Chapter 5
Spherical Varieties

Although the theory developed in the previous chapters applies to arbitrary homogeneous spaces of reductive groups, and even to more general group actions, it acquires its most complete and elegant form for spherical homogeneous spaces and their equivariant embeddings, called spherical varieties. A justification of the fact that spherical homogeneous spaces are a significant mathematical object is that they arise naturally in various fields, such as embedding theory, representation theory, symplectic geometry, etc. In §25 we collect various characterizations of spherical spaces, the most important being: the existence of an open B-orbit, the "multiplicity-free" property for spaces of global sections of line bundles, commutativity of invariant differential operators and of invariant functions on the cotangent bundle with respect to the Poisson bracket.

Then we examine the most interesting classes of spherical homogeneous spaces and spherical varieties in more detail. Algebraic symmetric spaces are considered in §26. We develop the structure theory and classification of symmetric spaces, compute the colored data required for the description of their equivariant embeddings, and study B-orbits and (co)isotropy representation. §27 is devoted to $(G \times G)$-equivariant embeddings of a reductive group G. A particular interest in this class is explained, for example, by an observation that linear algebraic monoids are nothing else but affine equivariant group embeddings. Horospherical varieties of complexity 0 are classified and studied in §28.

The geometric structure of toroidal varieties, considered in §29, is the best understood among all spherical varieties, since toroidal varieties are "locally toric". They can be defined by several equivalent properties: their fans are "colorless", they are spherical and pseudo-free, and the action sheaf on a toroidal variety is the log-tangent sheaf with respect to a G-stable divisor with normal crossings. An important property of toroidal varieties is that they are rigid as G-varieties. The so-called wonderful varieties are the most remarkable subclass of toroidal varieties. They are canonical completions with nice geometric properties of (certain) spherical homogeneous spaces. The theory of wonderful varieties is developed in §30. Applications include computation of the canonical divisor of a spherical variety and

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Luna's conceptual approach to the classification of spherical subgroups through the classification of wonderful varieties.

The concluding §31 is devoted to Frobenius splitting, a technique for proving geometric and algebraic properties (normality, rationality of singularities, cohomology vanishing, etc) in positive characteristic. However, this technique can be applied to zero characteristic using reduction $\mod p$ provided that reduced varieties are Frobenius split. This works for spherical varieties. As a consequence, one obtains the vanishing of higher cohomology of ample or numerically effective line bundles on complete spherical varieties, normality and rationality of singularities for G-stable subvarieties, etc. Some of these results can be proved by other methods, but Frobenius splitting provides a simple uniform approach.

25 Various Characterizations of Sphericity

25.1 Spherical Spaces. Spherical homogeneous spaces can be considered from diverse viewpoints: orbits and equivariant embeddings, representation theory and multiplicities, symplectic geometry, harmonic analysis, etc. The definition and some other implicit characterizations of this remarkable class of homogeneous spaces are already scattered in the text above. In this section, we review these issues and introduce other important properties of homogeneous spaces which are equivalent or closely related to sphericity.

As usual, G is a connected reductive group, O denotes a homogeneous G-space with the base point o, and $H = G_o$.

Definition–Theorem. *A* spherical homogeneous space O *(resp. a* spherical subgroup $H \subseteq G$, *a* spherical subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, *a* spherical pair (G,H) *or* $(\mathfrak{g},\mathfrak{h})$*) can be defined by any one of the following equivalent properties:*

(S1) $\Bbbk(O)^B = \Bbbk$.
(S2) *B has an open orbit in O.*
(S3) *H has an open orbit in G/B.*
(S4) $(\operatorname{char}\Bbbk = 0)$ $\exists g \in G:\ \mathfrak{b} + (\operatorname{Ad} g)\mathfrak{h} = \mathfrak{g}$.
(S5) $(\operatorname{char}\Bbbk = 0)$ *There exists a Borel subalgebra* $\widetilde{\mathfrak{b}} \subseteq \mathfrak{g}$ *such that* $\mathfrak{h} + \widetilde{\mathfrak{b}} = \mathfrak{g}$.
(S6) *H acts on G/B with finitely many orbits.*
(S7) *For any G-variety X and any $x \in X^H$, $\overline{Gx}$ contains finitely many G-orbits.*
(S8) *For any G-variety X and any $x \in X^H$, $\overline{Gx}$ contains finitely many B-orbits.*

The term "spherical homogeneous space" is traced back to Brion, Luna, and Vust [BLV], and "spherical subgroup" to Krämer [Krä], though the notions themselves appeared much earlier.

Proof. (S1) $\Longleftrightarrow$ (S2) B-invariant functions separate general B-orbits [PV, 2.3].
(S2) $\Longleftrightarrow$ (S3) Both conditions are equivalent to requiring that $B \times H : G$ has an open orbit, where B acts by left and H by right translations, or vice versa.

(S4) and (S5) are just reformulations of (S2) and (S3) in terms of tangent spaces. Note that in positive characteristic (S4) and (S5) are stronger than (S2) and (S3), because the orbit map onto the open B- or H-orbit may be inseparable.
(S2) $\Longrightarrow$ (S8) Gx satisfies (S2), too, and we conclude by Corollary 6.5.
(S8) $\Longrightarrow$ (S7) Obvious.
(S7) $\Longrightarrow$ (S2) Stems from Corollary 6.10.
(S8) $\Longrightarrow$ (S6) B acts on G/H with finitely many orbits, which are in bijection with $(B\times H)$-orbits on G and with H-orbits on G/B.
(S6) $\Longrightarrow$ (S3) Obvious. □

In particular, spherical spaces are characterized in the framework of embedding theory as those having finitely many orbits in the boundary of any equivariant embedding. The embedding theory of spherical spaces is considered in §15.

25.2 "Multiplicity-free" Property. Another important characterization of spherical spaces is in terms of representation theory, due to Kimelfeld and Vinberg [VK]. Recall from 2.6 that the multiplicity of a highest weight λ in a G-module M is

$$m_\lambda(M) = \dim \operatorname{Hom}_G(V(\lambda), M) = \dim M_\lambda^{(B)}.$$

In characteristic zero, $m_\lambda(M)$ is the multiplicity of the simple G-module $V(\lambda)$ in the decomposition of M. In positive characteristic, $V(\lambda)$ denotes the respective Weyl module. The module M is said to be multiplicity-free if all multiplicities in M are ≤ 1.

Theorem 25.1. *O is spherical if and only if the following equivalent conditions hold:*

(MF1) $\mathbb{P}(V^*(\lambda))^H$ *is finite for all* $\lambda \in \mathfrak{X}_+$.
(MF2) $\forall \lambda \in \mathfrak{X}_+,\ \chi \in \mathfrak{X}(H):\ \dim V^*(\lambda)_\chi^{(H)} \leq 1$.
(MF3) *For any G-line bundle $\mathscr{L}$ on O,* $\mathrm{H}^0(O,\mathscr{L})$ *is multiplicity-free.*

If O is quasiaffine, then the last two conditions can be weakened to

(MF4) $\forall \lambda \in \mathfrak{X}_+:\ \dim V^*(\lambda)^H \leq 1$.
(MF5) $\Bbbk[O]$ *is multiplicity-free.*

The spaces satisfying these conditions are called *multiplicity-free*.

Proof. (S1) $\Longleftrightarrow$ (MF3) If $m_\lambda(\mathscr{L}) \geq 2$, then there exist two non-proportional sections $\sigma_0, \sigma_1 \in \mathrm{H}^0(O,\mathscr{L})_\lambda^{(B)}$. Their ratio $f = \sigma_1/\sigma_0$ is a non-constant B-invariant function. Conversely, any $f \in \Bbbk(O)^B$ can be represented in this way: the G-line bundle $\mathscr{L}$ together with the canonical B-eigensection σ_0 is defined by a sufficiently big multiple of $\operatorname{div}_\infty f$ (cf. Corollary C.6).

Finally, if O is quasiaffine, then we may take for $\mathscr{L}$ the trivial bundle: for σ_0 take a sufficiently big power of any B-eigenfunction in $\mathscr{I}(D) \lhd \Bbbk[O]$, where $D \subset O$ is the support of $\operatorname{div}_\infty f$. Hence (S1) $\Longleftrightarrow$ (MF5).
(MF1) $\Longleftrightarrow$ (MF2) Stems from $\mathbb{P}(V^*(\lambda))^H = \mathbb{P}\left(V^*(\lambda)^{(H)}\right) = \bigsqcup_\chi \mathbb{P}\left(V^*(\lambda)_\chi^{(H)}\right)$ (a finite disjoint union).

