{js}|\varphi_j * f(x)|]^q \, dx \right\}^{1/q} < \infty, \tag{1.3}$$

where the supremum is taken over all dyadic cubes P with side length $l(P) \le 1$ and $j_P \equiv -\log_2 l(P)$; see [64].

(xxv) **Besov-Morrey spaces**. Let $s \in \mathbb{R}$, $q \in (0,\infty]$ and $0 < u \le p \le \infty$. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in (1.2). Then $\mathscr{N}^s_{pqu}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathscr{N}^s_{pqu}(\mathbb{R}^n)} \equiv \left\{ \sum_{j=0}^{\infty} 2^{jsq} \sup_B |B|^{q/p-q/u} \left(\int_B |\varphi_j * f(x)|^u \, dx \right)^{q/u} \right\}^{1/q} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$; see [88, 97].

(xxvi) **Triebel-Lizorkin-Morrey spaces**. Let $s \in \mathbb{R}$, $q \in (0,\infty]$ and $0 < u \le p \le \infty$, $u \ne \infty$. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in (1.2). The class $\mathscr{E}^s_{pqu}(\mathbb{R}^n)$ is defined to be the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathscr{E}^s_{pqu}(\mathbb{R}^n)} \equiv \sup_B |B|^{1/p-1/u} \left\{ \int_B \left(\sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f(x)|^q \right)^{u/q} dx \right\}^{1/u} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. We refer, e. g., to [88, 126, 139].

(xxvii) **Inhomogeneous Besov-type spaces**. Let $\tau, s \in \mathbb{R}$ and $p, q \in (0,\infty]$. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in (1.2). The *inhomogeneous Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \sup_{P\in\mathscr{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=(j_P\vee 0)}^{\infty} \left[\int_P (2^{js}|\varphi_j * f(x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty.$$

(xxviii) **Inhomogeneous Triebel-Lizorkin-type spaces**. Let $\tau, s \in \mathbb{R}$, $q \in (0,\infty]$ and $p \in (0,\infty)$. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in (1.2). The *inhomogeneous Triebel-Lizorkin-type space* $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \sup_{P\in\mathscr{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=(j_P\vee 0)}^{\infty} (2^{js}|\varphi_j * f(x)|)^q \right]^{p/q} dx \right\}^{1/p} < \infty.$$

A comment. The definitions of the spaces in (1.3) and (xxv)–(xxviii) are all of the same spirit. The major difference between Besov-Morrey and Triebel-Lizorkin-Morrey spaces on the one side and the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ on the other side consists in the starting index for the summation with respect to j. In (xxv) and (xxvi) the summation starts always with 0, whereas in the (xxvii) and (xxviii) the summation starts at $j_P \vee 0$. Comparing with (1.3) we find that this time there is a difference in the set of admissible cubes. The distribution spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ have some overlap with all 26 different classes we have introduced above; see the next subsection.

(xxix) **Homogeneous Besov-type spaces**. Let $\tau, s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Let $\varphi_j(x) \equiv 2^{jn}\varphi(2^j x)$, $j \in \mathbb{Z}$. The *Besov-type space* $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \sup_{P \in \mathscr{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P (2^{js}|\varphi_j * f(x)|)^p \, dx \right]^{q/p} \right\}^{1/q}$$

with suitable modifications made when $p = \infty$ or $q = \infty$.

(xxx) **Homogeneous Triebel-Lizorkin-type spaces**. Let $\tau, s \in \mathbb{R}$, $q \in (0, \infty]$ and $p \in (0, \infty)$. Let $\varphi_j(x) \equiv 2^{jn}\varphi(2^j x)$, $j \in \mathbb{Z}$. The *Triebel-Lizorkin-type space* $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \sup_{P \in \mathscr{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} (2^{js}|\varphi_j * f(x)|)^q \, dx \right]^{p/q} \right\}^{1/p}$$

with suitable modifications made when $q = \infty$.

