Preface

Arithmetic Geometry can be defined as the part of Algebraic Geometry connected with the study of algebraic varieties over arbitrary rings, in particular over non-algebraically closed fields. It lies at the intersection between classical algebraic geometry and number theory.

In recent years, significant progress has been achieved in this field, in several directions. More importantly, new links between arithmetic geometry and other branches of mathematics have been developed, and new powerful tools from geometry, complex analysis, differential equations and representation theory have been imported into number theory, thus putting arithmetic geometry at the crossroads of most of contemporary mathematics.

Some links between arithmetic geometry and classical algebraic geometry come from the classification of algebraic varieties, an old subject initiated by the Italian school in the case of surfaces and developed at a rapid pace in recent time.

As discovered by Osgood and Vojta about 20 years ago, there is a formal analogy between complex analysis and both diophantine approximation and arithmetic geometry. Such analogy has revealed itself as a fertile source of ideas and problems in both complex analysis and arithmetic geometry, and it has recently led to new achievements.

The algebraic theory of differential equations is also connected to arithmetic geometry, especially with algebraic geometry in positive characteristic; many authors, starting with the founders of transcendental number theory, stressed the role of differential equations in transcendence. Recently, the theory of algebraic foliations showed new relations between these topics and diophantine approximation.

The C.I.M.E. Summer School *Arithmetic Geometry*, held in Cetraro (Cosenza, Italy), September 10–15, aimed at presenting some of the most interesting new developments of arithmetic geometry. It consisted of four courses, given by some of the most eminent contributors to the field.

Here is an overview of the three courses which have been written up.

Section 1 Variétés presque rationnelles, leurs points rationnels et leurs dégénérescences, by Jean-Louis Colliot-Thélène.

This survey addresses the general question: Over a given type of field, is there a natural class of varieties which automatically have a rational point? Fields under consideration here include: finite fields, p-adic fields, function fields in one or two variables over an algebraically closed field, C_i-fields. Classical answers are given by the Chevalley-Warning theorem and by Tsen's theorem. More general answers were provided by a theorem of Graber, Harris and Starr and by a theorem of Esnault. The latter results apply to *rationally connected varieties*.

Colliot-Thélène discusses these varieties from various angles: weak approximation (see also Swinnerton-Dyer's contribution), R-equivalence on the set of rational points, Chow group of zero-cycles.

Loosely speaking, R-equivalence on the set of rational points of a variety defined over a given field is generated by the elementary relation: to be connected by a rational curve defined over the given field. Rationally connected varieties are varieties for which R-equivalence becomes trivial when one extends the ground field to an arbitrary algebraically closed field. Rationally connected varieties play an important rôle in the classification of algebraic varieties.

Ongoing work on "rationally simply connected" varieties over function fields in two variables is also mentioned. A common thread in this report is the study of the special fibre of a scheme over a discrete valuation ring: if the generic fibre has a simple geometry, what does it imply for the special fibre?

Many examples are presented in the course showing that, despite important recent advancements, still many questions remain open, keeping the subject strongly alive.

Section 2 Topics in diophantine equations, by Sir Peter Swinnerton-Dyer.

The notes by Swinnerton-Dyer address the main problem in the theory of diophantine equations: to decide whether a given algebraic equation has solutions in integer or rational numbers.

An obvious necessary condition for the existence of rational solutions to a diophantine equation is its solubility over the reals, and more generally over p-adic completions of $\mathbb{Q}$. Since an effective procedure to decide about solubility over local fields is known, such condition is very useful in many cases. Hence it is natural to ask for which class of diophantine equations the converse also holds:

1. If the equation is soluble over every local completion of the rational number field, is it soluble over the rationals?

