Chapter 1
Kinematics and Statics

Abstract *Chapter 1* is devoted to problems based on one and two dimensions. The use of various kinematical formulae and the sign convention are pointed out. Problems in statics involve force and torque, centre of mass of various systems and equilibrium.

1.1 Basic Concepts and Formulae

Motion in One Dimension

The notation used is as follows: u = initial velocity, v = final velocity, a = acceleration, s = displacement, t = time (Table 1.1).

Table 1.1 Kinematical equations

	U	V	A	S	t
(i) $v = u + at$	✓	✓	✓	X	✓
(ii) $s = ut + 1/2at^2$	✓	X	✓	✓	✓
(iii) $v^2 = u^2 + 2as$	✓	✓	✓	✓	X
(iv) $s = \frac{1}{2}(u + v)t$	✓	✓	X	✓	✓

In each of the equations u is present. Out of the remaining four quantities only three are required. The initial direction of motion is taken as positive. Along this direction u and s and a are taken as positive, t is always positive, v can be positive or negative. As an example, an object is dropped from a rising balloon. Here, the parameters for the object will be as follows:

u = initial velocity of the balloon (as seen from the ground)

$u = +$ve, $a = -g$. $t = +$ve, $v = +$ve or $-$ve depending on the value of t, $s = +$ve or $-$ve, if $s = -$ve, then the object is found below the point it was released.

Note that (ii) and (iii) are quadratic. Depending on the value of u, both the roots may be real or only one may be real or both may be imaginary and therefore unphysical.

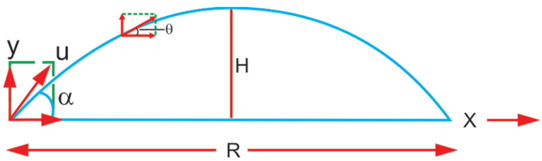
v–t and *a–t* Graphs

The area under the *v–t* graph gives the displacement (see prob. 1.11) and the area under the *a–t* graph gives the velocity.

Motion in Two Dimensions – Projectile Motion

$$\text{Equation: } y = x \tan \alpha - \frac{1}{2}\frac{gx^2}{u^2 \cos^2 \alpha} \tag{1.1}$$

Fig. 1.1 Projectile Motion



$$\text{Time of flight: } T = \frac{2u \sin \alpha}{g} \tag{1.2}$$

$$\text{Range: } R = \frac{u^2 \sin 2\alpha}{g} \tag{1.3}$$

$$\text{Maximum height: } H = \frac{u^2 \sin^2 \alpha}{2g} \tag{1.4}$$

$$\text{Velocity: } v = \sqrt{g^2 t^2 - 2ug \sin \alpha . t + u^2} \tag{1.5}$$

$$\text{Angle: } \tan \theta = \frac{u \sin \alpha - gt}{u \cos \alpha} \tag{1.6}$$

Relative Velocity

If v_A is the velocity of A and v_B that of B, then the relative velocity of A with respect to B will be

$$v_{AB} = v_A - v_B \tag{1.7}$$

Motion in Resisting Medium

In the absence of air the initial speed of a particle thrown upward is equal to that of final speed, and the time of ascent is equal to that of descent. However, in the presence of air resistance the final speed is less than the initial speed and the time of descent is greater than that of ascent (see prob. 1.21).

Equation of motion of a body in air whose resistance varies as the velocity of the body (see prob. 1.22).

Centre of mass is defined as

$$\boldsymbol{r}_{\text{cm}} = \frac{\Sigma m_i r_i}{\Sigma m_i} = \frac{1}{M}\Sigma m_i \boldsymbol{r}_i \tag{1.8}$$

Centre of mass velocity is defined as

$$\boldsymbol{V}_{\text{c}} = \frac{1}{M}\Sigma m_i \dot{r}_i \tag{1.9}$$

The centre of mass moves as if the mass of various particles is concentrated at the location of the centre of mass.

Equilibrium

A system will be in translational equilibrium if $\Sigma \boldsymbol{F} = 0$. In terms of potential $\frac{\partial V}{\partial x} = 0$, where V is the potential. The equilibrium will be stable if $\frac{\partial^2 V}{\partial x^2} < 0$. A system will be in rotational equilibrium if the sum of the external torques is zero, i.e. $\Sigma \tau_i = 0$

1.2 Problems

1.2.1 Motion in One Dimension

1.1 A car starts from rest at constant acceleration of $2.0\,\text{m/s}^2$. At the same instant a truck travelling with a constant speed of 10 m/s overtakes and passes the car.

(a) How far beyond the starting point will the car overtake the truck?
(b) After what time will this happen?
(c) At that instant what will be the speed of the car?

1.2 From an elevated point A, a stone is projected vertically upward. When the stone reaches a distance h below A, its velocity is double of what it was at a height h above A. Show that the greatest height obtained by the stone above A is $5h/3$.

[Adelaide University]

1.3 A stone is dropped from a height of 19.6 m, above the ground while a second stone is simultaneously projected from the ground with sufficient velocity to enable it to ascend 19.6 m. When and where the stones would meet.

1.4 A particle moves according to the law $x = A \sin \pi t$, where x is the displacement and t is time. Find the distance traversed by the particle in 3.0 s.

1.5 A man of height 1.8 m walks away from a lamp at a height of 6 m. If the man's speed is 7 m/s, find the speed in m/s at which the tip of the shadow moves.

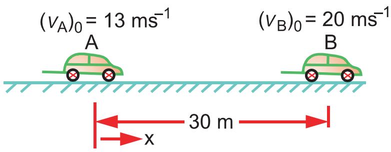
1.6 The relation $3t = \sqrt{3x} + 6$ describes the displacement of a particle in one direction, where x is in metres and t in seconds. Find the displacement when the velocity is zero.

1.7 A particle projected up passes the same height h at 2 and 10 s. Find h if $g = 9.8\,\text{m/s}^2$.

1.8 Cars A and B are travelling in adjacent lanes along a straight road (Fig. 1.2). At time, $t = 0$ their positions and speeds are as shown in the diagram. If car A has a constant acceleration of $0.6\,\text{m/s}^2$ and car B has a constant deceleration of $0.46\,\text{m/s}^2$, determine when A will overtake B.

[University of Manchester 2007]

Fig. 1.2



1.9 A boy stands at A in a field at a distance 600 m from the road BC. In the field he can walk at 1 m/s while on the road at 2 m/s. He can walk in the field along AD and on the road along DC so as to reach the destination C (Fig. 1.3). What should be his route so that he can reach the destination in the least time and determine the time.

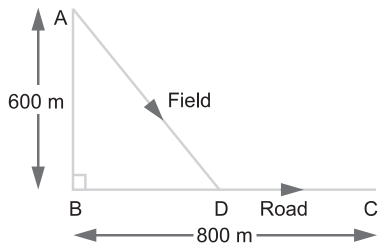
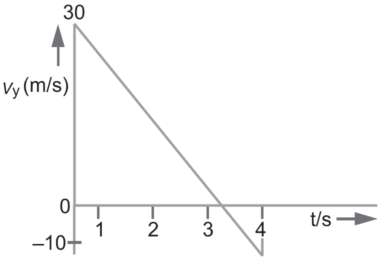


Fig. 1.3

1.10 Water drips from the nozzle of a shower onto the floor 2.45 m below. The drops fall at regular interval of time, the first drop striking the floor at the instant the third drop begins to fall. Locate the second drop when the first drop strikes the floor.

1.11 The velocity–time graph for the vertical component of the velocity of an object thrown upward from the ground which reaches the roof of a building and returns to the ground is shown in Fig. 1.4. Calculate the height of the building.

Fig. 1.4



1.12 A ball is dropped into a lake from a diving board 4.9 m above the water. It hits the water with velocity v and then sinks to the bottom with the constant velocity v. It reaches the bottom of the lake 5.0 s after it is dropped. Find

(a) the average velocity of the ball and
(b) the depth of the lake.

1.13 A stone is dropped into the water from a tower 44.1 m above the ground. Another stone is thrown vertically down 1.0 s after the first one is dropped. Both the stones strike the ground at the same time. What was the initial velocity of the second stone?

1.14 A boy observes a cricket ball move up and down past a window 2 m high. If the total time the ball is in sight is 1.0 s, find the height above the window that the ball rises.

1.15 In the last second of a free fall, a body covered three-fourth of its total path:

(a) For what time did the body fall?
(b) From what height did the body fall?

1.16 A man travelling west at 4 km/h finds that the wind appears to blow from the south. On doubling his speed he finds that it appears to blow from the southwest. Find the magnitude and direction of the wind's velocity.

1.17 An elevator of height h ascends with constant acceleration a. When it crosses a platform, it has acquired a velocity u. At this instant a bolt drops from the top of the elevator. Find the time for the bolt to hit the floor of the elevator.

1.18 A car and a truck are both travelling with a constant speed of 20 m/s. The car is 10 m behind the truck. The truck driver suddenly applies his brakes, causing the truck to decelerate at the constant rate of $2\ \text{m/s}^2$. Two seconds later the driver of the car applies his brakes and just manages to avoid a rear-end collision. Determine the constant rate at which the car decelerated.

1.19 Ship A is 10 km due west of ship B. Ship A is heading directly north at a speed of 30 km/h, while ship B is heading in a direction 60° west of north at a speed of 20 km/h.

(i) Determine the magnitude and direction of the velocity of ship B relative to ship A.
(ii) What will be their distance of closest approach?

[University of Manchester 2008]

1.20 A balloon is ascending at the rate of 9.8 m/s at a height of 98 m above the ground when a packet is dropped. How long does it take the packet to reach the ground?

1.2.2 Motion in Resisting Medium

1.21 An object of mass m is thrown vertically up. In the presence of heavy air resistance the time of ascent (t_1) is no longer equal to the time of descent (t_2). Similarly the initial speed (u) with which the body is thrown is not equal to the final speed (v) with which the object returns. Assuming that the air resistance F is constant show that

$$\frac{t_2}{t_1} = \sqrt{\frac{g + F/m}{g - F/m}}; \quad \frac{v}{u} = \sqrt{\frac{g - F/m}{g + F/m}}$$

1.22 Determine the motion of a body falling under gravity, the resistance of air being assumed proportional to the velocity.

1.23 Determine the motion of a body falling under gravity, the resistance of air being assumed proportional to the square of the velocity.

1.24 A body is projected upward with initial velocity u against air resistance which is assumed to be proportional to the square of velocity. Determine the height to which the body will rise.

1.25 Under the assumption of the air resistance being proportional to the square of velocity, find the loss in kinetic energy when the body has been projected upward with velocity u and return to the point of projection.

1.2.3 Motion in Two Dimensions

1.26 A particle moving in the xy-plane has velocity components $\mathrm{d}x/\mathrm{d}t = 6 + 2t$ and $\mathrm{d}y/\mathrm{d}t = 4 + t$
where x and y are measured in metres and t in seconds.

(i) Integrate the above equation to obtain x and y as functions of time, given that the particle was initially at the origin.
(ii) Write the velocity $\boldsymbol{v}$ of the particle in terms of the unit vectors $\hat{i}$ and $\hat{j}$.

