Contents

Series Foreword xvii
Preface xix

1 Introduction 1
1.1 Population Games 2
1.1.1 Modeling Interactions in Large Populations 2
1.1.2 Definitions and Classes of Population Games 5
1.2 Evolutionary Dynamics 6
1.2.1 Knowledge, Rationality, and Large Games 7
1.2.2 Foundations for Evolutionary Dynamics 8
1.2.3 Deterministic Evolutionary Dynamics 9
1.2.4 Orders of Limits for Stochastic Evolutionary Models 11
1.2.5 Stationary Distributions and Stochastic Stability 12
1.3 Remarks on History, Motivation, and Interpretation 13
Notes 15

I Population Games

2 Population Games 21
2.1 Population Games 22
2.1.1 Populations, Strategies, and States 22
2.1.2 Payoffs 23
2.1.3 Best Responses and Nash Equilibria 23
2.1.4 Prelude to Evolutionary Dynamics 24
2.2 Examples 25
2.2.1 Matching in Normal Form Games 25
2.2.2 Congestion Games 27
2.2.3 Two Simple Externality Models 28
2.3 The Geometry of Population Games and Nash Equilibria 29
2.3.1 Drawing Two-Strategy Games 29
2.3.2 Displacement Vectors and Tangent Spaces 31

2.3.3 *Orthogonal Projections* 34
2.3.4 *Drawing Three-Strategy Games* 36
2.3.5 *Tangent Cones and Normal Cones* 38
2.3.6 *Normal Cones and Nash Equilibria* 41
2.A Appendix: Affine Spaces, Tangent Spaces, and Projections 44
2.A.1 *Affine Spaces* 44
2.A.2 *Affine Hulls of Convex Sets* 45
2.A.3 *Orthogonal Projections* 46
2.A.4 *The Moreau Decomposition Theorem* 50
Notes 51

3 Potential Games, Stable Games, and Supermodular Games 53
3.1 Full Potential Games 53
3.1.1 *Full Population Games* 54
3.1.2 *Definition and Characterization* 54
3.1.3 *Examples* 55
3.1.4 *Nash Equilibria of Full Potential Games* 58
3.1.5 *The Geometry of Nash Equilibrium in Full Potential Games* 64
3.1.6 *Efficiency in Homogeneous Full Potential Games* 66
3.1.7 *Inefficiency Bounds for Congestion Games* 68
3.2 Potential Games 71
3.2.1 *Motivating Examples* 71
3.2.2 *Definition, Characterizations, and Examples* 72
3.2.3 *Potential Games and Full Potential Games* 76
3.2.4 *Passive Games and Constant Games* 77
3.3 Stable Games 79
3.3.1 *Definition* 79
3.3.2 *Examples* 81
3.3.3 *Invasion* 85
3.3.4 *Global Neutral Stability and Global Evolutionary Stability* 87
3.3.5 *Nash Equilibrium and Global Neutral Stability in Stable Games* 90
3.4 Supermodular Games 94
3.4.1 *Definition* 94
3.4.2 *Examples* 97
3.4.3 *Best Response Monotonicity in Supermodular Games* 98
3.4.4 *Nash Equilibria of Supermodular Games* 99
3.A Appendix: Multivariate Calculus 100
3.A.1 *Univariate Calculus* 100
3.A.2 *The Derivative as a Linear Map* 101
3.A.3 *Differentiation as a Linear Operation* 103
3.A.4 *The Product Rule and the Chain Rule* 103

3.A.5 *Homogeneity and Euler's Theorem* 104
3.A.6 *Higher-Order Derivatives* 105
3.A.7 *The Whitney Extension Theorem* 106
3.A.8 *Vector Integration and the Fundamental Theorem of Calculus* 107
3.A.9 *Potential Functions and Integrability* 108
3.B Appendix: Affine Calculus 108
3.B.1 *Linear Forms and the Riesz Representation Theorem* 109
3.B.2 *Dual Characterizations of Multiples of Linear Forms* 110
3.B.3 *Derivatives of Functions on Affine Spaces* 111
3.B.4 *Affine Integrability* 113
Notes 115

