Preface

The content of this monograph is situated in the intersection of important branches of mathematics like the theory of one complex variable, algebraic geometry, low dimensional topology and, from the point of view of the techniques used, combinatorial group theory. The main tool comes from the Uniformization Theorem for Riemann surfaces, which relates the topology of Riemann surfaces and holomorphic or antiholomorphic actions on them to the algebra of classical cocompact Fuchsian groups or, more generally, non-euclidean crystallographic groups. Foundations of this relationship were established by A. M. Macbeath in the early sixties and developed later by, among others, D. Singerman.

Another important result in Riemann surface theory is the connection between Riemann surfaces and their symmetries with complex algebraic curves and their real forms. Namely, there is a well known functorial bijective correspondence between compact Riemann surfaces and smooth, irreducible complex projective curves. The fact that a Riemann surface has a symmetry means, under this equivalence, that the corresponding complex algebraic curve has a real form, that is, it is the complexification of a real algebraic curve. Moreover, symmetries which are non-conjugate in the full group of automorphisms of the Riemann surface, correspond to real forms which are birationally non-isomorphic over the reals. Furthermore, the set of points fixed by a symmetry is homeomorphic to a projective smooth model of the real form.

The monograph consists of an extensive Introduction, a compilation of basic results in the Preliminaries, four principal Chapters and a short Appendix on asymmetric Riemann surfaces. After the Preliminaries, in Chap. 2, we focus our attention on the quantitative results concerning upper bounds for the number of conjugacy classes of symmetries. We divide our study into three cases, according to the nature of the set of points fixed by the symmetries. Namely we distinguish whether this set is empty or not and, accordingly, consider just symmetries with fixed points, just symmetries without fixed points and finally hybrid configurations allowing both types of symmetries simultaneously.

Chapter 3 can be seen as a variation on the classical Harnack theorem, that states that the set of points fixed by a symmetry of a Riemann surface of genus g has at most $g + 1$ connected components, all of them being closed Jordan curves, called ovals in Hilbert's terminology introduced in the nineteenth century. We first deal with the problem of finding the total number of ovals of a specified number of non-conjugate symmetries. We next consider the same problem for all the symmetries (conjugate or not) of a Riemann surface. We finally deal with the total number of ovals of a pair of symmetries in terms of the order of its product and the genus of the surface.

The monograph is actually devoted to the symmetries of Riemann surfaces of genus at least two since they are the ones uniformized by the hyperbolic plane. The theory of symmetries of the remaining surfaces, that is, the Riemann sphere and the tori, is well-known for a long time but, for the sake of completeness and the reader's convenience, we devote the main part of Chap. 4 to this subject. We also outline the classification of the symmetry types of hyperelliptic Riemann surfaces as being the double covers of the Riemann sphere.

Finally, Chap. 5 is dedicated to the symmetries of Riemann surfaces with large groups of automorphisms. Such surfaces are important since on the one hand they are determined by a 2-generator presentation of their groups of automorphisms, and on the other hand they can be defined over the algebraic numbers due to the celebrated theorem of Belyi from the late seventies. Furthermore, by a recent result of B. Köck and D. Singerman, these algebraic numbers can be chosen to be reals if the surface is symmetric. The foundations for the study of symmetries of such surfaces were established by Singerman, who found necessary and sufficient algebraic conditions in terms of the mentioned above generating pair for such a surface to be symmetric. In the first section, apart from Singerman's proof, we give formulae to compute the number of ovals of these symmetries, to which we refer as Singerman symmetries. Using these formulae we deal, in the next two sections, with the significant families of Macbeath-Singerman and Accola-Maclachlan and Kulkarni surfaces. Finally we describe the symmetries of the last two families by means of algebraic formulae.

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