(iii) Show that the acceleration of the particle may be written as $a = 2\hat{i} + \hat{j}$.
(iv) Find the magnitude of the acceleration and its direction with respect to the x-axis.

[University of Aberystwyth Wales 2000]

1.27 Two objects are projected horizontally in opposite directions from the top of a tower with velocities u_1 and u_2. Find the time when the velocity vectors are perpendicular to each other and the distance of separation at that instant.

1.28 From the ground an object is projected upward with sufficient velocity so that it crosses the top of a tower in time t_1 and reaches the maximum height. It then comes down and recrosses the top of the tower in time t_2, time being measured from the instant the object was projected up. A second object released from the top of the tower reaches the ground in time t_3. Show that $t_3 = \sqrt{t_1 t_2}$.

1.29 A shell is fired at an angle θ with the horizontal up a plane inclined at an angle α. Show that for maximum range, $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$.

1.30 A stone is thrown from ground level over horizontal ground. It just clears three walls, the successive distances between them being r and $2r$. The inner wall is 15/7 times as high as the outer walls which are equal in height. The total horizontal range is nr, where n is an integer. Find n.

[University of Dublin]

1.31 A boy wishes to throw a ball through a house via two small openings, one in the front and the other in the back window, the second window being directly behind the first. If the boy stands at a distance of 5 m in front of the house and the house is 6 m deep and if the opening in the front window is 5 m above him and that in the back window 2 m higher, calculate the velocity and the angle of projection of the ball that will enable him to accomplish his desire.

[University of Dublin]

1.32 A hunter directs his uncalibrated rifle toward a monkey sitting on a tree, at a height h above the ground and at distance d. The instant the monkey observes the flash of the fire of the rifle, it drops from the tree. Will the bullet hit the monkey?

1.33 If α is the angle of projection, R the range, h the maximum height, T the time of flight then show that
(a) $\tan\alpha = 4h/R$ and **(b)** $h = gT^2/8$

1.34 A projectile is fired at an angle of 60° to the horizontal with an initial velocity of 800 m/s:

(i) Find the time of flight of the projectile before it hits the ground
(ii) Find the distance it travels before it hits the ground (range)
(iii) Find the time of flight for the projectile to reach its maximum height

(iv) Show that the shape of its flight is in the form of a parabola $y = bx + cx^2$, where b and c are constants [acceleration due to gravity $g = 9.8\,\text{m/s}^2$].
[University of Aberystwyth, Wales 2004]

1.35 A projectile of mass 20.0 kg is fired at an angle of 55.0° to the horizontal with an initial velocity of 350 m/s. At the highest point of the trajectory the projectile explodes into two equal fragments, one of which falls vertically downwards with no initial velocity immediately after the explosion. Neglect the effect of air resistance:

(i) How long after firing does the explosion occur?
(ii) Relative to the firing point, where do the two fragments hit the ground?
(iii) How much energy is released in the explosion?
[University of Manchester 2008]

1.36 An object is projected horizontally with velocity 10 m/s. Find the radius of curvature of its trajectory in 3 s after the motion has begun.

1.37 A and B are points on opposite banks of a river of breadth a and AB is at right angles to the flow of the river (Fig. 1.4). A boat leaves B and is rowed with constant velocity with the bow always directed toward A. If the velocity of the river is equal to this velocity, find the path of the boat (Fig. 1.5).

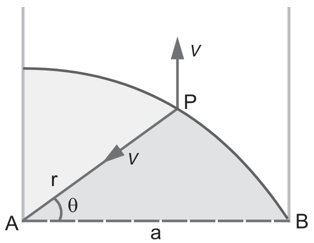


Fig. 1.5

1.38 A ball is thrown from a height h above the ground. The ball leaves the point located at distance d from the wall, at 45° to the horizontal with velocity u. How far from the wall does the ball hit the ground (Fig. 1.6)?

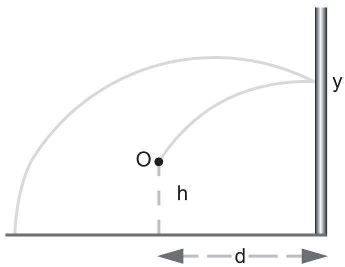


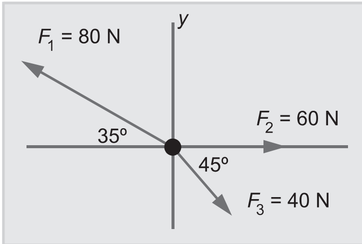
Fig. 1.6

1.2.4 Force and Torque

1.39 Three vector forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ act on a particle of mass $m = 3.80$ kg as shown in Fig. 1.7:

(i) Calculate the magnitude and direction of the net force acting on the particle.
(ii) Calculate the particle's acceleration.
(iii) If an additional stabilizing force $\boldsymbol{F}_4$ is applied to create an equilibrium condition with a resultant net force of zero, what would be the magnitude and direction of $\boldsymbol{F}_4$?

Fig. 1.7



1.40 **(a)** A thin cylindrical wheel of radius $r = 40$ cm is allowed to spin on a frictionless axle. The wheel, which is initially at rest, has a tangential force applied at right angles to its radius of magnitude 50 N as shown in Fig. 1.8a. The wheel has a moment of inertia equal to 20 kg m^2.

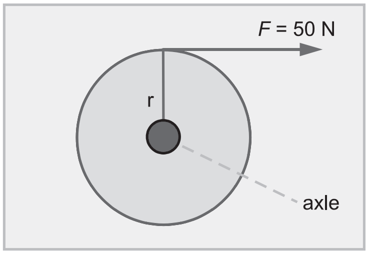


Fig. 1.8a

Calculate

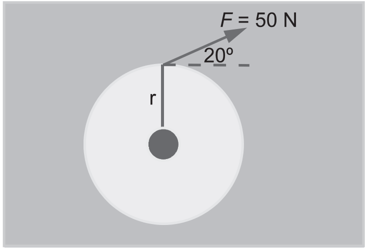
(i) The torque applied to the wheel
(ii) The angular acceleration of the wheel
(iii) The angular velocity of the wheel after 3 s
(iv) The total angle swept out in this time

(b) The same wheel now has the same force applied but inclined at an angle of 20° to the tangent as shown in Fig. 1.8b. Calculate

(i) The torque applied to the wheel
(ii) The angular acceleration of the wheel

[University of Aberystwyth, Wales 2005]

Fig. 1.8b



1.41 A container of mass 200 kg rests on the back of an open truck. If the truck accelerates at 1.5 m/s^2, what is the minimum coefficient of static friction between the container and the bed of the truck required to prevent the container from sliding off the back of the truck?

[University of Manchester 2007]

1.42 A wheel of radius r and weight W is to be raised over an obstacle of height h by a horizontal force F applied to the centre. Find the minimum value of F (Fig. 1.9).

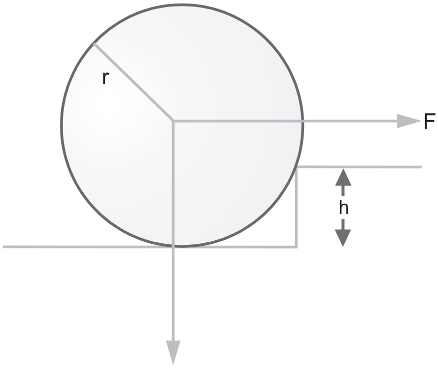


Fig. 1.9

1.2.5 Centre of Mass

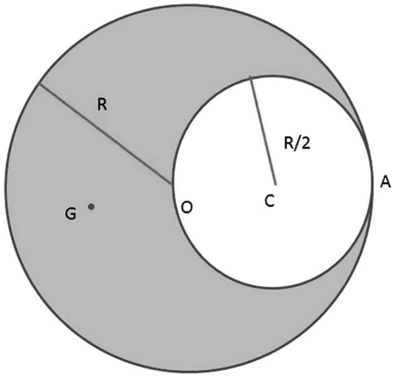
1.43 A thin uniform wire is bent into a semicircle of radius R. Locate the centre of mass from the diameter of the semicircle.

1.44 Find the centre of mass of a semicircular disc of radius R and of uniform density.

1.45 Locate the centre of mass of a uniform solid hemisphere of radius R from the centre of the base of the hemisphere along the axis of symmetry.

1.46 A thin circular disc of uniform density is of radius R. A circular hole of radius ½R is cut from the disc and touching the disc's circumference as in Fig. 1.10. Find the centre of mass.

Fig. 1.10



1.47 The mass of the earth is 81% the mass of the moon. The distance between the centres of the earth and the moon is 60 times the radius of earth $R = 6400\,\text{km}$. Find the centre of mass of the earth–moon system.

1.48 The distance between the centre of carbon and oxygen atoms in CO molecule is 1.13 Å. Locate the centre of mass of the molecule relative to the carbon atom.

1.49 The ammonia molecule NH_3 is in the form of a pyramid with the three H atoms at the corners of an equilateral triangle base and the N atom at the apex of the pyramid. The H–H distance = 1.014 Å and N–H distance = 1.628 Å. Locate the centre of mass of the NH_3 molecule relative to the N atom.

1.50 A boat of mass 100 kg and length 3 m is at rest in still water. A boy of mass 50 kg walks from the bow to the stern. Find the distance through which the boat moves.

1.51 At one end of the rod of length L, a body whose mass is twice that of the rod is attached. If the rod is to move with pure translation, at what fractional length from the loaded end should it be struck?

1.52 Find the centre of mass of a solid cone of height h.

1.53 Find the centre of mass of a wire in the form of an arc of a circle of radius R which subtends an angle 2α symmetrically at the centre of curvature.

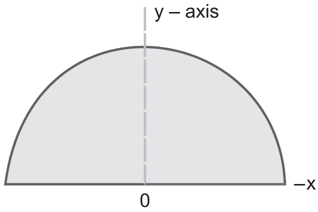
1.54 Five identical pigeons are flying together northward with speed v_0. One of the pigeons is shot dead by a hunter and the other four continue to fly with the same speed. Find the centre of mass speed of the rest of the pigeons which continue to fly with the same speed after the dead pigeon has hit the ground.

1.55 The linear density of a rod of length L is directly proportional to the distance from one end. Locate the centre of mass from the same end.

1.56 Particles of masses m, $2m$, $3m \dots nm$ are collinear at distances L, $2L$, $3L \dots nL$, respectively, from a fixed point. Locate the centre of mass from the fixed point.

1.57 A semicircular disc of radius R has density ρ which varies as $\rho = cr^2$, where r is the distance from the centre of the base and $c$is a constant. The centre of mass will lie along the y-axis for reasons of symmetry (Fig. 1.11). Locate the centre of mass from O, the centre of the base.

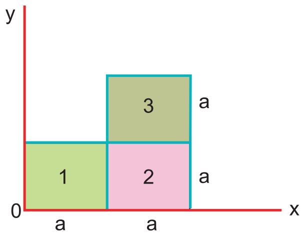
Fig. 1.11



1.58 Locate the centre of mass of a water molecule, given that the OH bond has length 1.77 Å and angle HOH is 105°.

1.59 Three uniform square laminas are placed as in Fig. 1.12. Each lamina measures 'a' on side and has mass m. Locate the CM of the combined structure.