(MF2) $\Longleftrightarrow$ (MF3) If $O = G/H$ is a quotient space, then this is the Frobenius reciprocity (2.2). Generally, there is a bijective purely inseparable morphism $G/H \to O$ (Remark 1.4), and O is spherical if and only if G/H is so. But we have already seen that the sphericity is equivalent to (MF3).
(MF4) $\Longleftrightarrow$ (MF5) is proved in the same way. □

25.3 Weakly Symmetric Spaces and Gelfand Pairs. The "multiplicity-free" property leads to an interpretation of sphericity in terms of automorphisms and group algebras associated with G. Since the complete reducibility of rational representations is essential here, we assume that $\operatorname{char}\Bbbk = 0$ up to the end of this section.

Recall from 2.5 the algebraic versions of the group algebra $\mathscr{A}(G)$ and the Hecke algebra $\mathscr{A}(O)$.

Theorem 25.2 ([AV], [Vin3]). *An affine homogeneous space O is spherical if and only if any of the following four equivalent conditions is satisfied:*

(GP1) $\mathscr{A}(O) = \mathscr{A}(G)^{H\times H}$ *is commutative.*
(GP2) $\mathscr{A}(V)^{H\times H}$ *is commutative for all G-modules V.*
(WS1) (Selberg condition) *The G-action on O extends to a cyclic extension $\widehat{G} = \langle G, s\rangle$ of G so that (sx, sy) is G-equivalent to (y,x) for general $x, y \in O$.*
(WS2) (Gelfand condition) *There exists $\theta \in \operatorname{Aut} G$ such that $\theta(H) = H$ and $\theta(g) \in Hg^{-1}H$ for general $g \in G$.*

The condition (GP1) is an algebraization of a similar commutativity condition for the group algebra of a Lie group, see [Gel], [Vin3, I.2], and 25.5. The condition (WS2) appeared in [Gel], and (WS1) was first introduced by Selberg in the seminal paper on the trace formula [Sel], and by Akhiezer and Vinberg [AV] in the context of algebraic geometry. The spaces satisfying (WS1)–(WS2) are called *weakly symmetric* and (G,H) is said to be a *Gelfand pair* if (GP1)–(GP2) hold.

Proof. (MF5) $\Longleftrightarrow$ (GP1) Stems from Schur's lemma.
(GP1) $\Longleftrightarrow$ (GP2) Obvious.
(MF5) $\Longrightarrow$ (WS1) There exists a Weyl involution $\theta \in \operatorname{Aut} G$, $\theta(H) = H$ [AV]. There is a conceptual argument for symmetric spaces and in general a case-by-case verification using the classification from 10.2. Define $s \in \operatorname{Aut} O$ by $s(go) = \theta(g)o$ and $\widehat{G} = G \rightthreetimes \langle s\rangle$ by $sgs^{-1} = \theta(g)$. The G-action on $O \times O$ is extended to $\widehat{G}$ by $s(x,y) = (sy, sx)$.

Consider the $(G\times G)$-isotypic decomposition

$$\Bbbk[O\times O] = \bigoplus_{\lambda,\mu\in\Lambda_+(O)} \Bbbk[O\times O]_{(\lambda,\mu)},$$

$$\text{where}\quad \Bbbk[O\times O]_{(\lambda,\mu)} = \Bbbk[O]_{(\lambda)} \otimes \Bbbk[O]_{(\mu)} \simeq V(\lambda)\otimes V(\mu).$$

Clearly, s twists the G-action by θ, and hence maps $\Bbbk[O\times O]_{(\lambda,\mu)}$ to $\Bbbk[O\times O]_{(\mu^*,\lambda^*)}$ and preserves each summand of

$$\Bbbk[O\times O]^G = \bigoplus_{\lambda\in\Lambda_+(O)} \Bbbk[O\times O]^G_{(\lambda,\lambda^*)}.$$

A $\widehat{G}$-invariant inner product $(f_1, f_2) \mapsto (f_1 f_2)^\natural$ on $\Bbbk[O]$ is non-degenerate. (Otherwise its kernel would be a non-trivial $\widehat{G}$-stable ideal in $\Bbbk[O]$.) Hence it induces a nonzero pairing between simple G-modules $\Bbbk[O]_{(\lambda^*)}$ and $\Bbbk[O]_{(\lambda)}$, whence by duality a $\widehat{G}$-invariant function in $\Bbbk[O \times O]_{(\lambda,\lambda^*)}$, which spans $\Bbbk[O \times O]^G_{(\lambda,\lambda^*)}$. It follows that s acts trivially on $\Bbbk[O \times O]^G$.

But general G-orbits in $O \times O$ are closed (Theorem 8.27), whence s preserves general orbits, which is exactly the Selberg condition.
(WS1) $\Longrightarrow$ (WS2) Multiplying s by $g \in G$ preserves the Selberg condition. Also, if $(sx, sy) \sim (y, x)$, then the same is true for any G-equivalent pair. Hence, without loss of generality, $so = o = x$. Define $\theta \in \operatorname{Aut} G$ by $\theta(g) = sgs^{-1}$; then $(so, sgo) = (o, \theta(g)o) \sim (go, o)$ for general $g \in G$. Hence $g'go = o$, $g'o = \theta(g)o$, i.e., $g'g = h \in H$, $\theta(g) = g'h' = hg^{-1}h'$ for some $h' \in H$.
(WS2) $\Longrightarrow$ (WS1) Since $\theta(H) = H$, there is a well-defined automorphism $s \in \operatorname{Aut} O$, $s(go) = \theta(g)o$. Put $\widehat{G} = G \leftthreetimes \langle s \rangle$, $sgs^{-1} = \theta(g)$. The Selberg condition is verified by reversing the previous arguments.
(WS2) $\Longrightarrow$ (GP1) Both θ and the inversion map $g \mapsto g^{-1}$ on G give rise to automorphisms of $\Bbbk[G]$ preserving $\Bbbk[G]^{H \times H}$, whose restrictions to $\Bbbk[G]^{H \times H}$ coincide. By the complete reducibility of $(H \times H)$-modules, $\mathscr{A}(G)^{H \times H} \simeq (\Bbbk[G]^{H \times H})^*$. Hence the antiautomorphism of $\mathscr{A}(G)^{H \times H}$ induced by the inversion coincides with the automorphism induced by θ. Therefore $\mathscr{A}(G)^{H \times H}$ is commutative. □

Remark 25.3. Already in the quasiaffine case the classes of weakly symmetric and spherical spaces are not contained in each other [Zor].

25.4 Commutativity. Now we characterize sphericity in terms of symplectic geometry.

Recall from 8.2 that the action $G : T^*O$ is Hamiltonian with respect to the natural symplectic structure. Thus we have a G-invariant Poisson bracket of functions on T^*O. Homogeneous functions on T^*O are locally the symbols of differential operators on O, and the Poisson bracket is induced by the commutator of differential operators.

The functions pulled back under the moment map $\Phi : T^*O \to \mathfrak{g}^*$ are called *collective*. They Poisson-commute with G-invariant functions on T^*O (Proposition 22.1).

Theorem 25.4. *O is spherical if and only if the following equivalent conditions hold:*

(WC1) *General orbits of $G : T^*O$ are coisotropic, i.e., $\mathfrak{g}\alpha \supseteq (\mathfrak{g}\alpha)^\angle$ for general $\alpha \in T^*O$.*
(WC2) *$\Bbbk(T^*O)^G$ is commutative with respect to the Poisson bracket.*
(CI) *There exists a complete system of collective functions in involution on T^*O.*

If O is affine, then these conditions are equivalent to

(Com) *$\mathscr{D}(O)^G$ is commutative.*

The theorem goes back to Guillemin, Sternberg [GS], and Mikityuk [Mik]. The spaces satisfying (WC1)–(WC2) are called *weakly commutative* and those satisfying (Com) are said to be *commutative*.

Proof. (S2) $\iff$ (WC1) By Theorem 8.17, $\operatorname{cork} T^*O = 2c(O)$ is zero if and only if O is spherical, and this means exactly that general orbits are coisotropic.
(WC1) $\iff$ (WC2) Skew gradients of $f \in \Bbbk(T^*O)^G$ at a point α of general position span $(\mathfrak{g}\alpha)^{\angle}$. All G-invariant functions Poisson-commute if and only if their skew gradients are skew-orthogonal to each other, i.e., if and only if $(\mathfrak{g}\alpha)^{\angle}$ is isotropic.
(GP1) $\iff$ (Com) If O is quasiaffine, then $\mathscr{D}(O)$ acts faithfully on $\Bbbk[O]$ by linear endomorphisms. Hence $\mathscr{D}(O)^G$ is a subalgebra in $\mathscr{A}(O)$. It remains to utilize the approximation of linear endomorphisms by differential operators.