(xxxi) **Besov-Hausdorff spaces and Triebel-Lizorkin-Hausdorff spaces**. The inhomogeneous classes $BH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $FH^{s,\tau}_{p,q}(\mathbb{R}^n)$ will be investigated in Chap. 7. For the homogeneous counterparts, see Sect. 8.4.
Let $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$, $\{\varphi_j\}_{j \in \mathbb{Z}_+}$ be as in (1.2). The *Besov-Hausdorff spaces* $BH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the *Triebel-Lizorkin-Hausdorff spaces* $FH^{s,\tau}_{p,q}(\mathbb{R}^n)$ $(q \neq 1)$ are defined, respectively, to be the sets of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{BH^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\{ \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j * f[\omega(\cdot, 2^{-j})]^{-1}\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty$$

and

$$\|f\|_{FH^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\| \left\{ \sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q [\omega(\cdot, 2^{-j})]^{-q} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

the infimums here are taken over all nonnegative Borel measurable functions ω on $\mathbb{R}^n \times (0, \infty)$ with

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p \vee q)'} \, d\Lambda^{(\infty)}_{n\tau(p \vee q)'}(x) \le 1,$$

and with the restriction that $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\varphi_j * f$ vanishes, where $N\omega$ is the nontangential maximal function of ω

and $\Lambda^{(\infty)}_{n\tau(p\vee q)'}$ is the $n\tau(p\vee q)'$-dimensional Hausdorff capacity; see Sect. 7.1 below.

(xxxii) There is a number of further spaces appearing in the book. But they will be of restricted importance.

1.4 A Table of Coincidences

As mentioned above there is some overlap of these different definitions. We are collecting some of these coincidence relations in what follows.

1.4.1 Besov-Morrey Spaces

(i) It holds $B^{s,0}_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$ for all s, p, and q; see Lemma 2.1. This implies

$$\begin{aligned} B^{s,0}_{\infty,\infty}(\mathbb{R}^n) &= B^s_{\infty,\infty}(\mathbb{R}^n) = \mathscr{Z}^s(\mathbb{R}^n), && s>0, \\ B^{s,0}_{\infty,\infty}(\mathbb{R}^n) &= B^s_{\infty,\infty}(\mathbb{R}^n) = C^s(\mathbb{R}^n), && s>0,\ s\notin\mathbb{N}, \\ B^{s,0}_{\infty,\infty}(\mathbb{R}^n) &= B^s_{\infty,\infty}(\mathbb{R}^n), && 0<s<1, \\ B^{s,0}_{p,p}(\mathbb{R}^n) &= B^s_{p,p}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n), && s>0,\ s\notin\mathbb{N},\ 1\le p<\infty \end{aligned}$$

(all in the sense of equivalent norms); see, e. g., [145, Sect. 2.2.2] and the references given there.

(ii) Let $s\in\mathbb{R}$, $0<u\le p\le\infty$ and $q\in(0,\infty]$. On the one hand we have

$$B^{s,0}_{p,q}(\mathbb{R}^n) = \mathscr{N}^s_{pqp}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad B^{s,1/u-1/p}_{u,\infty}(\mathbb{R}^n) = \mathscr{N}^s_{p\infty u}(\mathbb{R}^n)$$

(in the sense of equivalent quasi-norms), but on the other hand, it holds

$$B^{s,1/u-1/p}_{u,q}(\mathbb{R}^n) \supsetneq \mathscr{N}^s_{pqu}(\mathbb{R}^n) \qquad \text{if} \qquad 0<u<p<\infty \quad \text{and} \quad 0<q<\infty;$$

see Corollary 3.3.

(iii) Let $0<p<p_0<\infty$, $k\in\mathbb{N}$ and

$$s > \frac{k}{p} + n\max\left\{0, \frac{1}{p}-1\right\}.$$

Then

$$B^{s-k/p,\frac{1}{p}\frac{n+k}{n}}_{p,q}(\mathbb{R}^n) = \mathscr{Z}^s(\mathbb{R}^n) \qquad \text{if} \qquad p\le q\le\infty,$$

and

$$B_{p_0,q}^{s-\frac{k+n}{p}+\frac{n}{p_0},\frac{1}{p}\frac{n+k}{n}}(\mathbb{R}^n) = \mathscr{L}^s(\mathbb{R}^n) \qquad \text{if} \qquad p \le q \le \infty,$$

in the sense of equivalent quasi-norms; see Theorem 6.9 below.