Fig. 1.12



1.2.6 Equilibrium

1.60 Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(2x^3 - 5x^2 + 4x)$, where the constant $k > 0$. Show that the point $x = 1$ corresponds to a stable equilibrium position of the particle.

[University of Manchester 2007]

1.61 Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(x^2 - 4xl)$, where the constant $k > 0$. Show that the point $x = 2l$ corresponds to a stable equilibrium position of the particle.
Find the frequency of a small amplitude oscillation of the particle about the equilibrium position.

[University of Manchester 2006]

1.62 A cube rests on a rough horizontal plane. A tension parallel to the plane is applied by a thread attached to the upper surface. Show that the cube will slide or topple according to the coefficient of friction is less or greater than 0.5.

1.63 A ladder leaning against a smooth wall makes an angle α with the horizontal when in a position of limiting equilibrium. Show that the coefficient of friction between the ladder and the ground is $\frac{1}{2}\cot\alpha$.

1.3 Solutions

1.3.1 Motion in One Dimension

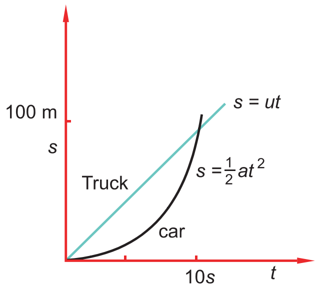
1.1 **(a)** Equation of motion for the truck: $s = ut$ (1)

Equation of motion for the car: $s = \dfrac{1}{2}at^2$ (2)

The graphs for (1) and (2) are shown in Fig. 1.13. Eliminating t between the two equations

$$s\left(1 - \frac{1}{2}\frac{as}{u^2}\right) = 0 \tag{3}$$

Fig. 1.13



Either $s = 0$ or $1 - \dfrac{1}{2}\dfrac{as}{u^2} = 0$. The first solution corresponds to the result that the truck overtakes the car at $s = 0$ and therefore at $t = 0$.

The second solution gives $s = \dfrac{2u^2}{a} = \dfrac{2 \times 10^2}{2} = 100\,\text{m}$

(b) $t = \dfrac{s}{u} = \dfrac{100}{10} = 10\,\text{s}$

(c) $v = at = 2 \times 10 = 20\,\text{m/s}$

1.2 When the stone reaches a height h above A

$$v_1^2 = u^2 - 2gh \tag{1}$$

and when it reaches a distance h below A

$$v_2^2 = u^2 + 2gh \tag{2}$$

since the velocity of the stone while crossing A on its return journey is again u vertically down.

Also, $v_2 = 2v_1$ (by problem) (3)

Combining (1), (2) and (3) $u^2 = \frac{10}{3}gh$ (4)

Maximum height

$$H = \frac{u^2}{2g} = \frac{10}{3}\frac{gh}{2g} = \frac{5h}{3}$$

1.3 Let the stones meet at a height s m from the earth after t s. Distance covered by the first stone

$$h - s = \frac{1}{2}gt^2 \tag{1}$$

where $h = 19.6\,\text{m}$. For the second stone

$$s = ut = \frac{1}{2}gt^2 \tag{2}$$

$$v^2 = 0 = u^2 - 2gh$$

$$u = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 19.6} = 19.6\,\text{m/s} \tag{3}$$

Adding (1) and (2)

$$h = ut,\ t = \frac{h}{u} = \frac{19.6}{19.6} = 1\,\text{s}$$

From (2),

$$s = 19.6 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 14.7\,\text{m}$$

1.4 $x = A \sin \pi t = A \sin \omega t$

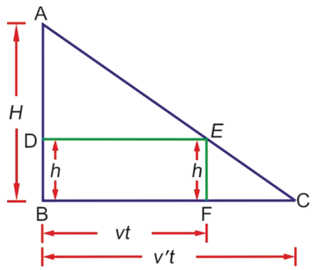
where ω is the angular velocity, $\omega = \pi$

Time period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi} = 2\,\text{s}$

In $\frac{1}{2}$s (a quarter of the cycle) the distance covered is A. Therefore in 3 s the distance covered will be $6A$.

1.5 Let the lamp be at A at height H from the ground, that is AB $= H$, Fig. 1.14. Let the man be initially at B, below the lamp, his height being equal to BD $= h$, so that the tip of his shadow is at B. Let the man walk from B to F in time t with speed v, the shadow will go up to C in the same time t with speed v':

Fig. 1.14



$\text{BF} = vt;\ \text{BC} = v't$

From similar triangles EFC and ABC

$$\frac{\text{FC}}{\text{BC}} = \frac{\text{EF}}{\text{AB}} = \frac{h}{H}$$

$$\frac{\text{FC}}{\text{BC}} = \frac{\text{EF}}{\text{AB}} = \frac{h}{H} \rightarrow \frac{v't - vt}{v't} = \frac{h}{H}$$

or

$$v' = \frac{Hv}{H - h} = \frac{6 \times 7}{(6 - 1.8)} = 10\,\text{m/s}$$

1.6 $\sqrt{3x} = 3t - 6$ (1)

Squaring and simplifying $x = 3t^2 - 12t + 12$ (2)

$$v = \frac{dx}{dt} = 6t - 12$$

$$v = 0 \text{ gives } t = 2\,\text{s} \tag{3}$$

Using (3) in (2) gives displacement $x = 0$

1.7 $$s = ut + \frac{1}{2}at^2 \tag{1}$$

$$\therefore \quad h = u \times 2 - \frac{1}{2}g \times 2^2 \tag{2}$$

$$h = u \times 10 - \frac{1}{2}g \times 10^2 \tag{3}$$

Solving (2) and (3) $h = 10g = 10 \times 9.8 = 98\,\text{m}$.

1.8 Take the origin at the position of A at $t = 0$. Let the car A overtake B in time t after travelling a distance s. In the same time t, B travels a distance $(s - 30)$ m:

$$s = ut + \frac{1}{2}at^2 \tag{1}$$

$$s = 13t + \frac{1}{2} \times 0.6\,t^2 \quad \text{(Car A)} \tag{2}$$

$$s - 30 = 20t - \frac{1}{2} \times 0.46\,t^2 \quad \text{(Car B)} \tag{3}$$

Eliminating s between (2) and (3), we find $t = 0.9\,\text{s}$.

1.9 Let BD $= x$. Time t_1 for crossing the field along AD is

$$t_1 = \frac{\text{AD}}{v_1} = \frac{\sqrt{x^2 + (600)^2}}{1.0} \tag{1}$$

Time t_2 for walking on the road, a distance DC, is

$$t_2 = \frac{\text{DC}}{v_2} = \frac{800 - x}{2.0} \tag{2}$$

$$\text{Total time } t = t_1 + t_2 = \sqrt{x^2 + (600)^2} + \frac{800 - x}{2} \tag{3}$$

Minimum time is obtained by setting $dt/dx = 0$. This gives us $x = 346.4\,\text{m}$. Thus the boy must head toward D on the round, which is 800–346.4 or 453.6 m away from the destination on the road.
The total time t is obtained by using $x = 346.4$ in (3). We find $t = 920\,\text{s}$.

1.10 Time taken for the first drop to reach the floor is

$$t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 2.45}{9.8}} = \frac{1}{\sqrt{2}}\,\text{s}$$

As the time interval between the first and second drop is equal to that of the second and the third drop (drops dripping at regular intervals), time taken by the second drop is $t_2 = \frac{1}{2\sqrt{2}}$ s; therefore, distance travelled by the second drop is

$$S = \frac{1}{2}gt_2^2 = \frac{1}{2} \times 9.8 \times \left(\frac{1}{2\sqrt{2}}\right)^2 = 0.6125\,\text{m}$$

1.11 Height h = area under the $\upsilon - t$ graph. Area above the t-axis is taken positive and below the t-axis is taken negative. h = area of bigger triangle minus area of smaller triangle.
Now the area of a triangle = base $\times$ altitude

$$h = \frac{1}{2} \times 3 \times 30 - \frac{1}{2} \times 1 \times 10 = 40\,\text{m}$$

1.12 **(a)** Time for the ball to reach water $t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 4.9}{9.8}} = 1.0\,\text{s}$
Velocity of the ball acquired at that instant $v = gt_1 = 9.8 \times 1.0 = 9.8\,\text{m/s}$.
Time taken to reach the bottom of the lake from the water surface

$$t_2 = 5.0 - 1.0 = 4.0\,\text{s}.$$

As the velocity of the ball in water is constant, depth of the lake,

$$d = vt_2 = 9.8 \times 4 = 39.2\,\text{m}.$$

(b) $$< v >= \frac{\text{total displacement}}{\text{total time}} = \frac{4.9 + 39.2}{5.0} = 8.82\,\text{m/s}$$

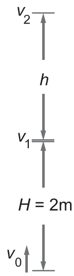
1.13 For the first stone time $t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 44.1}{9.8}} = 3.0\,\text{s}$.
Second stone takes $t_2 = 3.0 - 1.0 = 2.0\,\text{s}$ to strike the water

$$h = ut_2 + \frac{1}{2}gt_2^2$$

Using $h = 44.1\,\text{m}$, $t_2 = 2.0\,\text{s}$ and $g = 9.8\,\text{m/s}^2$, we find $u = 12.25\,\text{m/s}$

1.14 Transit time for the single journey = 0.5 s.
When the ball moves up, let υ_0 be its velocity at the bottom of the window, υ_1 at the top of the window and $\upsilon_2 = 0$ at height h above the top of the window (Fig. 1.15)

Fig. 1.15



$$v_1 = v_0 - gt = v_0 - 9.8 \times 0.5 = v_0 - 4.9 \quad (1)$$
$$v_1^2 = v_0^2 - 2gh = v_0^2 - 2 \times 9.8 \times 2 = v_0^2 - 39.2 \quad (2)$$

Eliminating v_1 between (1) and (2)

$$v_0 = 6.45\,\text{m/s} \quad (3)$$
$$v_2^2 = 0 = v_0^2 - 2g\,(H + h)$$
$$H + h = \frac{v_0^2}{2g} = \frac{(6.45)^2}{2 \times 9.8} = 2.1225\,\text{m}$$
$$h = 2.1225 - 2.0 = 0.1225\,\text{m}$$

Thus the ball rises 12.25 cm above the top of the window.

1.15 **(a)** $S_n = g\left(n - \frac{1}{2}\right)$ $\qquad S = \frac{1}{2}gn^2$

By problem $S_n = \frac{3s}{4}$

$$g\left(n - \frac{1}{2}\right) = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)gn^2$$

Simplifying $3n^2 - 8n + 4 = 0$, $n = 2$ or $\frac{2}{3}$

The second solution, $n = \frac{2}{3}$, is ruled out as $n < 1$.

