

Preface to the Second Edition

The theory of graphs can be roughly partitioned into two branches: the areas of undirected graphs and directed graphs (digraphs). While there are many books on undirected graphs with new ones coming out regularly, the first edition of *Digraphs*, which was published in 2000, is the only modern book on graph theory covering more than a small fraction of the theory of directed graphs.

Since we wrote the first edition, the theory of directed graphs has continued to evolve at a high speed; many important results, including some of the conjectures from the first edition, have been proved and new methods were developed. Hence a new completely revised version became necessary. Instead of merely adding some new results and deleting a number of old ones, we took the opportunity to reorganize the book and increase the number of chapters from 12 to 18. This allows us to treat, in separate chapters, important topics such as branchings, feedback arc and vertex sets, connectivity augmentations, sparse subdigraphs with prescribed connectivity, applications, as well as packing, covering and decompositions of digraphs. We have added a large number of open problems to the second edition and the book now contains more than 150 open problems and conjectures, almost twice as many as the first edition. In order to avoid the book becoming unacceptably long, we had to remove a significant portion of material from the first edition. We have tried to do this as carefully as possible so that most of the information in the first edition is still available, along with a very large number of new results.

Even though this book should not be seen as an encyclopedia on directed graphs, we included as many important results as possible. The book contains a considerable number of proofs, illustrating various approaches and techniques used in digraph theory and algorithms.

One of the main features of this book is the strong emphasis on algorithms. This is something which is regrettably omitted in many books on graphs. Algorithms on (directed) graphs often play an important role in problems arising in several areas, including computer science and operations research. Secondly, many problems on (directed) graphs are inherently algorithmic. Hence, whenever possible we give constructive proofs of the results in the book. From these proofs one can very often extract an efficient algorithm for the problem studied. Even though we describe many algorithms, partly

due to space limitations, we do not supply all the details necessary in order to implement these algorithms. The latter (often highly nontrivial step) is a science in itself and we refer the reader to books on data structures and algorithms.

Another important feature is the large number of exercises which not only helps the reader to improve his or her understanding of the material, but also complements the results introduced in the text by covering even more material. Attempting these exercises will help the reader to master the subject and its main techniques.

Through its broad coverage and the exercises, stretching from easy to quite difficult, the book will be useful for courses on subjects such as (di)graph theory, combinatorial optimization and graph algorithms. Furthermore, it can be used for more focused courses on topics such as flows, cycles and connectivity. The book contains a large number of illustrations. This will help the reader to understand otherwise difficult concepts and proofs.

To facilitate the use of this book as a reference book and as a graduate textbook, we have added comprehensive symbol and subject indexes. It is our hope that the organization of the book, as well as detailed subject index, will help many readers to find what they are looking for without having to read through whole chapters. Due to our experience, we think that the book will be a useful teaching and reference resource for several decades to come.

Highlights

We cover the majority of important topics on digraphs ranging from quite elementary to very advanced ones. One of the main features of the second edition is the focus on open problems and the book contains more than 150 open problems or conjectures, thus making it a rich source for future research. By organizing the book so as to single out important areas, we hope to make it easy for the readers to find results and problems of their interest.

Below we give a brief outline of some of the main highlights of this book. Readers who are looking for more detailed information are advised to consult the list of contents or the subject index at the end of the book.

Chapter 1 contains most of the terminology and notation used in this book as well as several basic results. These are not only used frequently in other chapters, but also serve as illustrations of digraph concepts.

Chapter 2 is devoted to describing several important classes of directed graphs, such as line digraphs, the de Bruijn and Kautz digraphs, digraphs of bounded tree-width, digraphs of bounded directed widths, planar digraphs and generalizations of tournaments. We concentrate on characterization, recognition and decomposition of these classes. Many properties of these classes are studied in more detail in the rest of the book.

Chapters 3 and 4 cover distances and flows in networks. Although the basic concepts of these two topics are elementary, both theoretical and al-

gorithmic aspects of distances in digraphs as well as flows in networks are of great importance, due to their high applicability to other problems on digraphs and large number of practical applications, in particular, as a powerful modelling tool.