II Deterministic Evolutionary Dynamics

4 Revision Protocols and Evolutionary Dynamics 119
4.1 The Basic Stochastic Evolutionary Model 120
4.1.1 *Inertia and Myopia* 120
4.1.2 *Revision Protocols* 121
4.2 Mean Dynamics 122
4.2.1 *Derivation* 122
4.2.2 *Target Protocols and Target Dynamics* 124
4.3 Examples 125
4.3.1 *Imitative Protocols and Dynamics* 126
4.3.2 *Direct Protocols and Dynamics* 128
4.4 Deterministic Evolutionary Dynamics 129
4.A Appendix: Ordinary Differential Equations 131
4.A.1 *Basic Definitions* 131
4.A.2 *Existence, Uniqueness, and Continuity of Solutions* 133
4.A.3 *Ordinary Differential Equations on Compact Convex Sets* 135
Notes 137

5 Deterministic Dynamics: Families and Properties 139
5.1 Information Requirements for Revision Protocols 141
5.2 Incentives and Aggregate Behavior 144
5.3 Families of Evolutionary Dynamics 148
5.4 Imitative Dynamics 153
5.4.1 *Definition* 153
5.4.2 *Examples* 154
5.4.3 *Biological Derivations of the Replicator Dynamic* 158
5.4.4 *Extinction and Invariance* 160
5.4.5 *Monotone Percentage Growth Rates and Positive Correlation* 162
5.4.6 *Rest Points and Restricted Equilibria* 164

5.5 Excess Payoff Dynamics 165
5.5.1 Definition and Interpretation *166*
5.5.2 Incentives and Aggregate Behavior *167*
5.6 Pairwise Comparison Dynamics 169
5.6.1 Definition *170*
5.6.2 Incentives and Aggregate Behavior *170*
5.6.3 Desiderata Revisited *172*
5.7 Multiple Revision Protocols and Hybrid Dynamics 173
Notes 175

6 Best Response and Projection Dynamics 177
6.1 The Best Response Dynamic 178
6.1.1 Definition and Examples *178*
6.1.2 Construction and Properties of Solution Trajectories *180*
6.1.3 Incentive Properties *186*
6.2 Perturbed Best Response Dynamics 187
6.2.1 Revision Protocols and Mean Dynamics *188*
6.2.2 Perturbed Optimization: A Representation Theorem *189*
6.2.3 Logit Choice and the Logit Dynamic *191*
6.2.4 Perturbed Incentive Properties via Virtual Payoffs *196*
6.3 The Projection Dynamic 198
6.3.1 Definition *198*
6.3.2 Solution Trajectories *200*
6.3.3 Incentive Properties *202*
6.3.4 Revision Protocols and Connections with the Replicator Dynamic *202*
6.A Appendix: Differential Inclusions 205
6.A.1 Basic Theory *205*
6.A.2 Differential Equations Defined by Projections *207*
6.B Appendix: The Legendre Transform 208
6.B.1 Legendre Transforms of Functions on Open Intervals *208*
6.B.2 Legendre Transforms of Functions on Multidimensional Domains *210*
6.C Appendix: Perturbed Optimization 212
6.C.1 Proof of the Representation Theorem *212*
6.C.2 Additional Results *215*
Notes 216

III Convergence and Nonconvergence of Deterministic Dynamics

7 Global Convergence of Evolutionary Dynamics 221
7.1 Potential Games 223
7.1.1 Potential Functions as Lyapunov Functions *223*
7.1.2 Gradient Systems for Potential Games *228*

7.2 Stable Games 232
7.2.1 *The Projection and Replicator Dynamics in Strictly Stable Games* 232
7.2.2 *Integrable Target Dynamics* 235
7.2.3 *Impartial Pairwise Comparison Dynamics* 246
7.2.4 *Summary* 248
7.3 Supermodular Games 249
7.3.1 *The Best Response Dynamic in Two-Player Normal Form Games* 251
7.3.2 *Stochastically Perturbed Best Response Dynamics* 253
7.4 Dominance Solvable Games 257
7.4.1 *Dominated and Iteratively Dominated Strategies* 258
7.4.2 *The Best Response Dynamic* 258
7.4.3 *Imitative Dynamics* 259
7.A Appendix: Limit and Stability Notions for Deterministic Dynamics 260
7.A.1 *ω-Limits and Notions of Recurrence* 261
7.A.2 *Stability of Sets of States* 262
7.B Appendix: Stability Analysis via Lyapunov Functions 262
7.B.1 *Lyapunov Stable Sets* 263
7.B.2 *ω-Limits and Attracting Sets* 263
7.B.3 *Asymptotically Stable and Globally Asymptotically Stable Sets* 265
7.C Appendix: Cooperative Differential Equations 266
Notes 268