(b) $s = \frac{1}{2}gn^2 = \frac{1}{2} \times 9.8 \times 2^2 = 19.6\,\text{m}$

1.16 In the triangle ACD, CA represents magnitude and apparent direction of wind's velocity w_1, when the man walks with velocity DC $= v = 4\,\text{km/h}$ toward west, Fig. 1.16. The side DA must represent actual wind's velocity because

$$\boldsymbol{W}_1 = \boldsymbol{W} - \boldsymbol{v}$$

When the speed is doubled, DB represents the velocity $2\boldsymbol{v}$ and BA represents the apparent wind's velocity $\boldsymbol{W}_2$. From the triangle ABD,

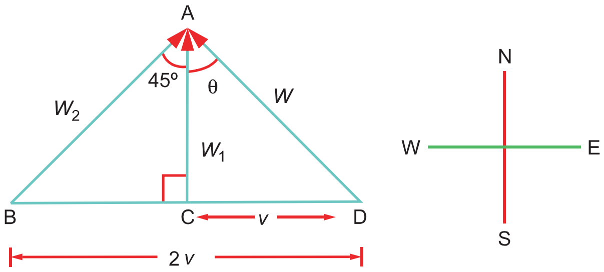


Fig. 1.16

$$\boldsymbol{W}_2 = \boldsymbol{W} - 2\boldsymbol{v}$$

By problem angle CAD $= \theta = 45°$. The triangle ACD is therefore an isosceles right angle triangle:

$$\text{AD} = \sqrt{2}\text{CD} = 4\sqrt{2}\,\text{km/h}$$

Therefore the actual speed of the wind is $4\sqrt{2}$ km/h from southeast direction.

1.17 Choose the floor of the elevator as the reference frame. The observer is inside the elevator. Take the downward direction as positive.
Acceleration of the bolt relative to the elevator is

$$a' = g - (-a) = g + a$$

$$h = \frac{1}{2}a't^2 = \frac{1}{2}(g + a)t^2 \qquad t = \sqrt{\frac{2h}{g + a}}$$

1.18 In 2 s after the truck driver applies the brakes, the distance of separation between the truck and the car becomes

$$d_{\text{rel}} = d - \frac{1}{2}\text{at}^2 = 10 - \frac{1}{2} \times 2 \times 2^2 = 6\,\text{m}$$

The velocity of the truck 2 becomes $20 - 2 \times 2 = 16$ m/s.
Thus, at this moment the relative velocity between the car and the truck will be

$$u_{\text{rel}} = 20 - 16 = 4\,\text{m/s}$$

Let the car decelerate at a constant rate of a_2. Then the relative deceleration will be

$$a_{\text{rel}} = a_2 - a_1$$

If the rear-end collision is to be avoided the car and the truck must have the same final velocity that is

$$v_{\text{rel}} = 0$$

Now $v_{\text{rel}}^2 = u_{\text{rel}}^2 - 2\,a_{\text{rel}}\,d_{\text{rel}}$

$$a_{\text{rel}} = \frac{v_{\text{rel}}^2}{2\,d_{\text{rel}}} = \frac{4^2}{2\times 6} = \frac{4}{3}\,\text{m/s}^2$$

$$\therefore \quad a_2 = a_1 + a_{\text{rel}} = 2 + \frac{4}{3} = 3.33\,\text{m/s}^2$$

1.19 $\boldsymbol{v}_{\text{BA}} = \boldsymbol{v}_{\text{B}} - \boldsymbol{v}_{\text{A}}$

From Fig. 1.17a

$$\boldsymbol{v}_{\text{BA}} = \sqrt{v_{\text{B}}^2 + v_{\text{A}}^2 - 2v_{\text{B}}v_{\text{A}}\cos 60^\circ}$$

$$= \sqrt{20^2 + 30^2 - 2\times 20\times 30\times 0.5} = 10\sqrt{7}\,\text{km/h}$$

The direction of $\boldsymbol{v}_{\text{BA}}$ can be found from the law of sines for ΔABC, Fig. 1.17a:

(i) $$\frac{\text{AC}}{\sin\theta} = \frac{\text{BC}}{\sin 60}$$

$$\text{or}\ \sin\theta = \frac{\text{AC}}{\text{BC}}\sin 60 = \frac{v_{\text{B}}}{v_{\text{BA}}}\sin 60^\circ = \frac{20\times 0.866}{10\sqrt{7}} = 0.6546$$

$$\theta = 40.9^\circ$$

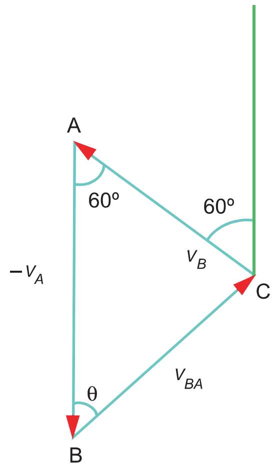


Fig. 1.17a

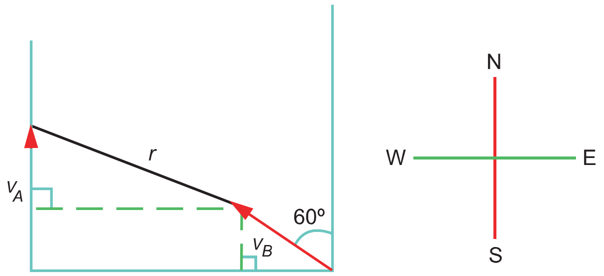


Fig. 1.17b

Thus $\boldsymbol{v}_{\mathrm{BA}}$ makes an angle 40.9° east of north.

(ii) Let the distance between the two ships be r at time t. Then from the construction of Fig. 1.17b

$$r = [(v_{\mathrm{A}}t - v_{\mathrm{B}}t\cos 60°)^2 + (10 - v_{\mathrm{B}}t\sin 60°)^2]^{1/2} \quad (1)$$

Distance of closest approach can be found by setting $\mathrm{d}r/\mathrm{d}t = 0$. This gives $t = \frac{\sqrt{3}}{7}h$. When $t = \frac{\sqrt{3}}{7}$ is inserted in (1) we get $r_{\min} = 20/\sqrt{7}$ or 7.56 km.

1.20 The initial velocity of the packet is the same as that of the balloon and is pointing upwards, which is taken as the positive direction. The acceleration due to gravity being in the opposite direction is taken negative. The displacement is also negative since it is vertically down:

$$u = 9.8\,\mathrm{m/s},\ a = -g = -9.8\,\mathrm{m/s^2};\ S = -98\,\mathrm{m}$$

$$s = ut + \frac{1}{2}at^2;\ -98 = 9.8t - \frac{1}{2} \times 9.8\,t^2 \quad \text{or} \quad t^2 - 2t - 20 = 0,$$

$$t = 1 \pm \sqrt{21}$$

The acceptable solution is $1 + \sqrt{21}$ or 5.58 s. The second solution being negative is ignored. Thus the packet takes 5.58 s to reach the ground.

1.3.2 Motion in Resisting Medium

1.21 Physically the difference between t_1 and t_2 on the one hand and v and u on other hand arises due to the fact that during ascent both gravity and air resistance act downward (friction acts opposite to motion) but during descent gravity and air resistance are oppositely directed. Air resistance F actually increases with the velocity of the object ($F \propto v$ or v^2 or v^3). Here for simplicity we assume it to be constant.

For upward motion, the equation of motion is

$$ma_1 = -(F + mg)$$

or

$$a_1 = -\left(\frac{F}{m} + g\right) \tag{1}$$

For downward motion, the equation of motion is

$$ma_2 = mg - F$$

or

$$a_2 = g - \frac{F}{m} \tag{2}$$

For ascent

$$v_1 = 0 = u + a_1 t = u - \left(\frac{F}{m} + g\right) t_1$$

$$t_1 = \frac{u}{g + \frac{F}{m}} \tag{3}$$

$$v_1{}^2 = 0 = u^2 + 2a_1 h$$

$$u = \sqrt{\frac{2h}{\left(g + \frac{F}{m}\right)}} \tag{4}$$

where we have used (1). Using (4) in (3)

$$t_1 = \sqrt{\frac{2h}{g + \frac{F}{m}}} \tag{5}$$

For descent $v^2 = 2a_2 h$

$$v = \sqrt{2h\left(g - \frac{F}{m}\right)} \tag{6}$$

where we have used (2)

$$t_2 = \frac{v}{a_2} = \sqrt{\frac{2h}{g - \frac{F}{m}}}, \tag{7}$$

where we have used (2) and (6)
From (5) and (7)

$$\frac{t_2}{t_1} = \sqrt{\frac{g + \frac{F}{m}}{g - \frac{F}{m}}} \tag{8}$$

It follows that $t_2 > t_1$, that is, time of descent is greater than the time of ascent. Further, from (4) and (6)

$$\frac{v}{u} = \sqrt{\frac{g - \frac{F}{m}}{g + \frac{F}{m}}} \tag{9}$$

It follows that $v < u$, that is, the final speed is smaller than the initial speed.

1.22 Taking the downward direction as positive, the equation of motion will be

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv \tag{1}$$

where k is a constant. Integrating

$$\int \frac{\mathrm{d}v}{g - kv} = \int \mathrm{d}t$$

$$\therefore \quad -\frac{1}{k} \ln\left(\frac{g - kv}{c}\right) = t$$

where c is a constant:

$$g - kv = ce^{-kt} \tag{2}$$

This gives the velocity at any instant.
As t increases e^{-kt} decreases and if t increases indefinitely $g - kv = 0$, i.e.

$$v = \frac{g}{k} \tag{3}$$

This limiting velocity is called the terminal velocity. We can obtain an expression for the distance x traversed in time t. First, we identify the constant c in (2). Since it is assumed that $v = 0$ at $t = 0$, it follows that $c = g$.
Writing $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ in (2) and putting $c = g$, and integrating

$$g - k\frac{\mathrm{d}x}{\mathrm{d}t} = g\mathrm{e}^{-kt}$$

$$\int g\mathrm{d}t - k\int \mathrm{d}x = g\int \mathrm{e}^{-kt}\mathrm{d}t + D$$

$$gt - kx = -\frac{g}{k}\mathrm{e}^{-kt} + D$$

At $x = 0, t = 0$; therefore, $D = \frac{g}{k}$

$$x = \frac{gt}{k} - \frac{g}{k^2}\left(1 - e^{-kt}\right) \tag{4}$$

1.23 The equation of motion is

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2 \tag{1}$$

$$\frac{dv}{dt} = g - kv^2 \tag{2}$$

$$\therefore \quad \frac{1}{k}\int \frac{dv}{\frac{g}{k} - v^2} = t + c \tag{3}$$

writing $V^2 = \frac{g}{k}$ and integrating

$$\ln \frac{V + v}{V - v} = 2kV(t + c) \tag{4}$$

If the body starts from rest, then $c = 0$ and

$$\ln \frac{V + v}{V - v} = 2kVt = \frac{2gt}{V}$$

$$\therefore \quad t = \frac{V}{2g} \ln \frac{V + v}{V - v} \tag{5}$$

which gives the time required for the particle to attain a velocity $\upsilon = 0$. Now

$$\frac{V + v}{V - v} = e^{2kVt}$$

$$\therefore \quad \frac{v}{V} = \frac{e^{2kVt} - 1}{e^{2kVt} + 1} = \tanh kVt \tag{6}$$

i.e.

$$v = V \tanh \frac{gt}{V} \tag{7}$$

The last equation gives the velocity υ after time t. From (7)

$$\frac{dx}{dt} = V \tanh \frac{gt}{V}$$

$$x = \frac{V^2}{g} \ln \cosh \frac{gt}{V} \tag{8}$$

$$x = \frac{V^2}{g} \ln \frac{e^{gt/v} + e^{-gt/v}}{2} \tag{9}$$

no additive constant being necessary since $x = 0$ when $t = 0$. From (6) it is obvious that as t increases indefinitely υ approaches the value V. Hence V is the terminal velocity, and is equal to $\sqrt{g/k}$.
The velocity v in terms of x can be obtained by eliminating t between (5) and (9).