Lemma 25.5. *Let X be a smooth affine G-variety.*

(1) *For any linear operator $\varphi : \Bbbk[X] \to \Bbbk[X]$ and any finite-dimensional subspace $M \subset \Bbbk[X]$ there exists $\partial \in \mathscr{D}(X)$ such that $\partial|_M = \varphi|_M$.*
(2) *If φ is G-equivariant, then one may assume that $\partial \in \mathscr{D}(X)^G$.*
(3) *Put $\mathscr{I} = \operatorname{Ann} M \lhd \mathscr{D}(X)$; then $\forall f \in \Bbbk[X] : \mathscr{I} f = 0 \implies f \in M$.*

We conclude by Lemma 25.5(2) that $\mathscr{A}(O)$ is commutative if and only if $\mathscr{D}(O)^G$ is so.

Proof of Lemma 25.5. (1) We deduce it from (3). Choose a basis $f_1, \dots, f_n$ of M. It suffices to construct $\partial \in \mathscr{D}(X)$ such that $\partial f_i = 0$, $\forall i < n$, $\partial f_n = 1$. By (3) there exists $\partial' \in \operatorname{Ann}(f_1, \dots, f_{n-1})$, $\partial' f_n \neq 0$. As $\Bbbk[X]$ is a simple $\mathscr{D}(X)$-module [MRo, 15.3.8] we may find $\partial'' \in \mathscr{D}(X)$, $\partial''(\partial' f_n) = 1$ and put $\partial = \partial''\partial'$.
(2) Without loss of generality, M is G-stable. Assertion (1) yields an epimorphism of G-$\Bbbk[X]$-modules $\mathscr{D}(X) \twoheadrightarrow \operatorname{Hom}(M, \Bbbk[X])$ given by restriction to M. But taking G-invariants is an exact functor.
(3) The assertion is trivial for $M = 0$ and we proceed by induction on $\dim M$. In the above notation, put $\mathscr{I}' = \operatorname{Ann}(f_1, \dots, f_{n-1})$. For any $\partial, \partial' \in \mathscr{I}'$ we have $(\partial f_n)\partial' - (\partial' f_n)\partial \in \mathscr{I}$, whence

$$(\partial f_n)(\partial' f) = (\partial' f_n)(\partial f). \tag{25.1}$$

Taking $\partial' = \xi\partial$, $\xi \in \mathrm{H}^0(X, \mathscr{T}_X)$, yields $\xi(\partial f / \partial f_n) = 0 \implies \partial f / \partial f_n = c_\partial = \text{const}$. Substituting this in (25.1) yields $c_{\partial'} = c_\partial = c$ (independent of ∂). Thus $\partial(f - c f_n) = 0 \implies f - c f_n \in \langle f_1, \dots, f_{n-1} \rangle \implies f \in M$. □

(Com) $\implies$ (WC2) If O is affine, then $\operatorname{gr} \mathscr{D}(O) = \Bbbk[T^*O]$. By complete reducibility, $\Bbbk[T^*O]^G = \operatorname{gr} \mathscr{D}(O)^G$ is Poisson-commutative. But general G-orbits in T^*O are closed (Remark 8.15), whence $\Bbbk(T^*O)^G = \operatorname{Quot} \Bbbk[T^*O]^G$ is Poisson-commutative as well.
(CI) $\iff$ (S2) This equivalence is due to Mikityuk [Mik] (for affine O).

A complete system of Poisson-commuting functions on M_O can be constructed by the method of argument shift [MF1]: choose a regular semisimple element $\xi \in \mathfrak{g}^*$ and consider the derivatives $\partial_\xi^n f$ of all $f \in \Bbbk[\mathfrak{g}^*]^G$. (Here ∂_ξ denotes the derivative in direction ξ.) The functions $\partial_\xi^n f$ Poisson-commute and produce a complete involutive system on $Gx \subset \mathfrak{g}^*$ for general ξ whenever $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$, where $\operatorname{ind}\mathfrak{h} = d_H(\mathfrak{h}^*)$ [Bol]. In the symplectically stable case, general points $x \in M_O$ are semisimple and $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$. Generally, the equality $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$ for all $x \in \mathfrak{g}^*$ was conjectured by Elashvili. It is easily reduced to the case of simple $\mathfrak{g}$ and nilpotent x. Elashvili's conjecture was proved by Yakimova for classical $\mathfrak{g}$ [Yak3] and verified by de Graaf for exceptional $\mathfrak{g}$ using computer calculations [Gra]. Recently a general proof (almost avoiding case-by-case considerations) was given by Charbonnel and Moreau [CM].

Since symplectic leaves of the Poisson structure on M_O are G-orbits, there are $(d_G(M_O) + \dim M_O)/2 = \dim O - c(O)$ independent Poisson-commuting collective functions. Thus a complete involutive system of collective functions exists if and only if $c(O) = 0$. □

Since $T^*O = G *_H \mathfrak{h}^\perp$, weak commutativity is readily reformulated in terms of the coadjoint representation [Mik], [Pan1], [Vin3, II.4.1].

Theorem 25.6. (G,H) *is a spherical pair if and only if general points* $\alpha \in \mathfrak{h}^\perp$ *satisfy any of the equivalent conditions:*

(Ad1) $\dim G\alpha = 2\dim H\alpha$.
(Ad2) *$H\alpha$ is a Lagrangian subvariety in $G\alpha$ with respect to the Kirillov form.*
(Ad3) (Richardson condition) $\mathfrak{g}\alpha \cap \mathfrak{h}^\perp = \mathfrak{h}\alpha$.

The Richardson condition means that $G\alpha \cap \mathfrak{h}^\perp$ is a finite union of open H-orbits [PV, 1.5].

Proof. (WC1) $\Longleftrightarrow$ (Ad1) Recall that the moment map $\Phi : G *_H \mathfrak{h}^\perp \to \mathfrak{g}^*$ is defined via replacing the $*$-action by the coadjoint action (Example 8.1). We have

$$d_G(T^*O) = \dim O - \dim H\alpha \qquad \text{and}$$
$$\operatorname{def} T^*O = \dim G_{\Phi(e*\alpha)}/G_{e*\alpha} = \dim G_\alpha/H_\alpha.$$

Hence

$$\operatorname{cork} T^*O = d_G(T^*O) - \operatorname{def} T^*O = \dim G\alpha - 2\dim H\alpha.$$

(Ad1) $\Longleftrightarrow$ (Ad2) The Kirillov form vanishes on $\mathfrak{h}\alpha$.
(Ad2) $\Longleftrightarrow$ (Ad3) Stems from $(\mathfrak{g}\alpha) \cap \mathfrak{h}^\perp = (\mathfrak{h}\alpha)^\angle$, the skew-orthocomplement with respect to the Kirillov form. □

Invariant functions on cotangent bundles of spherical homogeneous spaces have a nice structure.

Proposition 25.7 ([Kn1, 7.2]). *If $O = G/H$ is spherical, then $\Bbbk[T^*O]^G \simeq \Bbbk[\widetilde{L}_O] \simeq \Bbbk[\mathfrak{a}^*]^{W_O}$ is a polynomial algebra; there are similar isomorphisms for fields of rational functions.*

Proof. By Proposition 22.1 and (WC2), $\Bbbk(T^*O)^G \simeq \Bbbk(\widetilde{L}_O) \simeq \Bbbk(\mathfrak{a}^*)^{W_O}$. By Lemma 22.4, $\widetilde{\pi}_G\widetilde{\Phi} : T^*O \to \widetilde{L}_O$ is a surjective morphism of normal varieties. Therefore any $f \in \Bbbk(T^*O)^G$ having poles on $\widetilde{L}_O$ must have poles on T^*O, whence $\Bbbk[T^*O]^G = \Bbbk[\widetilde{L}_O]$. The latter algebra is polynomial for W_O is generated by reflections (Theorem 22.13). □

In other words, invariants of the coisotropy representation form a polynomial algebra $\Bbbk[\mathfrak{h}^\perp]^H \simeq \Bbbk[\mathfrak{a}^*]^{W_{G/H}}$ for any spherical pair (G,H).

Remark 25.8. A similar assertion in the non-commutative setup was proved in [Kn6]. Namely, all invariant differential operators on a spherical space O are completely regular, whence $\mathscr{D}(O)^G$ is a polynomial ring isomorphic to $\Bbbk[\rho + \mathfrak{a}^*]^{W_O}$ (see Remark 22.8). In particular, every spherical homogeneous space is commutative.