1.4.2 Triebel-Lizorkin-Morrey Spaces

(i) It holds $F_{u,q}^{s,0}(\mathbb{R}^n) = F_{p,q}^s(\mathbb{R}^n)$; see Lemma 2.1. This implies

$$\begin{aligned} F_{p,2}^{m,0}(\mathbb{R}^n) &= F_{p,2}^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n), \qquad m \in \mathbb{N},\ 1 < p < \infty, \\ F_{p,p}^{s,0}(\mathbb{R}^n) &= F_{p,p}^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n), \qquad s > 0,\ s \notin \mathbb{N},\ 1 \le p < \infty, \\ F_{p,2}^{0,0}(\mathbb{R}^n) &= F_{p,2}^0(\mathbb{R}^n) = h^p(\mathbb{R}^n), \qquad 0 < p < \infty, \\ F_{\infty,2}^0(\mathbb{R}^n) &= \mathrm{bmo}\,(\mathbb{R}^n) \end{aligned}$$

(all in the sense of equivalent norms); see, e. g. [145, Sect. 2.2.2] and the references given there.

(ii) Let $p \in (0,\infty)$ and $s \in (n \max\{0, \frac{1}{p} - 1\}, \infty)$. Then

$$F_{p,\infty}^{s,0}(\mathbb{R}^n) = F_{p,\infty}^s(\mathbb{R}^n) = C_s^p(\mathbb{R}^n);$$

see [130] and [146, Theorem 1.7.2].

(iii) Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Then

$$F_{p,q}^{s,1/p}(\mathbb{R}^n) = F_{\infty,q}^s(\mathbb{R}^n)$$

with equivalent quasi-norms; see [64] or Proposition 2.4 below. In particular,

$$F_{p,2}^{0,1/p}(\mathbb{R}^n) = F_{\infty,2}^0(\mathbb{R}^n) = \mathrm{bmo}\,(\mathbb{R}^n).$$

(iv) Let $q \in (0,\infty]$ and $0 < u \le p \le \infty$, $u \neq \infty$. Then

$$F_{u,q}^{s,1/u-1/p}(\mathbb{R}^n) = \mathscr{E}_{pqu}^s(\mathbb{R}^n).$$

For $s = 0$ and $1 < u \le p < \infty$ this yields

$$F_{u,2}^{0,1/u-1/p}(\mathbb{R}^n) = \mathscr{E}_{p2u}^0(\mathbb{R}^n) = \mathscr{M}_u^p(\mathbb{R}^n) \tag{1.4}$$

and with $1 < u = p < \infty$

$$F_{p,2}^{0,0}(\mathbb{R}^n) = \mathscr{E}_{p2p}^0(\mathbb{R}^n) = \mathscr{M}_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n),$$

all in the sense of equivalent quasi-norms; see Corollary 3.3 below.

(v) Let $\alpha \in (0,1)$ if $n \geq 2$ and $\alpha \in (0,1/2)$ if $n = 1$. Then we have

$$F_{2,2}^{\alpha,\frac{1}{2}-\frac{\alpha}{n}}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n) \cap L^2_{\frac{1}{2}-\frac{\alpha}{n}}(\mathbb{R}^n),$$

in the sense of equivalent norms; see Corollary 4.5 and Remark 4.7.

(vi) Let $0 < p < p_0 < \infty$, $k \in \mathbb{N}$ and

$$s > \frac{k}{p} + n \max\left\{0, \frac{1}{p} - 1\right\}.$$

Then

$$F_{p,q}^{s-k/p,\frac{1}{p}\frac{n+k}{n}}(\mathbb{R}^n) = \mathscr{Z}^s(\mathbb{R}^n) \qquad \text{if} \qquad p \leq q \leq \infty,$$

and

$$F_{p_0,q}^{s-\frac{k+n}{p}+\frac{n}{p_0},\frac{1}{p}\frac{n+k}{n}}(\mathbb{R}^n) = \mathscr{Z}^s(\mathbb{R}^n) \qquad \text{if} \qquad 0 < q \leq \infty,$$

in the sense of equivalent quasi-norms; see Theorem 6.9 below.

(vii) Pointwise multipliers. For a quasi-Banach space X of functions, the space $M(X)$ denotes the associated space of all pointwise multipliers; see Sect. 6.1. Let $s \in (0,1)$. Then

$$M(F^s_{1,1}(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n) \cap F^{s,\tau}_{1,1,\mathrm{unif}}(\mathbb{R}^n), \qquad \tau = 1 - s/n;$$

see Corollary 6.2 below.