From (9),

$$e^{kx} = \frac{e^{kVt} + e^{-kVt}}{2}$$

Squaring $4e^{2kx} = e^{2kVt} + e^{-2kVt} + 2$

$$= \frac{V+\upsilon}{V-\upsilon} + \frac{V-\upsilon}{V+\upsilon} + 2 \quad \text{from (5)}$$

$$= \frac{4V^2}{V^2 - \upsilon^2}$$

$$\therefore \quad \upsilon^2 = V^2(1 - e^{-2kx})$$

$$= V^2\left(1 - e^{-\frac{2gx}{V^2}}\right) \tag{10}$$

1.24 Measuring x upward, the equation of motion will be

$$\frac{d^2x}{dt^2} = -g - k\left(\frac{dx}{dt}\right)^2 \tag{1}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\therefore \quad v\frac{dv}{dx} = -g - kv^2 \tag{2}$$

$$\therefore \quad \frac{1}{2k}\int \frac{d\left(v^2\right)}{(g/k) + v^2} = -\int dx$$

Integrating, $\ln\left(\frac{(g/k) + v^2}{c}\right) = -2kx$

or $$\frac{g}{k} + v^2 = ce^{-2kx} \tag{3}$$

When $x = 0$, $v = u$; $\therefore c = \frac{g}{k} + u^2$ and writing $\frac{g}{k} = V^2$, we have

$$\frac{V^2 + v^2}{V^2 + u^2} = e^{-\frac{2gx}{V^2}} \tag{4}$$

$$\therefore \quad v^2 = (V^2 + u^2)e^{-\frac{2gx}{V^2}} - V^2 \tag{5}$$

The height h to which the particle rises is found by putting $\upsilon = 0$ at $x = h$ in (5)

$$\frac{V^2+u^2}{V^2} = \mathrm{e}^{\frac{2gh}{V^2}}$$

$$h = \frac{V^2}{2g}\ln\left(1+\frac{u^2}{V^2}\right) \tag{6}$$

1.25 The particle reaches the height h given by

$$h = \frac{V^2}{2g}\ln\left(1+\frac{u^2}{V^2}\right) \quad \text{(by prob. 1.24)}$$

The velocity at any point during the descent is given by

$$v^2 = V^2\left(1-\mathrm{e}^{-\frac{2gx}{V^2}}\right) \quad \text{(by prob. 1.23)}$$

The velocity of the body when it reaches the point of projection is found by substituting h for x:

$$\therefore \quad v^2 = V^2\left\{1-\frac{V^2}{V^2+u^2}\right\} = \frac{u^2V^2}{V^2+u^2}$$

$$\text{Loss of kinetic energy} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}mu^2\left\{1-\frac{V^2}{V^2+u^2}\right\} = \frac{1}{2}mu^2\left(\frac{u^2}{V^2+u^2}\right)$$

1.3.3 Motion in Two Dimensions

1.26 **(i)** $\frac{\mathrm{d}x}{\mathrm{d}t} = 6+2t$

$$\int \mathrm{d}x = 6\int \mathrm{d}t + 2\int t\,\mathrm{d}t$$

$$x = 6t + t^2 + C$$

$$x = 0,\ t = 0;\ C = 0$$

$$x = 6t + t^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4+t$$

$$\int \mathrm{d}y = 4\int \mathrm{d}t + \int t\,\mathrm{d}t$$

$$y = 4t + \frac{t^2}{2} + D$$

$$y = 0, t = u;\ D = u$$

$$y = u + 4t + \frac{t^2}{2}$$

(ii) $\vec{v} = (6 + 2t)\hat{i} + (4 + t)\hat{j}$

(iii) $\vec{a} = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 2\hat{i} + \hat{j}$

(iv) $a = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\tan\theta = \frac{1}{2};\ \theta = 26.565°$$

Acceleration is directed at an angle of $26°34'$ with the x-axis.

1.27 Take upward direction as positive, Fig. 1.18. At time t the velocities of the objects will be

$$\boldsymbol{v}_1 = u_1\hat{i} - gt\,\hat{j} \tag{1}$$

$$\boldsymbol{v}_2 = -u_2\hat{i} - gt\,\hat{j} \tag{2}$$

If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are to be perpendicular to each other, then $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, that is

$$\left(u_1\hat{i} - gt\,\hat{j}\right)\cdot\left(-u_2\hat{i} - gt\,\hat{j}\right) = 0$$

$$\therefore \quad -u_1u_2 + g^2t^2 = 0$$

$$\text{or} \quad t = \frac{1}{g}\sqrt{u_1u_2} \tag{3}$$

The position vectors are $\boldsymbol{r}_1 = u_1t\,\hat{i} - \frac{1}{2}gt^2\,\hat{j}, \quad \boldsymbol{r}_2 = -u_2t\,\hat{i} - \frac{1}{2}gt^2\,\hat{j}$.

The distance of separation of the objects will be

$$r_{12} = |\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2| = (u_1 + u_2)t$$

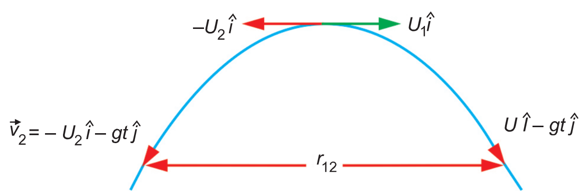


Fig. 1.18

or

$$r_{12} = \frac{(u_1 + u_2)}{g}\sqrt{u_1 u_2} \tag{4}$$

where we have used (2).

1.28 Consider the equation

$$s = ut + \frac{1}{2}at^2 \tag{1}$$

Taking upward direction as positive, $a = -g$ and let $s = h$, the height of the tower, (1) becomes

$$h = ut - \frac{1}{2}gt^2$$

or

$$\frac{1}{2}gt^2 - ut + h = 0 \tag{2}$$

Let the two roots be t_1 and t_2. Compare (2) with the quadratic equation

$$ax^2 + bx + c = 0 \tag{3}$$

The product of the two roots is equal to c/a. It follows that

$$t_1 t_2 = \frac{2h}{g} \text{ or } \sqrt{t_1 t_2} = \sqrt{\frac{2h}{g}} = t_3$$

which is the time taken for a free fall of an object from the height h.

1.29 Let the shell hit the plane at $p(x, y)$, the range being AP $= R$, Fig. 1.19. The equation for the projectile's motion is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \tag{1}$$

Now $\quad y = R\sin\alpha \quad (2)$

$$x = R\cos\alpha \tag{3}$$

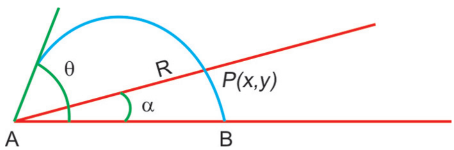


Fig. 1.19

Using (2) and (3) in (1) and simplifying

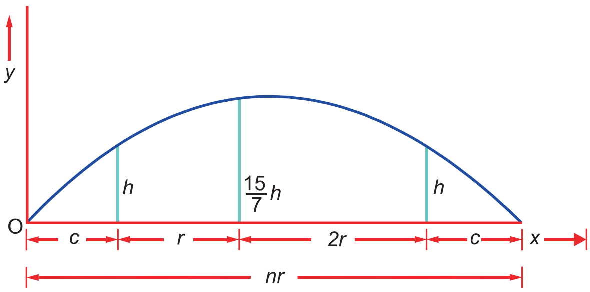
$$R = \frac{2u^2 \cos\theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

The maximum range is obtained by setting $\dfrac{dR}{d\theta} = 0$, holding u, α and g constant. This gives $\cos(2\theta - \alpha) = 0$ or $2\theta - \alpha = \dfrac{\pi}{2}$

$$\therefore\ \alpha = \frac{\theta}{2} + \frac{\pi}{4}$$

1.30 As the outer walls are equal in height (h) they are equally distant (c) from the extremities of the parabolic trajectory whose general form may be written as (Fig. 1.20)

Fig. 1.20



$$y = ax - bx^2 \tag{1}$$

$y = 0$ at $x = R = nr$, when R is the range

This gives $a = bnr$ (2)

The range $R = c + r + 2r + c = nr$, by problem

$$\therefore\quad c = (n-3)\frac{r}{2} \tag{3}$$

The trajectory passes through the top of the three walls whose coordinates are (c, h), $\left(c + r, \frac{15}{7}h\right)$, $(c + 3r, h)$, respectively. Using these coordinates in (1), we get three equations

$$h = ac - bc^2 \tag{4}$$

$$\frac{15h}{7} = a(c + r) - b(c + r)^2 \tag{5}$$

$$h = a(c + 3r) - b(c + 3r)^2 \tag{6}$$

Combining (2), (3), (4), (5) and (6) and solving we get $n = 4$.

1.31 The equation to the parabolic path can be written as

$$y = ax - bx^2 \tag{1}$$

$$\text{with } a = \tan\theta;\; b = \frac{g}{2u^2\cos^2\theta} \tag{2}$$

Taking the point of projection as the origin, the coordinates of the two openings in the windows are (5, 5) and (11, 7), respectively. Using these coordinates in (1) we get the equations

$$5 = 5a - 25b \tag{3}$$

$$7 = 11a - 121b \tag{4}$$

with the solutions, $a = 1.303$ and $b = 0.0606$. Using these values in (2), we find $\theta = 52.5°$ and $u = 14.8\,\text{m/s}$.

1.32 Let the rifle be fixed at A and point in the direction AB at an angle α with the horizontal, the monkey sitting on the tree top at B at height h, Fig. 1.21. The bullet follows the parabolic path and reaches point D, at height H, in time t.

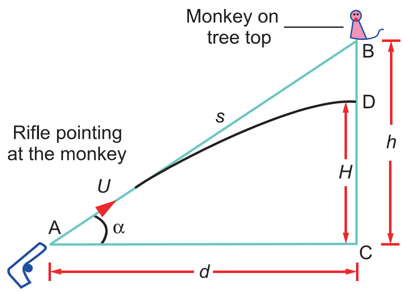


Fig. 1.21

The horizontal and initial vertical components of velocity of bullet are

$$u_x = u\cos\alpha;\; u_y = u\sin\alpha$$

Let the bullet reach the point D, vertically below B in time t, the coordinates of D being (d, H). As the horizontal component of velocity is constant

$$d = u_x t = (u\cos\alpha)t = \frac{udt}{s}$$

where $s = \text{AB}$:

$$t = \frac{s}{u}$$

The vertical component of velocity is reduced due to gravity.