Example 25.9 ([BPa, Ex. 4.3.3]). Let $G = \mathrm{Sp}_4$, $H = \Bbbk^\times \times \mathrm{Sp}_2 \subset \mathrm{Sp}_2 \times \mathrm{Sp}_2 \subset G$. The coisotropy representation of H is $\mathfrak{h}^\perp = \Bbbk^1 \oplus (\Bbbk^1)^* \oplus \Bbbk^2 \oplus (\Bbbk^2)^*$, where $\Bbbk^1, \Bbbk^2$ are the trivial and the tautological Sp_2-module acted on by $\Bbbk^\times$ via the characters $2\varepsilon, \varepsilon$, respectively, where $\mathfrak{X}(\Bbbk^\times) = \langle \varepsilon \rangle$. In the notation of Theorem 9.1, $H_* = Z(G)$, whence $A = \mathrm{Ad}_G T$. The algebra $\Bbbk[\widetilde{L}_O] \simeq \Bbbk[\mathfrak{h}^\perp]^H$ is freely generated by two quadratic invariants ab, $\langle x, y \rangle$, where $(a,b,x,y) \in \mathfrak{h}^\perp$. Hence $W_O \simeq \mathbf{Z}_2^2$ is a subgroup of W generated by the reflections along two orthogonal roots. This example shows that generally $W_X \neq W(\mathfrak{a}^*)$.

25.5 Generalizations. In our considerations G was always assumed to be reductive. However some of the concepts introduced above are reasonable even for non-reductive G assuming H be reductive instead. Some of the above results remain valid:

(1) If $O = G/H$ is weakly symmetric, then (G,H) is a Gelfand pair.
(2) O is commutative if and only if (G,H) is a Gelfand pair.
(3) A commutative space O is weakly commutative provided that $\Bbbk[\mathfrak{h}^\perp]^H$ separates general H-orbits in $\mathfrak{h}^\perp$.

The above proofs work in this case: O is affine, the functor $(\cdot)^G$ is exact on global sections of G-sheaves on O since $(\cdot)^H$ is exact on rational H-modules, and orbit separation in (3) guarantees $\Bbbk(T^*O)^G = \operatorname{Quot}\Bbbk[T^*O]^G$. The converse implication in (1) fails, the simplest counterexample being:

Example 25.10 ([Lau]). Put $H = \mathrm{Sp}_{2n}(\Bbbk)$, $G = H \leftthreetimes N$, where $N = \exp \mathfrak{n}$ is a unipotent group associated with the Heisenberg type Lie algebra $\mathfrak{n} = (\Bbbk^{2n} \oplus \Bbbk^{2n}) \oplus \Bbbk^3$, the commutator in $\mathfrak{n}$ being defined by the identification $\bigwedge^2(\Bbbk^{2n} \oplus \Bbbk^{2n})^{\mathrm{Sp}_{2n}(\Bbbk)} \simeq \Bbbk^3 = \mathfrak{z}(\mathfrak{n})$. Then (G,H) is a Gelfand pair, but O is not weakly symmetric.

Also, the implication (3) fails if the orbit separation is violated. The reason is that there may be too few invariant differential operators. For instance, in the previous example, replace H by $\Bbbk^\times$ acting on $\Bbbk^{2n}$ via a character $\chi \neq 0$ and on $\Bbbk^3$ via 2χ. Then O is not weakly commutative while $\mathscr{D}(O)^G = \Bbbk$.

The classes of weakly symmetric and (weakly) commutative homogeneous spaces were first introduced and examined in Riemannian geometry and harmonic analysis, see the survey [Vin3]. We shall review the analytic viewpoint now.

Quitting a somewhat restrictive framework of algebraic varieties, one may consider the above properties of homogeneous spaces in the category of Lie group actions, making appropriate modifications in formulations. For instance, instead of regular or rational functions one considers arbitrary analytic or differentiable functions. Some of these properties receive a new interpretation in terms of differential geometry, e.g., (CI) means that invariant Hamiltonian dynamic systems on T^*O are completely integrable in the class of Noether integrals [MF2], [Mik].

The situation where H is a compact subgroup of a real Lie group G, i.e., $O = G/H$ is a Riemannian homogeneous space, has attracted the main attention of researchers. Most of the above results were originally obtained in this setting.

The properties (MF4), (MF5) are naturally reformulated here in the category of unitary representations of G replacing $\Bbbk[O]$ by $L^2(O)$. In (GP1) one considers the algebra $\mathscr{A}(G)$ of complex measures with compact support on G. The conditions (WS1), (WS2) are formulated for *all* (not only general) points (which is equivalent for compact H); there is also an infinitesimal characterization of weak symmetry [Vin3, I.1.2].

There are the following implications:

$$\begin{array}{c}\text{weakly symmetric space}\\ \Downarrow\\ \text{Gelfand pair}\\ \Updownarrow\\ \text{multiplicity-free space}\\ \Updownarrow\\ \text{commutative space}\\ \Updownarrow\\ \text{weakly commutative space}\end{array}$$

The implication (WS2) $\Longrightarrow$ (GP1) is due to Gelfand [Gel] and (GP1) $\Longleftrightarrow$ (MF5) was proved in [BGGN]. The equivalence (GP1) $\Longleftrightarrow$ (Com) is due to Helgason [Hel2, Ch. IV, B13] and Thomas [Tho], for a proof see [Vin3, I.2.5]. The implication (Com) $\Longrightarrow$ (WC2) is easy [Vin3, I.4.2] and the converse was proved by Rybnikov [Ryb].

A classification of commutative Riemannian homogeneous spaces was obtained by Yakimova [Yak1], [Yak2] using partial results of Vinberg [Vin4] and the classification of affine spherical spaces from 10.2.

An algebraic homogeneous space $O = G/H$ over $\Bbbk = \mathbb{C}$ may be considered as a homogeneous manifold in the category of complex or real Lie group actions. At the same time, if (G,H) is defined over $\mathbb{R}$, then O has a real form $O(\mathbb{R})$ containing $G(\mathbb{R})/H(\mathbb{R})$ as an open orbit (in classical topology). Thus G/H may be regarded as the complexification of $G(\mathbb{R})/H(\mathbb{R})$, a homogeneous space of a real Lie group $G(\mathbb{R})$.

It is easy to see that G/H is commutative (resp. weakly commutative, multiplicity-free, weakly symmetric, satisfies (GP1), (CI)) if and only if $G(\mathbb{R})/H(\mathbb{R})$ is so. In other words, the above listed properties are stable under complexification and passing to a real form.

This observation leads to the following criterion of sphericity, which is a "real form" of Theorem 25.4.

By Chevalley's theorem, there exists a projective embedding $O \subseteq \mathbb{P}(V)$ for some G-module V. Assume that G is reductive and $K \subset G$ is a compact real form. Then V can be endowed with a K-invariant Hermitian inner product $(\cdot|\cdot)$, which induces a Kählerian metric on $\mathbb{P}(V)$ and on O (the *Fubini–Study metric*). The imaginary part of this metric is a real symplectic form. The action $K : \mathbb{P}(V)$ is Hamiltonian, the moment map $\Phi : \mathbb{P}(V) \to \mathfrak{k}^*$ being defined by the formula

$$\langle \Phi([v]), \xi \rangle = \frac{1}{2\mathrm{i}} \cdot \frac{(\xi v|v)}{(v|v)}, \qquad \forall v \in V,\ \xi \in \mathfrak{k}.$$

Theorem 25.11 ([Bri3], [HW], [Akh4, §13]). *O is spherical if and only if general K-orbits in O are coisotropic with respect to the Fubini–Study form or, equivalently, the algebra $C^\infty(O)^K$ of smooth K-invariant functions on O is Poisson-commutative.*

Proof. First note that general K-orbits in O are coisotropic if and only if

$$d_K(O) = \operatorname{def} O = \operatorname{rk} K - \operatorname{rk} K_*, \tag{25.2}$$

where K_* is the stabilizer of general position for $K : O$ [Vin3, II.3.1, 2.6]. The condition (25.2) does not depend on the symplectic structure.

If O is affine, then the assertion can be directly reduced to Theorem 25.4 by complexification. Without loss of generality $K \cap H$ is a compact real form of H. Using the Cartan decompositions $G = K \cdot \exp \mathrm{i}\mathfrak{k}$, $H = (K \cap H) \cdot \exp \mathrm{i}(\mathfrak{k} \cap \mathfrak{h})$, one obtains a K-diffeomorphism

$$O \simeq K *_{K \cap H} \mathrm{i}\mathfrak{k}/\mathrm{i}(\mathfrak{k} \cap \mathfrak{h}) \simeq T^*(K/K \cap H)$$

(see, e.g., [Kob, 2.7]). Complexifying the r.h.s. we obtain T^*O.

In the general case, it is more convenient to apply the theory of doubled actions (see 8.8).

There exists a Weyl involution θ of G commuting with the Hermitian conjugation $g \mapsto g^*$. The mapping $g \mapsto \overline{g} := \theta(g^*)^{-1}$ is a complex conjugation on G defining a split real form $G(\mathbb{R})$. There exists a $G(\mathbb{R})$-stable real form $V(\mathbb{R}) \subset V$ such that $(\cdot|\cdot)$ takes real values on $V(\mathbb{R})$. The complex conjugation on V, $\mathbb{P}(V)$, or G is defined by conjugating the coordinates or matrix entries with respect to an orthonormal basis in $V(\mathbb{R})$.