8 Local Stability under Evolutionary Dynamics 271
8.1 Non-Nash Rest Points of Imitative Dynamics 272
8.2 Local Stability in Potential Games 273
8.3 Evolutionarily Stable States 275
8.3.1 *Single-Population Games* 276
8.3.2 *Multipopulation Games* 280
8.3.3 *Regular Taylor ESS* 281
8.4 Local Stability via Lyapunov Functions 282
8.4.1 *The Replicator and Projection Dynamics* 282
8.4.2 *Target and Pairwise Comparison Dynamics: Interior ESS* 283
8.4.3 *Target and Pairwise Comparison Dynamics: Boundary ESS* 285
8.5 Linearization of Imitative Dynamics 290
8.5.1 *The Replicator Dynamic* 291
8.5.2 *General Imitative Dynamics* 295
8.6 Linearization of Perturbed Best Response Dynamics 297
8.6.1 *Deterministically Perturbed Best Response Dynamics* 297
8.6.2 *The Logit Dynamic* 298
8.A Appendix: Matrix Analysis 299
8.A.1 *Rank and Invertibility* 299

8.A.2 *Eigenvectors and Eigenvalues* 300
8.A.3 *Similarity, (Block) Diagonalization, and the Spectral Theorem* 302
8.A.4 *Symmetric Matrices* 304
8.A.5 *The Real Jordan Canonical Form* 305
8.A.6 *The Spectral Norm and Singular Values* 306
8.A.7 *Hines's Lemma* 307
8.B Appendix: Linear Differential Equations 308
8.B.1 *Examples* 308
8.B.2 *Solutions* 310
8.B.3 *Stability and Hyperbolicity* 312
8.C Appendix: Linearization of Nonlinear Differential Equations 313
Notes 315

9 Nonconvergence of Evolutionary Dynamics 319
9.1 Conservative Properties of Evolutionary Dynamics 320
9.1.1 *Constants of Motion in Null Stable Games* 320
9.1.2 *Preservation of Volume* 324
9.2 Games with Nonconvergent Evolutionary Dynamics 327
9.2.1 *Circulant Games* 327
9.2.2 *Continuation of Attractors for Parameterized Games* 330
9.2.3 *Mismatching Pennies* 333
9.2.4 *The Hypnodisk Game* 337
9.3 Chaotic Evolutionary Dynamics 341
9.4 Survival of Dominated Strategies 344
9.4.1 *A General Survival Theorem* 347
9.4.2 *Examples and Discussion* 351
9.A Appendix: Three Classical Theorems on Nonconvergent Dynamics 356
9.A.1 *Liouville's Theorem* 356
9.A.2 *The Poincaré-Bendixson and Bendixson-Dulac Theorems* 358
9.B Appendix: Attractors and Continuation 359
9.B.1 *Attractors and Repellors* 359
9.B.2 *Continuation of Attractors* 361
Notes 362

IV Stochastic Evolutionary Models

10 Stochastic Evolution and Deterministic Approximation 367
10.1 The Stochastic Evolutionary Process 368
10.2 Finite-Horizon Deterministic Approximation 369
10.2.1 *Kurtz's Theorem* 370
10.2.2 *Deterministic Approximation of the Stochastic Evolutionary Process* 372

10.3 Extensions 376
10.3.1 Discrete-Time Models 376
10.3.2 Finite-Population Adjustments 377
10.A Appendix: The Exponential and Poisson Distributions 378
10.A.1 Basic Properties 378
10.A.2 The Poisson Limit Theorem 381
10.B Appendix: Probability Models and Their Interpretation 382
10.B.1 Countable Probability Models 383
10.B.2 Uncountable Probability Models and Measure Theory 384
10.B.3 Distributional Properties and Sample Path Properties 386
10.C Appendix: Countable State Markov Chains and Processes 389
10.C.1 Countable State Markov Chains 389
10.C.2 Countable State Markov Processes: Definition and Construction 391
10.C.3 Countable State Markov Processes: Transition Probabilities 393
Notes 395

