Preface

This book is based on three developments in the theory of function spaces.

As *the first* we wish to mention **Besov** and **Triebel-Lizorkin spaces**. These scales

$$B^s_{p,q}(\mathbb{R}^n) \text{ and } F^s_{p,q}(\mathbb{R}^n)$$

allow a unified approach to various types of function spaces which have been known before like Hölder-Zygmund spaces, Sobolev spaces, Slobodeckij spaces and Bessel-potential spaces. Over the last 60 years these scales have proved their usefulness, there are hundreds of papers and many books using these scales in various connections. In a certain sense all these spaces are connected with the usual Lebesgue spaces $L^p(\mathbb{R}^n)$.

The *second source* we wish to mention is **Morrey** and **Campanato spaces**. Since several years there is an increasing interest in function spaces built on Morrey spaces and leading to generalizations of Campanato spaces. This interest originates, at least partly, in some applications in the field of Navier-Stokes equations.

The *third ingredient* is the so-called Q **spaces** (Q_α spaces). These spaces were originally defined as spaces of holomorphic functions on the unit disk, which are geometric in the sense that they transform naturally under conformal mappings. However, about 10 years ago, M. Essén, S. Janson, L. Peng and J. Xiao extended these spaces to the n-dimensional Euclidean space $\mathbb{R}^n$.

The aim of the book consists in giving a unified treatment of all these three types of spaces, i.e., we will define and investigate the scales

$$B^{s,\tau}_{p,q}(\mathbb{R}^n) \text{ and } F^{s,\tau}_{p,q}(\mathbb{R}^n)$$

generalizing the three types of spaces mentioned before. Such projects have been undertaken by various mathematicians during the last ten years, which have been investigating Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Let us mention only the names Kozono, Yamazaki, Mazzucato, El Baraka, Sawano, Tang, Xu and two of the authors (W.Y. and D.Y.) in this connection. A more detailed history will be given in the first chapter of the book; see Sect. 1.2.

Let us further mention the approach of Hedberg and Netrusov [70] to general spaces of Besov-Triebel-Lizorkin type. There is some overlap with our treatment. Details will be given in Sect. 4.5.

The real persons Besov, Lizorkin and Triebel never met Morrey or Campanato (which we learned from personal communications with Professor Besov and Professor Triebel). However, we hope at least, the meaning of the title is clear. We shall develop a theory of spaces of Besov-Triebel-Lizorkin type built on Morrey spaces.

A second aim of the book, just a byproduct of the first, will be a completion of the theory of the Triebel-Lizorkin spaces $F^s_{\infty,q}(\mathbb{R}^n)$. By looking into the series of monographs written by Triebel over the last 30 years, these spaces play an exceptional role, in most of the cases they are even not treated. The only exception is the monograph [145], where they are introduced essentially as the dual spaces of $F^s_{1,q}(\mathbb{R}^n)$ (with some restrictions in q). Also after Jawerth and Frazier [64] have found a more appropriate definition, there have been no further contributions developing the theory of these spaces further, e. g., by establishing characterizations by differences or local oscillations (at least we do not know about).

In Chaps. 4–6 we shall prove characterizations by differences, local oscillations, and wavelets as well as assertions on the boundedness of pseudo-differential operators, nonlinear composition operators and pointwise multipliers.

In this book we only treat unweighted isotropic spaces, with other words, all directions and all points in $\mathbb{R}^n$ are of equal value. This means anisotropic and/or weighted spaces are not treated here. Further, we also do not deal with spaces of generalized smoothness or smoothness parameters depending on x (variable exponent spaces). However, some basic properties of corresponding spaces of Besov-Triebel-Lizorkin type are known in all these situations, we refer to

- Anisotropic spaces: [3, 13, 14, 148].
- Spaces of dominating mixed smoothness: [4, 128, 129, 151].
- Weighted spaces: [120, 129].
- Spaces of generalized smoothness: [57].
- Spaces of variable exponent: [47, 152].

Further investigations could be based also on a generalization of the underlying Morrey spaces, we refer to [29–31]. We believe that our methods could be applied also in these more general situations. But nothing is done at this moment.

The book contains eight chapters. Because of the generality of the spaces we use Chap. 1 for helping the reader to get an overview in various directions. First of all we summarize the contents of Chaps. 2–8. Second, we give a list of definitions of the function spaces which occur in the book. Third, we collect the various known coincidences of these spaces. Finally, we add a short history. Chapters 2–6 deal with the definition and basic properties of the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$. Chapter 7 is devoted to the study of Besov-Hausdorff and Triebel-Lizorkin-Hausdorff spaces. Finally, in Chap. 8, parts of the theory of the *homogeneous* counterparts,

$$\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) \text{ and } \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n),$$

of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ are discussed.

The book is essentially self-contained. However, sometimes we carry over some results originally obtained for the homogeneous spaces, mainly from [163–165]. The papers [163–165] supplement the book in a certain sense. Most of the results are new in this generality and have been published never before.

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