In the same time, the y-coordinate at D is given by

$$y = H = u_y t - \frac{1}{2}gt^2 = u(\sin\alpha)t - \frac{1}{2}gt^2$$

$$H = u\left(\frac{h}{s}\right)\left(\frac{s}{u}\right) - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2$$

$$\text{or} \quad h - H = \frac{1}{2}gt^2$$

$$\therefore \quad t = \sqrt{\frac{2(h-H)}{g}}$$

But the quantity $(h–H)$ represents the height through which the monkey drops from the tree and the right-hand side of the last equation gives the time for a free fall. Therefore, the bullet would hit the monkey independent of the bullet's initial velocity.

1.33 $R = \dfrac{u^2 \sin 2\alpha}{g}, \quad h = \dfrac{u^2 \sin^2\alpha}{g}, \quad T = \dfrac{2u\sin\alpha}{g}$

(a) $\dfrac{h}{R} = \dfrac{1}{4}\tan\alpha \rightarrow \tan\alpha = \dfrac{4h}{R}$

(b) $\dfrac{h}{T^2} = \dfrac{g}{8} \rightarrow h = \dfrac{gT^2}{8}$

1.34 **(i)** $T = \dfrac{2u\sin\alpha}{g} = \dfrac{2 \times 800 \sin 60°}{9.8} = 141.4\,\text{s}$

(ii) $R = \dfrac{u^2\sin 2\alpha}{g} = \dfrac{(800)^2\sin(2\times 60)}{9.8} = 5.6568 \times 10^4\text{m} = 56.57\,\text{km}$

(iii) Time to reach maximum height $= \frac{1}{2}T = \frac{1}{2} \times 141.4 = 707\,\text{s}$

(iv) $x = (u\cos\alpha)t$ (1)

$$y = (u\sin\alpha)t - \frac{1}{2}gt^2 \quad (2)$$

Eliminating t between (1) and (2) and simplifying

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \quad (3)$$

which is of the form $y = bx + cx^2$, with $b = \tan\alpha$ and $c = -\dfrac{1}{2}\dfrac{g}{u^2\cos^2\alpha}$.

1.35 **(i)** $T = \dfrac{u\sin\alpha}{g} = \dfrac{350\ \sin 55°}{9.8} = 29.25\,\text{s}$

(ii) At the highest point of the trajectory, the velocity of the particle is entirely horizontal, being equal to $u\cos\alpha$. The momentum of this particle at the highest point is $p = mu\cos\alpha$, when m is its mass. After the explosion, one fragment starts falling vertically and so does not carry any momentum initially. It would fall at half of the range, that is

$$\frac{R}{2} = \frac{1}{2}\frac{u^2 \sin 2\alpha}{g} = \frac{(350)^2 \sin(2 \times 55°)}{2 \times 9.8} = 5873\,\text{m, from the firing point.}$$

The second part of mass $\frac{1}{2}m$ proceeds horizontally from the highest point with initial momentum p in order to conserve momentum. If its velocity is v then

$$p = \frac{m}{2}v = mu\cos\alpha$$
$$v = 2u\cos\alpha = 2 \times 350\cos 55° = 401.5\,\text{m/s}$$

Then its range will be

$$R' = v\sqrt{\frac{2h}{g}} \tag{1}$$

But the maximum height

$$h = \frac{u^2 \sin^2\alpha}{2g} \tag{2}$$

Using (2) in (1)

$$R' = \frac{vu\sin\alpha}{g} = \frac{(401.5)(350)(\sin 55°)}{9.8} = 11746\,\text{m}$$

The distance form the firing point at which the second fragment hits the ground is

$$\frac{R}{2} + R' = 5873 + 11746 = 17619\,\text{m}$$

(iii) Energy released = (kinetic energy of the fragments) − (kinetic energy of the particle) at the time of explosion

$$= \frac{1}{2}\frac{m}{2}v^2 - \frac{1}{2}m(u\cos\alpha)^2$$
$$= \frac{20}{4} \times (401.5)^2 - \frac{20}{2}(350\cos\ 55°)^2 = 4.03 \times 10^5\,\text{J}$$

1.36 The radius of curvature

$$\rho = \frac{\left[1 + (\mathrm{d}y/\mathrm{d}x)^2\right]^{3/2}}{\mathrm{d}^2y/\mathrm{d}x^2} \quad (1)$$

$$x = v_o t = 10 \times 3 = 30\,\mathrm{m}$$

$$y = \frac{1}{2}gt^2 = \frac{1}{2} \times 9.8 \times 3^2 = 44.1\,\mathrm{m}.$$

$$\therefore y = \frac{1}{2}g\frac{x^2}{v_0^2}$$

$$v_0^2 = \frac{9.8 \times 30}{10^2} = 2.94 \quad (2)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{g}{v_0^2} = \frac{9.8}{10^2} = 0.098 \quad (3)$$

Using (2) and (3) in (1) we find $\rho = 305\,\mathrm{m}$.

1.37 Let P be the position of the boat at any time, Let AP $= r$, angle $B\hat{A}P = \theta$, and let v be the magnitude of each velocity, Fig. 1.5:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -v + v\sin\theta$$

$$\text{and} \quad \frac{r\mathrm{d}\theta}{\mathrm{d}t} = v\cos\theta$$

$$\therefore \quad \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{-1 + \sin\theta}{\cos\theta}$$

$$\therefore \quad \int \frac{\mathrm{d}r}{r} = \int [-\sec\theta + \tan\theta]\,\mathrm{d}\theta$$

$$\therefore \quad \ln r = -\ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \ln\cos\theta + \ln C \text{ (a constant)}$$

When $\theta = 0, r = a,$ so that $C = a$

$$\therefore \quad r = \frac{a}{\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\cos\theta}$$

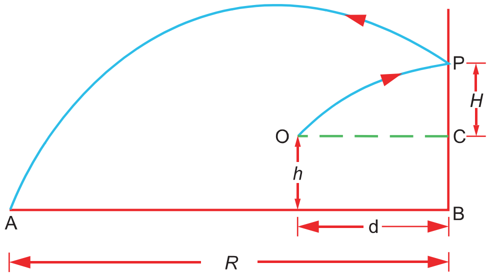
The denominator can be shown to be equal to $1 + \sin\theta$:

$$\therefore \quad r = \frac{a}{1 + \sin\theta}$$

This is the equation of a parabola with AB as semi-latus rectum.

1.38 Take the origin at O, Fig. 1.22. Draw the reference line OC parallel to AB, the ground level. Let the ball hit the wall at a height H above C. Initially at O,

Fig. 1.22



$$u_x = u\cos\alpha = u\cos 45^\circ = \frac{u}{\sqrt{2}}$$
$$u_y = u\sin\alpha = u\sin 45^\circ = \frac{u}{\sqrt{2}}$$

When the ball hits the wall, $y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$
Using $y = H$, $x = d$ and $\alpha = 45^\circ$

$$H = d\left(1 - \frac{gd}{u^2}\right) \tag{1}$$

If the collision of the ball with the wall is perfectly elastic then at P, the horizontal component of the velocity (u'_x) will be reversed, the magnitude remaining constant, while both the direction and magnitude of the vertical component v'_y are unaltered. If the time taken for the ball to bounce back from P to A is t and the range BA = R

$$y = v'_y t - \frac{1}{2}gt^2 \tag{2}$$
$$\text{Using} \quad t = \frac{R}{u\cos 45^\circ} = \sqrt{2}\frac{R}{u} \tag{3}$$
$$y = -(H + h) \tag{4}$$
$$v'_y t = u\sin 45^\circ - g\frac{d}{u\cos 45^\circ} = \frac{u}{\sqrt{2}} - \sqrt{2}\frac{gd}{u} \tag{5}$$

Using (3), (4) and (5) in (2), we get a quadratic equation in R which has the acceptable solution

$$R = \frac{u^2}{2g} + \sqrt{\frac{u^2}{4g^2} + H + h}$$

1.3.4 Force and Torque

1.39 Resolve the force into x- and y-components:

$$F_x = -80\cos 35^\circ + 60 + 40\cos 45^\circ = 22.75\,\text{N}$$
$$F_y = 80\sin 35^\circ + 0 - 40\sin 45^\circ = 17.6\,\text{N}$$

(i) $\boldsymbol{F}_{\text{net}} = \sqrt{F_x^2 + F_y^2} = \sqrt{(22.75)^2 + (17.6)^2} = 28.76\,\text{N}$

$$\tan\theta = \frac{F_y}{F_x} = \frac{17.6}{22.75} = 0.7736 \rightarrow \theta = 37.7^\circ$$

The vector $\boldsymbol{F}_{\text{net}}$ makes an angle of 37.7° with the x-axis.

(ii) $a = \dfrac{\boldsymbol{F}_{\text{net}}}{m} = \dfrac{28.76\,N}{3.8\,kg} = 7.568\,\text{m/s}^2$

(iii) $\boldsymbol{F}_4$ of magnitude 28.76 N must be applied in the opposite direction to $\boldsymbol{F}_{\text{net}}$

1.40 **(a)** **(i)** $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$

$$\tau = rF\sin\theta = (0.4\,\text{m})(50\,\text{N})\sin 90^\circ = 20\,\text{N} - \text{m}$$

(ii) $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{20}{20} = 1.0\,\text{rad/s}^2$$

(iii) $\omega = \omega_0 + \alpha t = 0 + 1 \times 3 = 3\,\text{rad/s}$

(iv) $\omega^2 = \omega_0^2 + 2\alpha\theta,\ \theta = \frac{3^2-0}{2\times 1} = 4.5\,\text{rad}$

(b) **(i)** $\tau = 0.4 \times 50 \times \sin(90 + 20) = 18.794\,\text{N m}$

(ii) $\alpha = \dfrac{\tau}{I} = \dfrac{18.794}{20} = 0.9397\,\text{rad/s}^2$

1.41 Force applied to the container $F = ma$
Frictional force = $F_\text{r} = \mu\, mg$

$$F_\text{r} = F$$
$$\mu\, mg = ma$$
$$\mu = \frac{a}{g} = \frac{1.5}{9.8} = 0.153$$

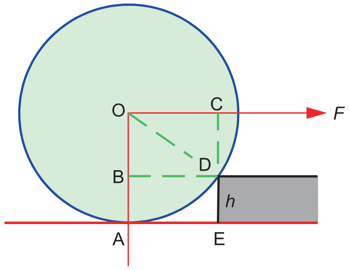


Fig. 1.23

1.42 Taking torque about D, the corner of the obstacle, $(F)\mathrm{CD} = (W)\mathrm{BD}$ (Fig. 1.23)

$$F = W\frac{\mathrm{BD}}{\mathrm{CD}} = \sqrt{\frac{\mathrm{OD}^2 - \mathrm{OB}^2}{\mathrm{CE} - \mathrm{DE}}}$$
$$= \sqrt{\frac{r^2 - (r-h)^2}{r-h}} = \frac{\sqrt{h(2r-h)}}{r-h}$$

1.3.5 Centre of Mass

1.43 Let λ be the linear mass density (mass per unit length) of the wire. Consider an infinitesimal line element $\mathrm{d}s = R\,\mathrm{d}\theta$ on the wire, Fig. 1.24. The corresponding mass element will be $\mathrm{d}m = \lambda \mathrm{d}s = \lambda R\,\mathrm{d}\theta$. Then

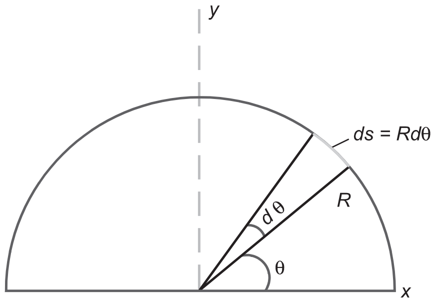
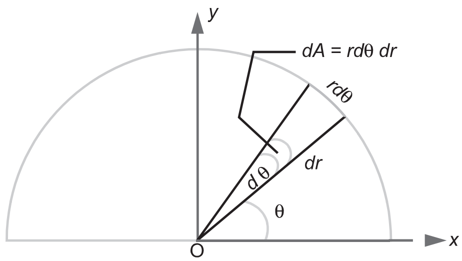


Fig. 1.24

$$y_{\mathrm{CM}} = \frac{\int y\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^\pi (R\sin\theta)(\lambda R\,\mathrm{d}\theta)}{\int_0^\pi \lambda R\,\mathrm{d}\theta}$$
$$= \frac{\lambda R^2 \int_0^\pi \sin\theta\,\mathrm{d}\theta}{\lambda R \int_0^\pi \mathrm{d}\theta} = \frac{2R}{\pi}$$

1.44 Let the x-axis lie along the diameter of the semicircle. The centre of mass must lie on y-axis perpendicular to the flat base of the semicircle and through O, the centre of the base, Fig. 1.25.