1.4.3 Morrey-Campanato Spaces

(i) Let $0 < u \leq p \leq \infty$. Then

$$\mathscr{M}^u_u(\mathbb{R}^n) = L^u(\mathbb{R}^n) \qquad \text{and} \qquad \mathscr{M}^\infty_u(\mathbb{R}^n) = L^\infty(\mathbb{R}^n).$$

(ii) Let $p \in [1,\infty)$ and $\lambda \in (n, n+p)$. Then $\mathscr{L}^{p,n}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$,

$$\mathscr{L}^{p,n}(\mathbb{R}^n) = \mathscr{Z}^{\frac{\lambda-n}{p}}(\mathbb{R}^n) \qquad \text{and} \qquad \mathscr{L}^{p,n+p}(\mathbb{R}^n) = \mathrm{Lip}\,1(\mathbb{R}^n);$$

see [34, 36] and [89, Theorem 2.4.6.1].

(iii) Let $p \in [1,\infty)$. Then

$$T_p^{-n/p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \qquad \text{and} \qquad T^0_p(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n).$$

(iv) Let $p \in [1,\infty)$ and $s \in (-n/p, 0)$. Then

$$\mathscr{L}^{p,\lambda}(\mathbb{R}^n) = \mathscr{M}_p^{-n/s}(\mathbb{R}^n) = T^s_p(\mathbb{R}^n), \qquad s = \frac{\lambda - n}{p};$$

see [89, Theorem 2.4.6.1] and [146, Sect. 1.7.2].

(v) Let $p \in [1,\infty)$ and $s \in (0,\infty)$. Then

$$T_p^s(\mathbb{R}^n) = \mathscr{Z}^s(\mathbb{R}^n);$$

see [146, Sect. 1.7.2] and the references given there.

1.4.4 Homogeneous Spaces

Here we make use of the following interpretation. When comparing a class of functions, which is defined modulo polynomials of a certain order, with the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, then we always associate to an element of the first space the equivalence class

$$[f] \equiv \{g: \ g = f + p, \quad p \quad \text{is an arbitrary polynomial}\}.$$

By means of this interpretation the following relations are known.

(i) We have

$$\dot{F}^0_{\infty,2}(\mathbb{R}^n) = \mathrm{BMO}\,(\mathbb{R}^n).$$

(ii) Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Then

$$\dot{F}^{s,1/p}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{\infty,q}(\mathbb{R}^n)$$

with equivalent quasi-norms. In particular,

$$\dot{F}^{0,1/p}_{p,2}(\mathbb{R}^n) = \dot{F}^0_{\infty,2}(\mathbb{R}^n) = \mathrm{BMO}\,(\mathbb{R}^n).$$

(iii) Let $\alpha \in (0,1)$ if $n \geq 2$ and $\alpha \in (0,1/2)$ if $n = 1$. Then we have

$$\dot{F}^{\alpha,\frac{1}{2}-\frac{\alpha}{n}}_{2,2}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)$$

in the sense of equivalent norms; see [164].

(iv) Let $\lambda \in [0, n+2)$. Then

$$\dot{F}^{0,2\lambda/n}_{2,2}(\mathbb{R}^n) = \mathscr{L}^{2,\lambda}(\mathbb{R}^n);$$

see [50].

1.5 Notation

At the end of this chapter, we make some conventions on notation.

Throughout this book, C denotes unspecified positive constants, possibly different at each occurrence; the symbol $X \lesssim Y$ means that there exists a positive constant

C such that $X \le CY$, and $X \sim Y$ means $C^{-1}Y \le X \le CY$. We also use $C(\gamma, \beta, \cdots)$ to denote a positive constant depending on the indicated parameters γ, β, $\cdots$.

The real numbers are denoted by $\mathbb{R}$. Many times we shall use the abbreviations

$$a_+ \equiv \max(0, a),$$

$\lfloor a \rfloor$ for the integer part of the real number a, and $a^* \equiv a - \lfloor a \rfloor$. The symbol χ_E is used to denote the *characteristic function* of set $E \subset \mathbb{R}^n$. If $q \in [1, \infty]$ then by q' we mean its *conjugate index*, i. e., $1/q + 1/q' = 1$. Further we shall use the abbreviations

$$p \vee q \equiv \max\{p, q\}$$

and

$$p \wedge q \equiv \min\{p, q\}.$$

When dealing with the classes $A^{s,\tau}_{p,q}(\mathbb{R}^n)$, then four restrictions for the set of parameters s, p, q, τ will occur relatively often. They are connected with the quantities

$$\sigma_p \equiv \max\{n(1/p - 1), 0\} \quad \text{and} \quad \sigma_{p,q} \equiv \max\{n(1/\min\{p,q\} - 1), 0\}, \tag{1.5}$$