The main part of Chapter 3 is devoted to minimization and maximization of distance parameters in digraphs. In the self-contained Chapter 4, which may be used for a course on flows in networks, we cover basic topics on flows in networks, including a number of important applications to other (di)graph problems. Although there are several comprehensive books on flows, we believe that our fairly short and yet quite detailed account of the topic will give the majority of readers sufficient knowledge of the area. The reader who masters the techniques described in this chapter will be well equipped for solving many problems arising in practice.

Connectivity in (di)graphs is a very important topic. It contains numerous deep and beautiful results and has applications to other areas of graph theory and mathematics in general. It has various applications to other areas of research as well. We give a comprehensive account of connectivity topics and devote Chapters 5, 10, 12, 14 and parts of Chapter 11 to different aspects of connectivity.

Chapter 5 contains basic topics such as ear-decompositions, Menger's theorem, algorithms for determining the connectivity of a digraph as well as advanced topics such as properties of minimally k -(arc)-strong digraphs and critically k -strong digraphs.

Chapter 10 deals with problems concerning (arc-)disjoint linkings with prescribed initial and terminal vertices in digraphs. We prove that the 2-linkage problem is \mathcal{NP} -complete for arbitrary digraphs, but polynomially solvable for acyclic digraphs. Results on linkings in planar digraphs, eulerian digraphs as well as several generalizations of tournaments are discussed.

In Chapter 12 we study the problem of finding, in a k -(arc)-strong digraph, a small set of arcs (called a certificate) so that these arcs alone show that the digraph has the claimed connectivity. These problems are generally \mathcal{NP} -hard, so we give various approximation algorithms as well as polynomial algorithms for special classes of digraphs. We illustrate an application due to Cheriyan and Thurimella of Mader's results on minimally k -(arc)-strong digraphs to the problem of finding a small certificate for k -(arc)-strong connectivity. Finally, we also discuss recent results due to Gabow et al. on directed multigraphs.

In Chapter 14 we describe the splitting technique due to Mader and Lovász and illustrate its usefulness by giving an algorithm, due to Frank, for finding a minimum cardinality set of new arcs whose addition to a digraph D increases its arc-strong connectivity to a prescribed number. We also discuss a number of results related to increasing the connectivity by reversing arcs.

In Chapter 11 the famous theorem by Nash-Williams on orientations preserving a high degree of local arc-strong connectivity is described and the

weak version dealing with uniform arc-strong connectivities is proved using splitting techniques. We also discuss extensions of these results, including recent ones by Király and Szigeti. We give a proof of Jordán's result that every 18-connected graph has a 2-strong orientation. Submodular flows form a powerful generalization of circulations in networks. We introduce submodular flows and illustrate how to use this tool to obtain (algorithmic) proofs of many important results in graph theory. Finally we describe in detail an application, due to Frank, of submodular flows to the problem of orienting a mixed graph in order to maintain a prescribed degree of arc-strong connectivity.

In Chapter 6 we give a detailed account of results concerning the existence of hamiltonian paths and cycles in digraphs. Many results of this chapter deal with generalizations of tournaments. The reader will see that several of these much larger classes of digraphs share various nice properties with tournaments. In particular the hamiltonian path and cycle problems are polynomially solvable for most of these classes. The chapter illustrates various methods (such as the multi-insertion technique) for proving hamiltonicity.

In Chapter 7 we describe a number of interesting topics related to restricted hamiltonian paths and cycles. These include hamiltonian paths with prescribed end-vertices and orientations of hamiltonian paths and cycles in tournaments. We cover one of the main ingredients in the proof by Havet and Thomassé of Rosenfeld's conjecture on orientations of hamiltonian paths in tournaments.

Chapter 8 describes results on (generally) non-hamiltonian cycles in digraphs. We cover pancyclicity and the colour-coding technique by Alon, Yuster and Zwick and its application to yield polynomial algorithms for finding paths and cycles of 'logarithmic' length. We discuss the even cycle problem, including Thomassen's even cycle theorem. We also cover short cycles in multipartite tournaments, the girth of a digraph, chords of cycles and Adám's conjecture. The chapter features various proof techniques including several algebraic, algorithmic, combinatorial and probabilistic methods.