Fig. 1.25



For continuous mass distribution

$$y_{\mathrm{CM}} = \frac{1}{M}\int y\,\mathrm{d}m$$

Let σ be the surface density (mass per unit area), so that

$$M = \frac{1}{2}\pi R^2\sigma$$

In polar coordinates $\mathrm{d}m = \sigma\,\mathrm{d}A = \sigma r\,\mathrm{d}\theta\,\mathrm{d}r$
where $\mathrm{d}A$ is the element of area. Let the centre of mass be located at a distance y_{CM} from O along y-axis for reasons of symmetry:

$$y_{\mathrm{CM}} = \frac{1}{\frac{1}{2}\pi R^2\sigma}\int_0^R\int_0^\pi (r\sin\theta)(\sigma r\,\mathrm{d}\theta\,\mathrm{d}r) = \frac{2}{\pi R^2}\int_0^R r^2\int_0^\pi \sin\theta\,\mathrm{d}\theta = \frac{4R}{3\pi}$$

1.45 Let O be the origin, the centre of the base of the hemisphere, the z-axis being perpendicular to the base. From symmetry the CM must lie on the z-axis, Fig. 1.26. If ρ is the density, the mass element, $\mathrm{d}m = \rho\,\mathrm{d}V$, where $\mathrm{d}V$ is the volume element:

$$Z_{\mathrm{CM}} = \frac{1}{M}\int Z\,\mathrm{d}m = \frac{1}{M}\int Z\rho\,\mathrm{d}V \tag{1}$$

In polar coordinates, $Z = r\cos\theta$ (2)

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r \tag{3}$$

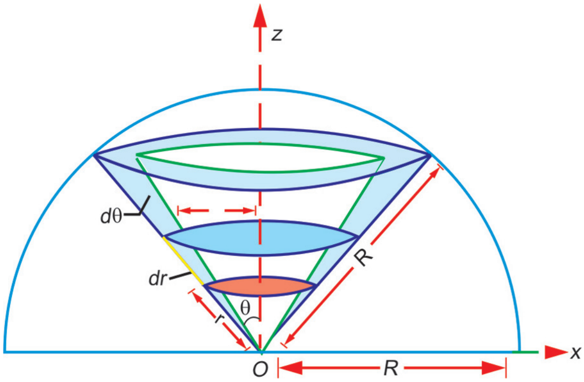
$$0 < r < R;\ 0 < \theta < \frac{\pi}{2}; \qquad 0 < \phi < 2\pi$$

The mass of the hemisphere

$$M = \rho\frac{2}{3}\pi R^3 \tag{4}$$

Using (2), (3) and (4) in (1)

Fig. 1.26



$$Z_{\mathrm{CM}} \frac{\int_0^R r^3 \mathrm{d}r \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi}{\frac{2\pi R^3}{3}} = \frac{3R}{8}$$

1.46 The mass of any portion of the disc will be proportional to its surface area. The area of the original disc is πR^2, that corresponding to the hole is $\frac{1}{4}\pi R^2$ and that of the remaining portion is $\pi R^2 - \frac{\pi R^2}{4} = \frac{3}{4}\pi R^2$.

Let the centre of the original disc be at O, Fig. 1.10. The hole touches the circumference of the disc at A, the centre of the hole being at C. When this hole is cut, let the centre of mass of the remaining part be at G, such that

$$\mathrm{OG} = x \text{ or } \mathrm{AG} = \mathrm{AO} + \mathrm{OG} = R + x$$

If we put back the cut portion of the hole and fill it up then the centre of the mass of this small disc (C) and that of the remaining portion (G) must be located at the centre of the original disc at O

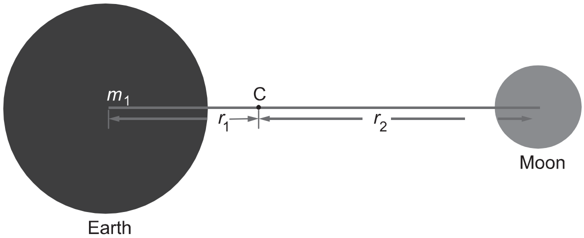
$$\mathrm{AO} = R = \frac{\mathrm{AC}\pi(R^2/4) + \mathrm{AG}\frac{3\pi}{4}R^2}{\pi R^2/4 + 3\pi R^2/4} = \frac{R}{8} + \frac{3}{4}(R + x)$$

$$\therefore \quad x = \frac{R}{6}$$

Thus the C:M of the remaining portion of the disc is located at distance $R/6$ from O on the left side.

1.47 Let m_1 be the mass of the earth and m_2 that of the moon. Let the centre of mass of the earth–moon system be located at distance r_1 from the centre of the earth and at distance r_2 from the centre of the moon, so that $r = r_1 + r_2$ is the distance between the centres of earth and moon, Fig. 1.27. Taking the origin at the centre of mass

Fig. 1.27



$$\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = 0$$

$$m_1 r_1 - m_2 r_2 = 0$$

$$r_1 = \frac{m_2 r_2}{m_1} = \frac{m_2(r - r_1)}{81 m_2} = \frac{60R - r_1}{81}$$

$$r_1 = 0.7317R = 0.7317 \times 6400 = 4683\,\text{km}$$

along the line joining the earth and moon; thus, the centre of mass of the earth–moon system lies within the earth.

1.48 Let the centre of mass be located at a distance r_c from the carbon atom and at r_0 from the oxygen atom along the line joining carbon and oxygen atoms. If r is the distance between the two atoms, m_c and m_o the mass of carbon and oxygen atoms, respectively

$$m_c r_c = m_o r_o = m_o(r - r_c)$$

$$r_c = \frac{m_o r}{m_o + m_o} = \frac{16 \times 1.13}{12 + 16} = 0.646\,\text{Å}$$

1.49 Let C be the centroid of the equilateral triangle formed by the three H atoms in the xy-plane, Fig. 1.28. The N–atom lies vertically above C, along the z-axis. The distance r_{CN} between C and N is

$$r_{CN} - \sqrt{r^2_{NH_3} - r^2_{CH_3}}$$

$$r_{CN} = \frac{r^2_{H_1H_2}}{\sqrt{3}} = \frac{1.628}{1.732} = 0.94\,\text{Å}$$

$$r_{CN} = \sqrt{(1.014)^2 - (0.94)^2} = 0.38\,\text{Å}$$

Now, the centre of mass of the three H atoms $3m_H$ lies at C. The centre of mass of the NH_3 molecule must lie along the line of symmetry joining N and C and is located below N atom at a distance

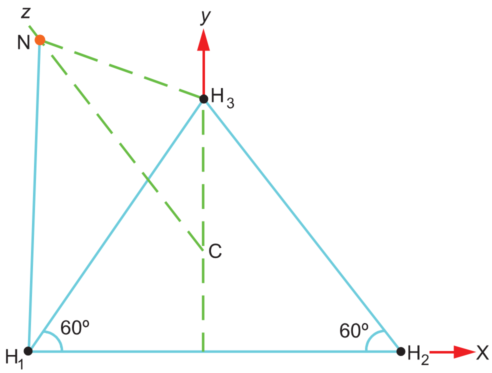


Fig. 1.28 Centre of mass of NH_3 molecule

$$Z_{CM} = \frac{3m_H}{3m_H + m_N} \times r_{CN} = \frac{3m_H}{3m_H + 14m_H} \times 0.38 = 0.067\,\text{Å}$$

1.50 Take the origin at A at the left end of the boat, Fig. 1.29. Let the boy of mass m be initially at B, the other end of the boat. The boat of mass M and length L has its centre of mass at C. Let the centre of mass of the boat + boy system be located at G, at a distance x from the origin. Obviously AC = 1.5 m:

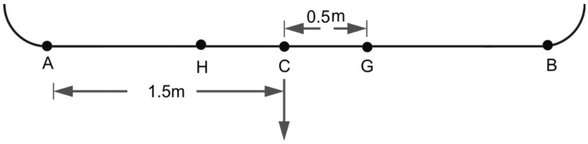


Fig. 1.29

$$\text{AG} = x = \frac{M\text{AC} + m\text{AB}}{M + m}$$
$$= \frac{100 \times 1.5 + 50 \times 3}{100 + 50} = 2\,\text{m}$$

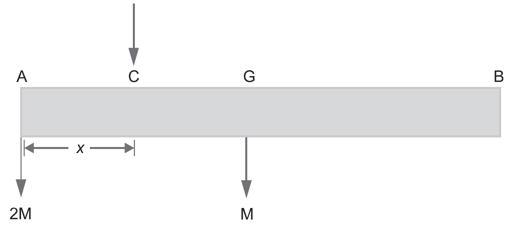
Thus CG = AG − AC

= 2.0 − 1.5 = 0.5 m

When the boy reaches A, from symmetry the CM of boat + boy system would have moved to H by a distance of 0.5 m on the left side of C. Now, in the absence of external forces, the centre of mass should not move, and so to restore the original position of the CM the boat moves towards right so that the point H is brought back to the original mark G. Since HG = 0.5 + 0.5 = 1.0, the boat in the mean time moves through 1.0 m toward right.

1.51 If the rod is to move with pure translation without rotation, then it should be struck at C, the centre of mass of the loaded rod. Let C be located at distance x from A so that
GC $= \frac{1}{2}L - x$, Fig. 1.30. Let M be the mass of the rod and $2M$ be attached at A. Take torques about C

Fig. 1.30



$$2Mx = M\left(\frac{L}{2} - x\right) \quad \therefore x = \frac{L}{6}$$

Thus the rod should be struck at a distance $\frac{L}{6}$ from the loaded end.