It follows that the complex conjugate variety $\overline{O}$ is naturally embedded in $\mathbb{P}(V)$ as a G-orbit. Complexifying the action $K : O$ we obtain the diagonal action $G : O \times \overline{O}$, $g(x,\overline{y}) = (gx, \theta(g)\overline{y})$, $\forall g \in G,\ x,y \in O$. This action differs slightly from the doubled action, but Theorems 8.24–8.25 remain valid, together with the proofs. Now it follows from (8.3)–(8.4) that O is spherical if and only if

$$d_G(O \times \overline{O}) = \operatorname{rk} G - \operatorname{rk} G_*,$$

where $G_* = K_*(\mathbb{C})$ is the stabilizer of general position for $G : O \times \overline{O}$. The latter condition coincides with (25.2). □

26 Symmetric Spaces

26.1 Algebraic Symmetric Spaces. The concept of a Riemannian symmetric space was introduced by É. Cartan [Car1], [Car2]. A (globally) symmetric space is defined as a connected Riemannian manifold O such that for any $x \in O$ there exists an isometry s_x of O inverting the geodesics passing through x. Symmetric spaces form a very important class of Riemannian spaces including all classical geometries. The theory of Riemannian symmetric spaces is well developed, see [Hel1].

In particular, it is easy to see that a symmetric space O is homogeneous with respect to the unity component G of the full isometry group, so that $O = G/H$, where $H = G_o$ is the stabilizer of a fixed base point. The geodesic symmetry $s = s_o$ is an involutive automorphism of O normalizing G. It defines an involution $\theta \in \operatorname{Aut} G$ by $\theta(g) = sgs^{-1}$. From the definition of a geodesic symmetry one deduces that $(G^\theta)^0 \subseteq H \subseteq G^\theta$. Furthermore, reducing G to a smaller transitive isometry group if necessary, one may assume that $\mathfrak{g}$ is a reductive Lie algebra. This leads to the following algebraic definition of a symmetric space, which we accept in our treatment.

Definition 26.1. An (algebraic) *symmetric space* is a homogeneous algebraic variety $O = G/H$, where G is a connected reductive group equipped with a non-identical involution $\theta \in \operatorname{Aut} G$, and $(G^\theta)^0 \subseteq H \subseteq G^\theta$.

Riemannian symmetric spaces are locally isomorphic to real forms (with compact isotropy subgroups) of algebraic symmetric spaces over $\mathbb{C}$.

It is reasonable to impose a restriction $\operatorname{char} \Bbbk \neq 2$ on the ground field.

Remark 26.2. If G is semisimple simply connected, then G^θ is connected [St, 8.2], whence $H = G^\theta$. On the other hand, if G is adjoint, then $G^\theta = N_G(H)$ [Vu2, 2.2].

The differential of θ, denoted by the same letter by abuse of notation, induces a $\mathbf{Z}_2$-grading

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{26.1}$$

where $\mathfrak{h}, \mathfrak{m}$ are the (± 1)-eigenspaces of θ.

The subgroup H is reductive [St, §8], and hence O is an affine algebraic variety. More specifically, consider a morphism $\tau : G \to G$, $\tau(g) = \theta(g)g^{-1}$. Observe that τ is the orbit map at e for the G-action on G by *twisted conjugation*: $g \circ x = \theta(g)xg^{-1}$. It is not hard to prove the following result.

Proposition 26.3 ([Sp1, 2.2]). *$\tau(G) \simeq G/G^\theta$ is a connected component of* $\{x \in G \mid \theta(x) = x^{-1}\}$.

Example 26.4. Let $G = \mathrm{GL}_n(\Bbbk)$ and let θ be defined by $\theta(x) = (x^\top)^{-1}$. Then $G^\theta = \mathrm{O}_n(\Bbbk)$ and $\tau(G) = \{x \in G \mid \theta(x) = x^{-1}\}$ is the set of non-degenerate symmetric matrices, which is isomorphic to $\mathrm{GL}_n(\Bbbk)/\mathrm{O}_n(\Bbbk)$.

However, if θ is an inner involution, i.e., the conjugation by a matrix of order 2, then the set of matrices x such that $\theta(x) = x^{-1}$ is disconnected. The connected components are determined by the collection of eigenvalues of x, which are ± 1.

26.2 θ-stable Tori. The local and global structure of symmetric spaces is examined in [KoR], [Hel1] (transcendental methods), [Vu1], [Vu2] ($\operatorname{char}\Bbbk = 0$), [Ri2], [Sp1]. We follow these sources in our exposition. The starting point is an analysis of θ-stable tori.

Lemma 26.5. *Every Borel subgroup $B \subseteq G$ contains a θ-stable maximal torus T.*

Proof. The group $B \cap \theta(B)$ is connected, solvable, and θ-stable. By [St, 7.6] it contains a θ-stable maximal torus T, which is a maximal torus in G, too. □

Corollary 26.6. *Every θ-stable torus $S \subseteq G$ is contained in a θ-stable maximal torus T.*

Proof. Take for T any θ-stable maximal torus in $Z_G(S)$. □

A θ-stable torus T decomposes into an almost direct product $T = T_0 \cdot T_1$, where $T_0 \subseteq H$ and T_1 is *θ-split*, which means that θ acts on T_1 as the inversion.

Let Δ denote the root system of G with respect to T and $\mathfrak{g}^\alpha \subset \mathfrak{g}$ the root subspace corresponding to $\alpha \in \Delta$. One may choose root vectors $e_\alpha \in \mathfrak{g}^\alpha$ in such a way that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ form an $\mathfrak{sl}_2$-triple for all $\alpha \in \Delta$. Clearly, θ acts on $\mathfrak{X}(T)$ leaving Δ stable. Choosing e_α in a compatible way allows us to subdivide all roots into *complex*, *real*, and *imaginary* (*compact* or *non-compact*) ones, according to Table 26.1.

Table 26.1 Root types with respect to an involution

α	complex	real	imaginary	
			compact	non-compact
$\theta(\alpha)$	$\neq \pm\alpha$	$-\alpha$	α	α
$\theta(e_\alpha)$	$e_{\theta(\alpha)}$	$e_{-\alpha}$	e_α	$-e_\alpha$

We fix an inner product on $\mathfrak{X}(T) \otimes \mathbb{Q}$ invariant under the Weyl group $W = N_G(T)/T$ and θ. Then $\mathfrak{X}(T) \otimes \mathbb{Q}$ is identified with $\mathfrak{X}^*(T) \otimes \mathbb{Q}$ and with the orthogonal sum of $\mathfrak{X}(T_0) \otimes \mathbb{Q}$ and $\mathfrak{X}(T_1) \otimes \mathbb{Q}$. The coroots $\alpha^\vee \in \Delta^\vee$ (for $\alpha \in \Delta$) are identified with $2\alpha/(\alpha,\alpha)$. Let $\langle \alpha | \beta \rangle = \langle \alpha^\vee, \beta \rangle = 2(\alpha,\beta)/(\alpha,\alpha)$ denote the Cartan pairing on $\mathfrak{X}(T)$ and let $r_\alpha(\beta) = \beta - \langle \alpha | \beta \rangle \alpha$ be the reflection of β along α.

Two opposite classes of θ-stable maximal tori are of particular importance in the theory of symmetric spaces.

26.3 Maximal θ-fixed Tori.

Lemma 26.7. *If* $\dim T_0$ *is maximal possible, then* T_0 *is a maximal torus in* H *and* $Z_G(T_0) = T$*. Moreover,* T *is contained in a* θ*-stable Borel subgroup* $B \subseteq G$ *such that* $(B^\theta)^0$ *is a Borel subgroup in* H*.*

Proof. If $Z_G(T_0) \neq T$, then the commutator subgroup $Z_G(T_0)'$ and $(Z_G(T_0)')^\theta$ have positive dimension. Hence T_0 can be extended by a subtorus in $(Z_G(T_0)')^\theta$, a contradiction. Now choose a Borel subgroup of H containing T_0 and extend it to a Borel subgroup B of G. Then $B \supseteq T$. If B were not θ-stable, then there would exist a root $\alpha \in \Delta_+$ such that $\theta(\alpha) \in \Delta_-$. Then $e_{\pm\alpha} + \theta(e_{\pm\alpha})$ are opposite root vectors in $\mathfrak{h}$ outside the Borel subalgebra $\mathfrak{b}^\theta$, a contradiction. □

In particular, if T_0 is maximal, then there are no real roots, and θ preserves the set Δ_+ of positive roots (with respect to B) and induces a diagram involution $\overline{\theta}$ of the set $\Pi \subseteq \Delta_+$ of simple roots. If G is of simply connected type, then $\overline{\theta}$ extends to an automorphism of G so that $\theta = \overline{\theta} \cdot \theta_0$, where θ_0 is an inner automorphism defined by an element of T_0.