11 Stationary Distributions and Infinite-Horizon Behavior 397
11.1 Irreducible Evolutionary Processes 399
11.1.1 Full Support Revision Protocols 399
11.1.2 Stationary Distributions and Infinite-Horizon Behavior 401
11.1.3 Reversibility 402
11.2 Stationary Distributions for Two-Strategy Games 403
11.2.1 Birth and Death Processes 403
11.2.2 The Stationary Distribution of the Evolutionary Process 405
11.2.3 Examples 407
11.3 Waiting Times and Infinite-Horizon Prediction 412
11.3.1 Examples 412
11.3.2 Discussion 415
11.4 Model Adjustments for Finite Populations 417
11.4.1 Finite-Population Games 418
11.4.2 Clever Payoff Evaluation 419
11.4.3 Committed Agents and Imitative Protocols 421
11.5 Potential Games and Exponential Protocols 423
11.5.1 Finite-Population Potential Games 424
11.5.2 Exponential Revision Protocols 428
11.5.3 Reversibility and Stationary Distributions 430
11.A Appendix: Long-Run Behavior of Markov Chains and Processes 434
11.A.1 Communication, Recurrence, and Irreducibility 434
11.A.2 Periodicity 436
11.A.3 Hitting Times and Hitting Probabilities 437
11.A.4 The Perron-Frobenius Theorem 440

11.A.5 *Stationary Distributions for Markov Chains* 441
11.A.6 *Reversible Markov Chains* 442
11.A.7 *Stationary Distributions and Reversibility for Markov Processes* 444
11.A.8 *Convergence in Distribution* 445
11.A.9 *Ergodicity* 448
Notes 449

12 Limiting Stationary Distributions and Stochastic Stability 451
12.1 Definitions of Stochastic Stability 454
12.1.1 *Small Noise Limits* 454
12.1.2 *Large Population Limits* 455
12.1.3 *Double Limits* 458
12.1.4 *Double Limits: A Counterexample* 460
12.2 Exponential Protocols and Potential Games 463
12.2.1 *Direct Exponential Protocols: The Small Noise Limit* 464
12.2.2 *Direct Exponential Protocols: The Large Population Limit* 465
12.2.3 *Direct Exponential Protocols: Double Limits* 472
12.2.4 *Imitative Exponential Protocols with Committed Agents* 472
12.3 Noisy Best Response Protocols in Two-Strategy Games 474
12.3.1 *Noisy Best Response Protocols and Their Cost Functions* 474
12.3.2 *The Small Noise Limit* 477
12.3.3 *The Large Population Limit* 479
12.3.4 *Double Limits* 480
12.3.5 *Stochastic Stability: Examples* 482
12.3.6 *Risk Dominance, Stochastic Dominance, and Stochastic Stability* 486
12.4 Imitative Protocols in Two-Strategy Games 491
12.4.1 *Imitative Protocols with Mutations* 491
12.4.2 *Imitative Protocols with Committed Agents* 494
12.4.3 *Imitative Protocols, Mean Dynamics, and Stochastic Stability* 496
12.5 Small Noise Limits 499
12.5.1 *Noisy Best Response Protocols and Cost Functions* 500
12.5.2 *Limiting Stationary Distributions via Trees* 502
12.5.3 *Two-Strategy Games and Risk Dominance* 504
12.5.4 *The Radius-Coradius Theorem* 510
12.5.5 *Half-Dominance* 512
12.6 Large Population Limits 515
12.6.1 *Convergence to Recurrent States of the Mean Dynamic* 515
12.6.2 *Convergence to Stable Rest Points of the Mean Dynamic* 517
12.A Appendix: Trees, Escape Paths, and Stochastic Stability 519
12.A.1 *The Markov Chain Tree Theorem* 520
12.A.2 *Limiting Stationary Distributions via Trees* 521

12.A.3 Limiting Stationary Distributions via Trees on Recurrent Classes 524
12.A.4 Radius-Coradius Theorems 526
12.A.5 Lenient Transition Costs and Weak Stochastic Stability 529
12.B Appendix: Stochastic Approximation Theory 531
12.B.1 Convergence to the Birkhoff Center 532
12.B.2 Sufficient Conditions for Convergence to Stable Rest Points 533
Notes 537

References 541
Notation Index 565
Index 575