1.52 Volume of the cone, $V = \frac{1}{3}\pi R^2 h$ where R is the radius of the base and h is its height, Fig. 1.31. The volume element at a depth z below the apex is $\mathrm{d}V = \pi r^2 \mathrm{d}z$, the mass element $\mathrm{d}m = \rho \mathrm{d}V = \pi r^2 \mathrm{d}z f$

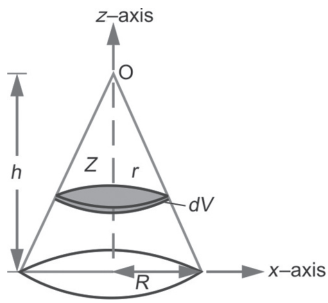


Fig. 1.31

$$\mathrm{d}m = \rho \mathrm{d}v = \rho\pi r^2 \mathrm{d}z$$

$$\frac{z}{r} = \frac{h}{R} \quad \therefore \mathrm{d}z = \frac{h}{R}\mathrm{d}r$$

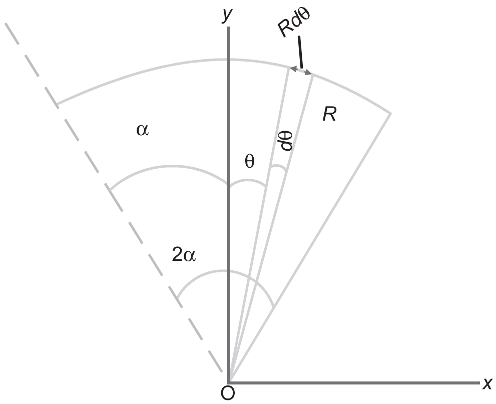
For reasons of symmetry, the centre of mass must lie on the axis of the cone. Take the origin at O, the apex of the cone:

$$Z_{\mathrm{CM}} = \frac{\int Z\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^R \left(\frac{hr}{R}\right)\rho\pi r^2 \left(\frac{h}{r}\,\mathrm{d}r\right)}{\frac{1}{3}\pi R^2 h\rho} = \frac{3h}{4}$$

Thus the CM is located at a height $h - \frac{3}{4}h = \frac{1}{4}h$ above the centre of the base of the cone.

1.53 Take the origin at O, Fig. 1.32. Let the mass of the wire be M. Consider mass element dm at angles θ and $\theta + \mathrm{d}\theta$

Fig. 1.32



$$\mathrm{d}m = \frac{MR\,\mathrm{d}\theta}{2\alpha R} = \frac{M\,\mathrm{d}\theta}{2\alpha} \qquad (1)$$

From symmetry the CM of the wire must be on the y-axis.
The y-coordinate of dm is $y = R\sin\theta$

$$y_{\mathrm{CM}} = \frac{1}{M}\int y_{dm} = \int_{90-\alpha}^{90+\alpha} \frac{R\sin\theta\,\mathrm{d}\theta}{2\alpha} = \frac{R\sin\alpha}{\alpha}$$

Note that the results of prob. (1.43) follow for $\alpha = \frac{1}{2}\pi$.

1.54 $V_{\mathrm{CM}} = \dfrac{\Sigma m_i v_i}{\Sigma m_i} = \dfrac{4mv_0 + (m)(0)}{5m} = \dfrac{4v_0}{5}$

1.55 $\rho = cx (c = \text{constant});\ \mathrm{d}m = \rho\,\mathrm{d}x = cx\,\mathrm{d}x$

$$x_{\mathrm{CM}} = \frac{\int x\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^L xcx\,\mathrm{d}x}{\int_0^L cx\,\mathrm{d}x} = \frac{2}{3}L$$

1.56 $x_{\mathrm{CM}} = \dfrac{\Sigma m_i x_i}{\Sigma m_i} = \dfrac{mL + (2m)(2L) + (3m)(3L) + \cdots + (nm)(nL)}{m + 2m + 3m + \cdots + nm}$

$$= \frac{(1 + 4 + 9 + \cdots + n^2)L}{1 + 2 + 3 + \cdots + n} = \frac{(\text{sum of squares of natural numbers})L}{\text{sum of natural numbers}}$$

$$= \frac{n(n+1)(2n+1)L/6}{n(n+1)/2} = (2n+1)\frac{L}{3}$$

1.57 The diagram is the same as for prob. (1.44)

$$y = r \sin\theta$$
$$\mathrm{d}A = r\ \mathrm{d}\theta \mathrm{d}r$$
$$\mathrm{d}m = r\ \mathrm{d}\theta \mathrm{d}r\rho = r\mathrm{d}\theta\mathrm{d}r cr^2 = cr^3\ \mathrm{d}r\ \mathrm{d}\theta$$

$$\text{Total mass} \quad M = \int \mathrm{d}m = c\int_0^R r^3 \mathrm{d}r \int_0^\pi \mathrm{d}\theta = \frac{\pi c R^4}{4} \qquad (1)$$

$$y_{\mathrm{CM}} = \frac{1}{M}\int y\ \mathrm{d}m = \frac{1}{M}\int\int (r\sin\theta)cr^3\ \mathrm{d}r\mathrm{d}\theta$$
$$= \frac{C}{M}\int_0^R r^4 \mathrm{d}r \int_0^\pi \sin\theta\ \mathrm{d}\theta$$
$$= \frac{C}{M}\frac{2}{5}R^5 = \frac{8a}{5\pi} \qquad (2)$$

where we have used (1).

1.58 The CM of the two H atoms will be at G the midpoint joining the atoms, Fig. 1.33. The bisector of $\widehat{\mathrm{HOH}}$

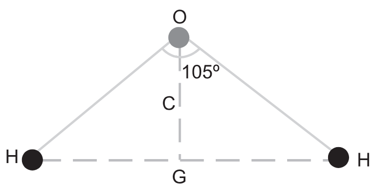


Fig. 1.33

$$\mathrm{OG} = (\mathrm{OH})\cos\left(\frac{105°}{2}\right) = 1.77 \times 0.06088 = 1.0775\ \text{Å}$$

Let the CM of the O atom and the two H atoms be located at C at distance y_{CM} from O on the bisector of angle $\mathrm{H\hat{O}H}$

$$y_{\mathrm{CM}} = \frac{2M_{\mathrm{H}}}{M_0} \times OG = \frac{2\times 1}{16} \times 1.0775 = 0.1349\ \text{Å}$$

1.59 The CM coordinates of three individual laminas are

$$\mathrm{CM}(1) = \left(\frac{a}{2}, \frac{a}{2}\right), \mathrm{CM}(2) = \left(\frac{3a}{2}, \frac{a}{2}\right), \mathrm{CM}(3) = \left(\frac{3a}{2}, \frac{3a}{2}\right)$$

The CM coordinates of the system of these three laminas will be

$$x_{\mathrm{CM}} = \frac{m\frac{a}{2} + m\frac{3a}{2} + m\frac{3a}{2}}{m+m+m} = \frac{7a}{6} \quad y_{\mathrm{CM}} = \frac{m\frac{a}{2} + m\frac{a}{2} + m\frac{3a}{2}}{m+m+m} = \frac{5a}{6}$$

1.3.6 Equilibrium

1.60 $U(x) = k(2x^3 - 5x^2 + 4x)$ (1)

$$\frac{dU(x)}{dx} = k(6x^2 - 10x + 4) \quad (2)$$

$$\therefore \quad \frac{dU(x)}{dx}\Big|_{x=1} = k(6x^2 - 10x + 4)\Big|_{x=1} = 0$$

which is the condition for maximum or minimum. For stable equilibrium position of the particle it should be a minimum. To this end we differentiate (2) again:

$$\frac{d^2U(x)}{dx^2} = k(12x - 10)$$

$$\therefore \quad \frac{d^2U(x)}{dx^2}\Big|_{x=1} = +2k$$

This is positive because k is positive, and so it is minimum corresponding to a stable equilibrium.

1.61 $U(x) = k(x^2 - 4xl)$ (1)

$$\frac{dU(x)}{dx} = 2k(x - 2l) \quad (2)$$

At $x = 2l$, $\frac{dU(x)}{dx} = 0$ (3)

Differentiating (2) again

$$\frac{d^2U}{dx^2} = 2k$$

which is positive. Hence it is a minimum corresponding to a stable equilibrium. Force

$$F = -\frac{dU}{dx} = -2k(x - 2l)$$

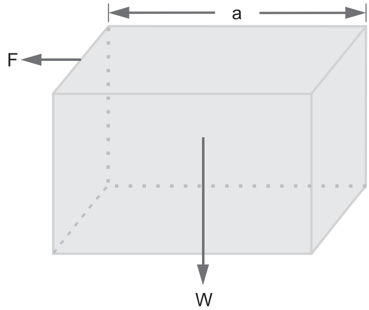
Put $X = x - 2l$, $\ddot{X} = \ddot{x}$

acceleration $\ddot{X} = \frac{F}{m} = -\frac{2k}{m}X = -\omega^2 X$

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$$

1.62 Let 'a' be the side of the cube and a force F be applied on the top surface of the cube, Fig. 1.34. Take torques about the left-hand side of the edge. The condition that the cube would topple is

Fig. 1.34



Counterclockwise torque > clockwise torque

$$Fa > W\frac{a}{2}$$

or

$$F > 0.5W \quad (1)$$

Condition for sliding is

$$F > \mu W \quad (2)$$

Comparing (1) and (2), we conclude that the cube will topple if $\mu > 0.5$ and will slide if $\mu < 0.5$.

1.63 In Fig. 1.35 let the ladder AB have length L, its weight mg acting at G, the CM of the ladder (middle point). The weight mg produces a clockwise torque τ_1 about B:

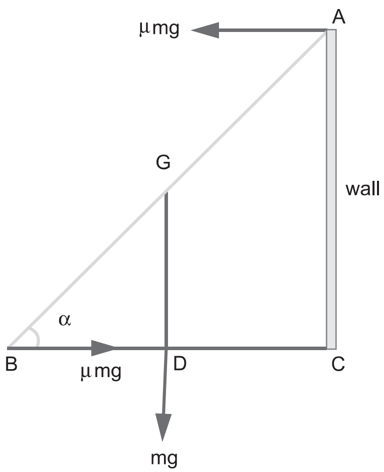


Fig. 1.35

$$\tau_1 = (mg)(\mathrm{BD}) = (mg)\left(\frac{\mathrm{BD}}{\mathrm{BG}}\mathrm{BG}\right) = mg\frac{L}{2}\cos\alpha \tag{1}$$

The friction with the ground, which acts toward right produces a counterclockwise torque τ_2:

$$\tau_2 = (\mu\, mg)\mathrm{AC} = \mu mg\frac{\mathrm{AC}}{\mathrm{AB}}\mathrm{AB} = \mu mgL\sin\alpha \tag{2}$$

For limiting equilibrium $\tau_1 = \tau_2$

$$\therefore \quad mg\frac{L}{2}\cos\alpha = \mu mgL\sin\alpha$$

$$\therefore \quad \mu = \frac{1}{2}\cot\alpha$$