Consider the set $\overline{\Delta} = \{\overline{\alpha} = \alpha|_{T_0} \mid \alpha \in \Delta\} \subset \mathfrak{X}(T_0)$. Clearly, $\overline{\Delta}$ consists of the roots of H with respect to T_0 and the nonzero weights of $T_0 : \mathfrak{m}$. The restrictions of complex roots belong to both subsets, the eigenvectors being $e_\alpha + \theta(e_\alpha) \in \mathfrak{h}$, $e_\alpha - \theta(e_\alpha) \in \mathfrak{m}$, whereas (non-)compact roots restrict to roots of H (resp. weights of $\mathfrak{m}$).

Lemma 26.8. $\overline{\Delta}$ *is a (possibly non-reduced) root system with base* $\overline{\Pi} = \{\overline{\alpha} \mid \alpha \in \Pi\}$*. The simple roots of* H *and the (nonzero) lowest weights of* $H : \mathfrak{m}$ *form an affine simple root system* $\widetilde{\Pi}$*, i.e.,* $\langle \overline{\alpha} | \overline{\beta} \rangle \in \mathbb{Z}_-$ *for distinct* $\overline{\alpha}, \overline{\beta} \in \widetilde{\Pi}$*.*

Proof. Note that the restriction of $\alpha \in \Delta$ to T_0 is the orthogonal projection to $\mathfrak{X}(T_0) \otimes \mathbb{Q}$, so that $\overline{\alpha} = (\alpha + \theta(\alpha))/2$. If α is complex, then $\langle \alpha | \theta(\alpha) \rangle = 0$ or -1 (otherwise $\alpha - \theta(\alpha)$ would be a real root), In the second case, $2\overline{\alpha} = \alpha + \theta(\alpha)$ is a non-compact root with a root vector $e_{\alpha+\theta(\alpha)} = [e_\alpha, \theta(e_\alpha)]$.

A direct computation shows that $\langle \overline{\alpha} | \overline{\beta} \rangle \in \mathbb{Z}$, $\forall \alpha, \beta \in \Delta$, and the reflections $r_{\overline{\alpha}}$ preserve $\overline{\Delta}$, see Table 26.2. Hence $\overline{\Delta}$ is a root system. The subset $\overline{\Pi}$ is linearly independent. (Otherwise there would be a linear dependence between $2\overline{\alpha} = \alpha + \theta(\alpha)$, where $\alpha, \theta(\alpha) \in \Pi$, i.e., between roots in Π.) Restricting to T_0 the expression of $\alpha \in \Delta$ as a linear combination of Π with integer coefficients of the same sign yields a similar expression of $\overline{\alpha}$ in terms of $\overline{\Pi}$. Thus $\overline{\Pi}$ is a base of $\overline{\Delta}$.

Table 26.2 Cartan numbers and reflections for restricted roots

Case	$\langle \overline{\alpha} \| \overline{\beta} \rangle$	$r_{\overline{\alpha}}(\overline{\beta})$
$\alpha = \theta(\alpha)$	$\langle \alpha \| \beta \rangle$	$\overline{r_\alpha(\beta)}$
$\langle \alpha \| \theta(\alpha) \rangle = 0$	$\langle \alpha \| \beta \rangle + \langle \theta(\alpha) \| \beta \rangle$	$\overline{r_\alpha r_{\theta(\alpha)}(\beta)}$
$\langle \alpha \| \theta(\alpha) \rangle = -1$	$2\langle \alpha \| \beta \rangle + 2\langle \theta(\alpha) \| \beta \rangle$	$r_{2\overline{\alpha}}(\overline{\beta}) = \overline{r_{\alpha+\theta(\alpha)}(\beta)}$

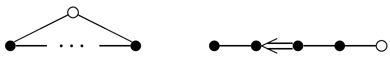
Note that $\overline{\alpha} = \overline{\beta}$ if and only if $\alpha = \beta$ or $\theta(\alpha) = \beta$. (Otherwise $\alpha - \beta$ or $\theta(\alpha) - \beta$ would be a real root, depending on whether $\langle\alpha|\beta\rangle > 0$ or $\langle\theta(\alpha)|\beta\rangle > 0$.) Therefore the nonzero weights occur in $\mathfrak{m}$ with multiplicity 1.

To prove the second assertion, it suffices to consider the Cartan numbers $\langle\overline{\alpha}|\overline{\beta}\rangle$ of lowest weights of $\mathfrak{m}$. The assumption $\langle\overline{\alpha}|\overline{\beta}\rangle > 0$ yields without loss of generality $\langle\alpha|\beta\rangle > 0$, whence $\gamma = \alpha - \beta \in \Delta$, $e_\alpha = [e_\beta, e_\gamma]$. If β is non-compact, then $[e_\beta, e_\gamma + \theta(e_\gamma)] = e_\alpha - \theta(e_\alpha)$. If β is complex, then γ is also complex. (This is clear if α is non-compact; otherwise $(\overline{\gamma},\overline{\gamma}) = \min\{(\overline{\alpha},\overline{\alpha}),(\overline{\beta},\overline{\beta})\} < \min\{(\alpha,\alpha),(\beta,\beta)\} = (\gamma,\gamma)$.) Then $\beta + \theta(\gamma), \theta(\beta) + \gamma \notin \Delta$, whence $[e_\beta - \theta(e_\beta), e_\gamma + \theta(e_\gamma)] = e_\alpha - \theta(e_\alpha)$. In both cases, either $\overline{\beta}$ or $\overline{\alpha}$ is not a lowest weight, a contradiction. □

Remark 26.9. If some of the Cartan numbers of Δ vanish in $\Bbbk$, then the previous argument concerning lowest weights does not work. The assertion on $\widetilde{\Pi}$ is true only if one interprets lowest weights in the combinatorial sense as those weights of $\mathfrak{m}$ which cannot be obtained from other weights by adding simple roots of H. However this happens only for $G = \mathbf{G}_2$ (if $\operatorname{char}\Bbbk = 3$), where the unique (up to conjugation) involution is easy to describe by hand.

In a usual way, the system $\widetilde{\Pi}$ together with the respective Cartan numbers is encoded by an (affine) Dynkin diagram. Marking the nodes corresponding to the simple roots of H by black, and those corresponding to the lowest weights of $\mathfrak{m}$ by white, one obtains the so-called *Kac diagram* of the involution θ, or of the symmetric space O. From the Kac diagram one easily recovers $\mathfrak{h}$ and (at least in characteristic zero) the (co)isotropy representation $H^0 : \mathfrak{m}$.

Example 26.10. Let H be diagonally embedded in $G = H \times H$, where θ permutes the factors. Here the Kac diagram is the affine Dynkin diagram of H with the white nodes corresponding to the lowest roots, e.g.:



26.4 Maximal θ-split Tori. Now consider an opposite class of θ-stable maximal tori.

Lemma 26.11. *There exist non-trivial θ-split tori.*

Proof. In the converse case θ acts identically on every θ-stable torus. Lemma 26.5 implies that all Borel subgroups are θ-stable. Then all maximal tori are θ-stable and even pointwise fixed, whence θ is identical. □

Lemma 26.12. *If T_1 is a maximal θ-split torus, then $L = Z_G(T_1)$ decomposes into an almost direct product $L = L_0 \cdot T_1$, where $L_0 = L \cap H$.*

Proof. Clearly, L and the commutator subgroup L' are θ-stable. If $L' \not\subseteq H$, then T_1 could be extended by a non-trivial θ-split torus in L' by Lemma 26.11, a contradiction. The assertion follows from $L' \subseteq H$. □

Corollary 26.13. *Every maximal torus $T \supseteq T_1$ is θ-stable.*

Choose a general one-parameter subgroup $\gamma \in \mathfrak{X}^*(T_1)$ and consider the associated parabolic subgroup $P = P(\gamma)$ with the Lie algebra $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\langle\alpha,\gamma\rangle \geq 0} \mathfrak{g}^\alpha$. Clearly, $L \subseteq P$ is a Levi subgroup and $\mathfrak{p}_u = \bigoplus_{\langle\alpha,\gamma\rangle > 0} \mathfrak{g}^\alpha$. Note that $\theta(P) = P^-$ (since $\langle\theta(\alpha),\gamma\rangle = -\langle\alpha,\gamma\rangle$, $\forall\alpha \in \Delta$). In fact, all minimal parabolics having this property are obtained as above [Vu1, 1.2]. It follows that $\mathfrak{h}$ is spanned by $\mathfrak{l}_0$ and $e_\alpha + \theta(e_\alpha)$ over all $\alpha \in \Delta$ such that $\langle\alpha,\gamma\rangle > 0$. This yields:

Iwasawa decomposition. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}_1 \oplus \mathfrak{p}_u$.