Chapter 9 is devoted to branchings, a very important structure generalizing spanning trees in graphs. Branchings are not only of interest by themselves, they also play an important role in many proofs on digraphs. We prove Tutte's Matrix-Tree theorem on the number of distinct out-branchings in a digraph. We give a number of recent results on branchings with bounds on the degrees or extremal number of leaves. Edmonds' theorem on arc-disjoint branchings is proved and several applications of this important result are described. The problem of finding a minimum cost out-branching in a weighted digraph generalizes the minimum spanning tree problem. We describe algorithms for finding such a branching.

Chapter 13 covers a number of very important results related to packing, covering and decompositions of digraphs. We prove the Lucchesi-Younger theorem on arc-disjoint directed cuts, and give a number of results on arc-

disjoint hamiltonian paths and cycles. We discuss results on decomposing highly connected tournaments into many spanning strong subdigraphs, including a conjecture which generalizes the famous Kelly conjecture. We give a number of results on cycle factors with a prescribed number of cycles. Finally, we give a full proof, due to Bessy and Thomassé, of Gallai's conjecture on the minimum number of cycles needed to cover all vertices in a strong digraph.

Chapter 15 deals with another very important topic, namely, how to destroy all cycles in a digraph by removing as few vertices or arcs as possible. One of the main results in the chapter is that the feedback arc set problem is \mathcal{NP} -hard already for tournaments. This was conjectured by Bang-Jensen and Thomassen in 1992, but was proved only recently by three different sets of authors. Another major result is the proof by Chen, Liu, Lu, O'Sullivan and Razgon and that the feedback arc set problem and the feedback vertex set problems are both fixed-parameter tractable. We also include a scheme of a solution of Younger's conjecture, by Reed, Robertson, Seymour and Thomas, on the relation between the number of disjoint cycles and the size of a minimum feedback vertex set in a digraph.

Chapter 16 deals with edge-coloured graphs, a topic which has a strong relation to directed graphs. Alternating cycles in 2-edge-coloured graphs generalize the concept of cycles in digraphs. Certain results on cycles in bipartite digraphs, such as the characterization of hamiltonian bipartite tournaments, are special cases of results for edge-coloured complete graphs. There are useful implications in the other direction as well. In particular, using results on hamiltonian cycles in bipartite tournaments, we prove a characterization of those 2-edge-coloured complete graphs which have an alternating hamiltonian cycle. We briefly describe one important recent addition to the topic, i.e., a characterization by Feng, Giesen, Guo, Gutin, Jensen and Rafiey of edge-coloured complete multigraphs containing properly coloured Hamilton paths. This characterization was conjectured in the first edition.

Chapter 17 describes a number of different applications of directed and edge-coloured graphs. Through the topics treated we illustrate the diversity of the applications including some in quantum mechanics, bioinformatics, embedded computing and the traveling salesman problem.

Chapter 18 is included to make the book self-contained, by giving the reader a collection of the most relevant definitions and methods on algorithms and related areas.

Technical Remarks

We have tried to rank exercises according to their expected difficulty. Marks range from $(-)$ to $(++)$ in order of increasing difficulty. The majority of exercises have no mark, indicating that they are of moderate difficulty. An exercise marked $(-)$ requires not much more than the understanding of the

main definitions and results. A (+) exercise requires a non-trivial idea, or involves substantial work. Finally, the few exercises which are ranked (++) require several deep ideas. Inevitably, this labelling is subjective and some readers may not agree with this ranking in certain cases. Some exercises have a header in boldface, which means that they cover an important or useful result not discussed in the text in detail.

We use the symbol \square to denote the end of a proof, or to indicate that either no proof will be given or the assertion is left as an exercise.

A few sections of the book require some basic knowledge of linear programming, while a few others require basic knowledge of probability theory.

We would be grateful to receive comments on the book. They may be sent to us by email to jbj@imada.sdu.dk or gutin@cs.rhul.ac.uk. We plan to have a web page containing information about misprints and other information about the book; see <http://www.cs.rhul.ac.uk/books/dbook/>

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