This is called the Hasse principle. It is known that its does not hold for an arbitrary equation. An obstruction for its validity was discovered by Manin in the seventies and is nowadays called the Brauer-Manin obstruction. The notes briefly describe this obstruction, and then address the second natural question:

2. Is the Brauer-Manin obstruction the only obstruction to the Hasse principle?

In the case when a given equation is known to be soluble, one may be interested in the distribution of its solutions, i.e., of rational points on the algebraic variety V defined by that equation. When such points are Zariski-dense, one would like to "measure" their density. There are at least two very distinct notions of density. First: for every positive integer H, we let $N(H)$ be the number of rational points of height less than H. We ask:

3. Can one estimate the growth of $N(H)$, for $H \to \infty$, in terms of the geometry of V?

Secondly: embed $V(\mathbb{Q})$ in the product $\prod_p V(\mathbb{Q}_p)$ and consider the corresponding product topology.

4. (Weak approximation) Is the image of $V(\mathbb{Q})$ dense in every finite product as above?

These problems and questions are related with many other aspects of arithmetic and geometry, and the author illustrates these links in the first chapters of his text, which can be viewed as an introduction to most of twentieth century Arithmetic Geometry.

In the second part of the notes, answers are given to the above mentioned questions in many concrete nontrivial cases, especially for surfaces. The methods employed have been pioneered by Swinnerton-Dyer himself and his collaborators in the last ten years; here a panoramic view of these methodologies is given. Also, several new examples are presented for the first time, in particular for the most important case of elliptic and rational surfaces.

Section 3 Diophantine approximation and Nevanlinna theory, by Paul Vojta.

In the eighties, P. Vojta discovered striking analogies between Nevanlinna theory in complex analysis, diophantine approximation, some results on entire curves and the distribution of integral and rational points on algebraic varieties.

Suppose that X is a projective variety defined over a field K of characteristic zero. If K is a number field we are interested in the structure of the set $X(K)$ of its rational points. If $K = \mathbb{C}$ we are interested in the image of analytic maps $f : \mathbb{C} \to X$.

We may ask the following questions in the two cases:

(1_{ar}) Is the set $X(K)$ Zariski dense?
(1_{an}) May we find maps $f : \mathbb{C} \to X$ with Zariski dense image?
(2_{ar}) Is there a finite extension L/K such that $X(L)$ is Zariski dense?
(2_{an}) Is there a finite covering $h : Y \to \mathbb{C}$ with a map $f : Y \to X$ with Zariski dense image?
(3_{ar}) May we control the size of the rational points in $X(K)$ outside of a proper Zariski closed set?
(3_{an}) Is it possible to control the order of growth of a map $f : \mathbb{C} \to X$ with Zariski dense image, in terms of the geometry of X?

Analogous question can be asked for open subsets $Y \subset X$ of algebraic varieties, namely:

(4_{ar}) Let $\mathscr{O}_K$ be the ring of integers of K. Is the set $Y(\mathscr{O}_K)$ Zariski dense?
(4_{an}) Does there exist a map $f : \mathbb{C} \to Y$ with Zariski dense image?

Many other similar questions may be asked.

The notes by P. Vojta begin by formalizing the language needed to attack these questions: In the arithmetic context, the theory of height and Weil functions is described, while in the analytic context, the appropriate Nevanlinna theory is used.

Vojta shows how, using an appropriate dictionary, the two theories have striking similarities. Also he shows how his “dictionary” can be used as a source of problems in both theories. In particular, the analogies between Roth’s theorem in diophantine approximation and Nevanlinna’s Second Main Theorem, between Schmidt’s subspace theorem in diophantine approximation and Cartan’s Theorem in Nevanlinna theory are presented, and this leads to the natural analogy between Griffiths’ conjecture in complex analysis and his own conjecture on rational points.

After showing the classical results on the distribution of rational and integral points in their historical perspective, he presents some of the recent developments obtained from Schmidt’s subspace theorem (and from Cartan theorem in the Nevanlinna setting), to give nontrivial answers to questions (4_{ar}) and (4_{an}) in certain cases. In the last part of the course, he explains the relations of these theories with different versions of the famous *abc* conjecture of Masser and Oesterlé, and gives some ideas on recent developments obtained by McQuillan and Yamanoi on the so-called $1+\varepsilon$ conjecture, in the function field case. Finally, he formulates some new conjectures in arithmetic, which are strongly inspired by the work of McQuillan on the *abc* conjecture over function fields.

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