As a consequence, we obtain

Theorem 26.14. *Symmetric spaces are spherical.*

Indeed, choosing a Borel subgroup $B \subseteq P$, $B \supseteq T$ yields (S5). There are many other ways to verify this fact. For instance, it is easy to verify the Richardson condition (Ad3): for any $\xi \in \mathfrak{m} \simeq \mathfrak{h}^\perp$ one has $[\mathfrak{g},\xi] \cap \mathfrak{m} = [\mathfrak{h},\xi]$, because $[\mathfrak{m},\xi] \subseteq \mathfrak{h}$. One can also check the Gelfand condition (WS2) for elements in a dense subset $\tau(G)H \subseteq G$: $g = xh$, $x \in \tau(G)$, $h \in H \implies \theta(g) = x^{-1}h = hg^{-1}h$. The multiplicity-free property (for compact Riemannian symmetric spaces and unitary representations) was established already by É. Cartan [Car3, n°17].

The Iwasawa decomposition clarifies the local structure of a symmetric space. Namely, O contains a dense orbit $P \cdot o \simeq P/L_0 \simeq P_u \times A$, where $A = T/T \cap H$ is the quotient of T_1 by an elementary Abelian 2-group $T_1 \cap H$. We have $\mathfrak{a} \simeq \mathfrak{t}_1$, $\Lambda(O) = \mathfrak{X}(A)$, $r(O) = \dim \mathfrak{a}$. The notation here agrees with Theorem 4.7 and Subsection 7.2.

Lemma 26.15. *All maximal θ-split tori are H^0-conjugate.*

Proof. In the above notation, PH is open in G, whence the H^0-orbit of P is open in G/P. Since P coincides with the normalizer of the open B-orbit in O, all such parabolics are G-conjugate and therefore H^0-conjugate. Hence the Levi subgroups $L = P \cap \theta(P)$ and finally the maximal θ-split tori $T_1 = (Z(L)^0)_1$ are H^0-conjugate. □

If T_1 is maximal, then every imaginary root is compact and θ maps positive complex or real roots to negative ones. Compact (simple) roots form (the base of) the root system of L.

The endomorphism $\iota = -w_L\theta$ of $\mathfrak{X}(T)$ preserves Δ_+ and induces a diagram involution of the set Π of simple roots. (Here w_L is the longest element in the Weyl group of L.) Since $w_G w_L \theta$ preserves Δ_+ and differs from θ by an inner automorphism, it coincides with the diagram automorphism $\overline{\theta}$, whence $\iota(\lambda) = \overline{\theta}(\lambda)^*$, $\forall\lambda \in \mathfrak{X}(T)$.

Consider the set $\Delta_O \subset \mathfrak{X}(T_1)$ and the subset $\Pi_O \subset \Delta_O$ consisting of the restrictions $\overline{\alpha} = \alpha|_{T_1}$ of complex and real roots $\alpha \in \Delta$ (resp. $\alpha \in \Pi$) to T_1.

Lemma 26.16. *Δ_O is a (possibly non-reduced) root system with base Π_O, called the* (little) root system *of the symmetric space O.*

Proof. The proof is similar to that of Lemma 26.8. The restriction of $\alpha \in \Delta$ to T_1 is identified with the orthogonal projection to $\mathfrak{X}(T_1) \otimes \mathbb{Q}$ given by $\overline{\alpha} = (\alpha - \theta(\alpha))/2$. We have $\alpha + \theta(\alpha) \notin \Delta$, $\forall\alpha \in \Delta$. (Otherwise $\alpha + \theta(\alpha)$ would be a non-compact root.) The involution ι coincides with $-\theta$ modulo the root lattice of L. One easily deduces that $\overline{\alpha} = \overline{\beta}$ if and only if $\alpha = \beta$ or $\iota(\alpha) = \beta$, $\forall \alpha, \beta \in \Pi$, and that Π_O is linearly independent. Taking these remarks into account, the proof repeats that of Lemma 26.8 with θ replaced by $-\theta$. □

The Dynkin diagram of Π with the "compact" nodes marked by black and the remaining nodes by white, where the white nodes transposed by ι are joined by two-headed arrows, is called the *Satake diagram* of the involution θ, or of the symmetric space O. The Satake diagram encodes the embedding of $\mathfrak{h}$ into $\mathfrak{g}$. Besides, it contains information on the weight lattice (semigroup) of the symmetric space (see Propositions 26.22, 26.24).

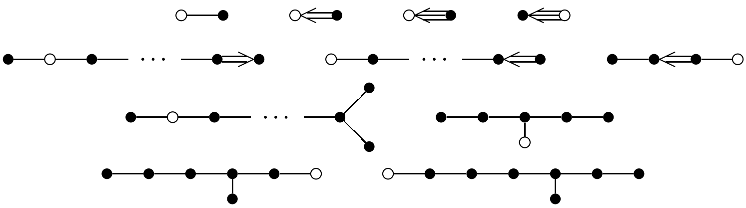
Example 26.17. The Satake diagram of the symmetric space $O = H \times H / \operatorname{diag} H$ of Example 26.10 consists of two Dynkin diagrams of H, so that all nodes are white and each node of the first diagram is joined with the respective node of the second diagram, e.g.:

26.5 Classification. The classification of symmetric spaces goes back to Cartan. To describe it, first note that θ preserves the connected center and either preserves or transposes the simple factors of G. Hence every symmetric space is locally isomorphic to a product of a torus $Z/Z \cap H$, of symmetric spaces $H \times H / \operatorname{diag} H$ with H simple, and of symmetric spaces of simple groups.

Thus the classification reduces to simple G. It can be obtained using either Kac diagrams [Hel1, X.5], [GOV, Ch. 3, §3] or Satake diagrams [Sp2], [GOV, Ch. 4, §4]. For simple G both Kac and Satake diagrams are connected.

Further analysis shows that the underlying affine Dynkin diagram for the Kac diagram of θ depends only on the diagram involution $\overline{\theta}$. This diagram is easily recovered from the Dynkin diagram of Π and from $\overline{\theta}$ using Table 26.2. Since the weight system of $T_0 : \mathfrak{m}$ is symmetric, for each "white" root $\overline{\alpha} \in \widetilde{\Pi}$ there exists a "white" root $\overline{\alpha}_0$ and "black" roots $\overline{\alpha}_1, \dots, \overline{\alpha}_r$ such that $-\overline{\alpha} = \overline{\alpha}_0 + \overline{\alpha}_1 + \cdots + \overline{\alpha}_r$. As $\widetilde{\Pi}$ is bound by a unique linear dependence, the coefficients being positive integers, there exists either a unique "white" root, with the coefficient 1 or 2, or exactly two "white" roots, with the coefficients 1. The first possibility occurs exactly for outer involutions, because in this case the weight system contains the zero weight, while the other two possibilities correspond to inner involutions. Using these observations, it is easy to write down all possible Kac diagrams, see Table 26.3.

On the other hand, all a priori possible Satake diagrams can also be classified. One verifies that a Satake diagram cannot be one of the following:



In the first seven cases, the sum of all simple roots would be a complex root α such that $\alpha + \theta(\alpha) \in \Delta$ is a non-compact root, a contradiction. In the remaining four cases, θ would be an inner involution represented by an element $s \in S = Z_G(L_0)^0$. The group S is a simple SL_2-subgroup corresponding to the highest root $\delta \in \Delta$ and T_1 is a maximal torus in S. Replacing T_1 by another maximal torus containing s, one obtains $\delta(s) = -1$. However the unique $\alpha \in \Pi$ such that $\alpha(s) = -1$ occurs in the decomposition of δ with coefficient 2, a contradiction.

By a *fragment* of a Satake diagram we mean an ι-stable subdiagram such that no one of its nodes is joined with a black node outside the fragment. A fragment is the Satake diagram of a Levi subgroup in G. It follows that a Satake diagram cannot contain the above listed fragments. Also, if a Satake diagram contains a fragment •— ··· —• of length > 1, then there are no other black nodes and ι is non-trivial. Having this in mind, it is easy to write down all possible Satake diagrams, see Table 26.3.

Both Kac and Satake diagrams uniquely determine the involution θ, up to conjugation. All a priori possible diagrams are realized for simply connected G. It follows that symmetric spaces of simple groups are classified, up to a local isomorphism, by Kac or Satake diagrams.

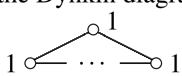
The classification is presented in Table 26.3. $\mathrm{S}(\mathrm{L}_m \times \mathrm{L}_{n-m})$ in the column "H" denotes the group of unimodular block-diagonal matrices with blocks of size m and $n-m$. The column "θ" describes the involution for classical G in matrix terms. Here

$$I_{n,m} = \begin{pmatrix} -E_m & 0 \\ 0 & E_{n-m} \end{pmatrix}, \quad K_{n,m} = \begin{pmatrix} I_{n,m} & 0 \\ 0 & I_{n,m} \end{pmatrix}, \quad \text{and } \Omega_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

is the matrix of a standard symplectic form fixed by $\mathrm{Sp}_{2n}(\Bbbk)$, where E_k is the unit $k \times k$ matrix.