(restrictions for s) and

$$\tau_{s,p} \equiv \frac{1}{p} + \begin{cases} \frac{1-(\sigma_p+n-s)^*}{n} & \text{if} \quad s \le \sigma_p\,, \\ \frac{s-\sigma_p}{n} & \text{if} \quad s > \sigma_p\,, \end{cases} \tag{1.6}$$

$$\tau_{s,p,q} \equiv \frac{1}{p} + \begin{cases} \frac{1-(\sigma_{p,q}+n-s)^*}{n} & \text{if} \quad s \le \sigma_{p,q}\,, \\ \frac{s-\sigma_{p,q}}{n} & \text{if} \quad s > \sigma_{p,q} \end{cases} \tag{1.7}$$

(restrictions for τ). Also, set $\mathbb{N} \equiv \{1, 2, \cdots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. By $C^\infty_c(\mathbb{R}^n)$ we denote the set of all infinitely differentiable and compactly supported functions on $\mathbb{R}^n$. The symbol $\mathscr{S}(\mathbb{R}^n)$ is used in place of the set of all *Schwartz functions* φ on $\mathbb{R}^n$, i. e., φ is infinitely differentiable and

$$\|\varphi\|_{\mathscr{S}_M} \equiv \sup_{\gamma \in \mathbb{Z}^n_+, |\gamma| \le M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma \varphi(x)| (1 + |x|)^{n+M+|\gamma|} < \infty$$

for all $M \in \mathbb{N}$. The topological dual of $\mathscr{S}(\mathbb{R}^n)$, the set of *tempered distributions*, will be denoted by $\mathscr{S}'(\mathbb{R}^n)$.

For $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, Q_{jk} denotes the dyadic cube

$$Q_{jk} \equiv \{(x_1, \cdots, x_n) : \; k_i \le 2^j x_i < k_i + 1 \text{ for } i = 1, \cdots, n\}.$$

For the collection of all such cubes we use

$$\mathscr{Q} \equiv \left\{Q_{jk} : \; j \in \mathbb{Z}, k \in \mathbb{Z}^n\right\}.$$

Furthermore, we denote by x_Q the *lower left-corner* $2^{-j}k$ of $Q = Q_{jk}$. When the dyadic cube Q appears as an index, such as $\sum_{Q\in\mathscr{Q}}$ and $\{\cdot\}_{Q\in\mathscr{Q}}$, it is understood that Q runs over all *dyadic cubes* in $\mathbb{R}^n$. For each cube Q, we denote its *side length* by $l(Q)$, its *center* by c_Q, and for $r > 0$, we denote by rQ the *cube concentric* with Q having the side length $rl(Q)$. Further, the abbreviation $j_Q \equiv -\log_2 l(Q)$ is used.

For $j \in \mathbb{Z}$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set $\widetilde{\varphi}(x) \equiv \overline{\varphi(-x)}$,

$$\widehat{\varphi}(x) \equiv \mathscr{F}\varphi(x) \equiv \int_{\mathbb{R}^n} \varphi(\xi)e^{-ix\cdot\xi}\, d\xi\,,$$

$\varphi_j(x) \equiv 2^{jn}\varphi(2^j x)$, and

$$\varphi_Q(x) \equiv |Q|^{-1/2}\varphi(2^j x - k) = |Q|^{1/2}\varphi_j(x - x_Q) \qquad \text{if} \quad Q = Q_{jk}\,.$$

For a dyadic cube Q, we shall work also with the $L^2(\mathbb{R}^n)$-normalized version

$$\widetilde{\chi}_Q(x) \equiv |Q|^{-1/2}\chi_Q(x)\,.$$

Let E denote a class of tempered distributions. Then E_{loc} denotes the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that the product $\varphi \cdot f$ belongs to E for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$. Furthermore, if E is in addition quasi-normed, then E_{unif} is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{E_{\mathrm{unif}}} \equiv \sup_{\lambda\in\mathbb{R}^n} \|\psi(\cdot - \lambda)f(\cdot)\|_E < \infty.$$

Here ψ is a nontrivial function in $C_c^\infty(\mathbb{R}^n)$.