Example 26.18. Let us describe the symmetric spaces of $G = \mathrm{SL}_n(\Bbbk)$. Take the standard Borel subgroup of upper-triangular matrices $B \subset G$ and the standard diagonal torus $T \subset B$. By $\varepsilon_1, \dots, \varepsilon_n$ denote the weights of the tautological representation in $\Bbbk^n$ (i.e., the diagonal entries of T).

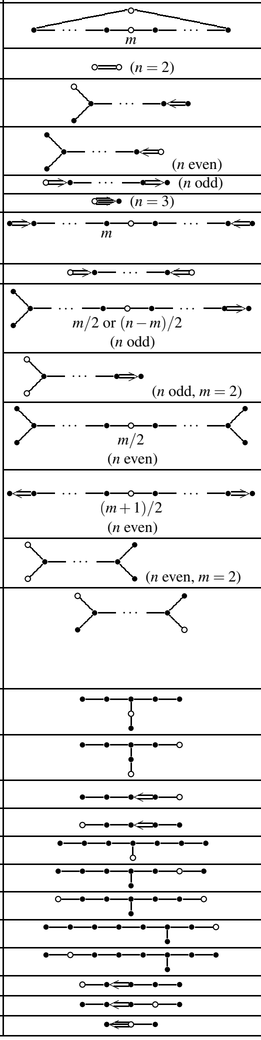
If θ is inner, then $\overline{\Delta} = \Delta$ and the Dynkin diagram of $\widetilde{\Pi}$ is the following one:

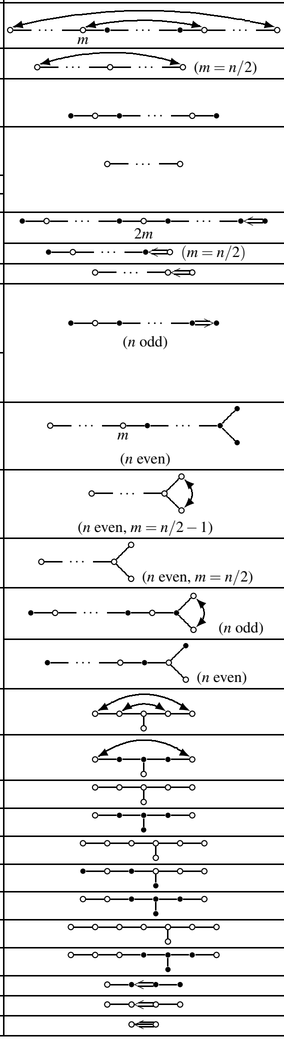


The coefficients of the unique linear dependence on $\widetilde{\Pi}$ are indicated at the diagram. It follows that there are exactly two white nodes in the Kac diagram. The involution ι is non-trivial, whence there is at most one black fragment in the Satake diagram, which is located in the middle. Thus we obtain No. 1 of Table 26.3.

The involution θ is the conjugation by an element of order 2 in $\mathrm{GL}_n(\Bbbk)$. In a certain basis, $\theta(g) = I_{n,m} \cdot g \cdot I_{n,m}$. Then $T_0 = T$, $H = \mathrm{S}(\mathrm{L}_m \times \mathrm{L}_{n-m})$ is embedded in G by the two diagonal blocks, the simple roots being $\varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $i \ne m$, and $\mathfrak{m} = \Bbbk^m \otimes (\Bbbk^{n-m})^* \oplus (\Bbbk^m)^* \otimes \Bbbk^{n-m}$ is embedded in $\mathfrak{g}$ by the two antidiagonal blocks, the lowest weights of the summands being $\varepsilon_m - \varepsilon_{m+1}$, $\varepsilon_n - \varepsilon_1$, in accordance with the Kac diagram.

Table 26.3 Symmetric spaces of simple groups





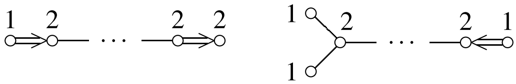
No.	G	H	θ	Kac diagram	Satake diagram	$\Delta_{G/H}$
1	SL_n	$S(L_m \times L_{n-m})$ $(m \le n/2)$	$g \mapsto I_{n,m} g I_{n,m}$	m	m	$\mathbf{BC}_m$
				$(n=2)$	$(m=n/2)$	$\mathbf{C}_{n/2}$
2	SL_{2n}	Sp_{2n}	$g \mapsto \Omega_n (g^\top)^{-1} \Omega_n^\top$			$\mathbf{A}_{n-1}$
3	SL_n	SO_n	$g \mapsto (g^\top)^{-1}$	$(n$ even$)$		$\mathbf{A}_{n-1}$
				$(n$ odd$)$		
				$(n=3)$		
4	Sp_{2n}	$Sp_{2m} \times Sp_{2(n-m)}$ $(m \le n/2)$	$g \mapsto K_{n,m} g K_{n,m}$	m	$2m$	$\mathbf{BC}_m$
					$(m=n/2)$	$\mathbf{C}_{n/2}$
5	Sp_{2n}	GL_n	$g \mapsto I_{2n,n} g I_{2n,n}$			$\mathbf{C}_n$
6	SO_n	$SO_m \times SO_{n-m}$ $(m \le n/2)$	$g \mapsto I_{n,m} g I_{n,m}$	$m/2$ or $(n-m)/2$ $(n$ odd$)$	$(n$ odd$)$	$\mathbf{B}_m$
				$(n$ odd, $m=2)$		
				$m/2$ $(n$ even$)$	m $(n$ even$)$	$\mathbf{B}_m$
				$(m+1)/2$ $(n$ even$)$	$(n$ even, $m=n/2-1)$	$\mathbf{B}_{n/2-1}$
				$(n$ even, $m=2)$	$(n$ even, $m=n/2)$	$\mathbf{D}_{n/2}$
7	SO_{2n}	GL_n	$g \mapsto \Omega_n g \Omega_n^\top$		$(n$ odd$)$	$\mathbf{BC}_{[n/2]}$
					$(n$ even$)$	$\mathbf{C}_{n/2}$
8	$\mathbf{E}_6$	$\mathbf{A}_5 \times \mathbf{A}_1$				$\mathbf{F}_4$
9	$\mathbf{E}_6$	$\mathbf{D}_5 \times \Bbbk^\times$				$\mathbf{BC}_2$
10	$\mathbf{E}_6$	$\mathbf{C}_4$				$\mathbf{E}_6$
11	$\mathbf{E}_6$	$\mathbf{F}_4$				$\mathbf{A}_2$
12	$\mathbf{E}_7$	$\mathbf{A}_7$				$\mathbf{E}_7$
13	$\mathbf{E}_7$	$\mathbf{D}_6 \times \mathbf{A}_1$				$\mathbf{F}_4$
14	$\mathbf{E}_7$	$\mathbf{E}_6 \times \Bbbk^\times$				$\mathbf{C}_3$
15	$\mathbf{E}_8$	$\mathbf{D}_8$				$\mathbf{E}_8$
16	$\mathbf{E}_8$	$\mathbf{E}_7 \times \mathbf{A}_1$				$\mathbf{F}_4$
17	$\mathbf{F}_4$	$\mathbf{B}_4$				$\mathbf{BC}_1$
18	$\mathbf{F}_4$	$\mathbf{C}_3 \times \mathbf{A}_1$				$\mathbf{F}_4$
19	$\mathbf{G}_2$	$\mathbf{A}_1 \times \mathbf{A}_1$				$\mathbf{G}_2$

In another basis, $\theta(g) = J_{n,m} \cdot g \cdot J_{n,m}$, where

$$J_{n,m} = \begin{pmatrix} \mathbf{0} & & \overbrace{\begin{matrix}0 & & 1\\ & \iddots & \\ 1 & & 0\end{matrix}}^{m} \\ & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & \\ \underbrace{\begin{matrix}0 & & 1\\ & \iddots & \\ 1 & & 0\end{matrix}}_{m} & & \mathbf{0} \end{pmatrix}$$

Now $T_1 = \{t = \mathrm{diag}(t_1,\dots,t_m,1,\dots,1,t_m^{-1},\dots,t_1^{-1})\}$ is a maximal θ-split torus and the (compact) imaginary roots are $\varepsilon_i - \varepsilon_j$, $m < i \neq j \leq n-m$, in accordance with the Satake diagram. The little root system Δ_O consists of the nonzero restrictions $\overline{\varepsilon}_i - \overline{\varepsilon}_j$, $1 \leq i, j \leq n$, i.e., of $\pm\overline{\varepsilon}_i \pm \overline{\varepsilon}_j$, $\pm 2\overline{\varepsilon}_i$, and $\pm\overline{\varepsilon}_i$ unless $m = n/2$, $1 \leq i \neq j \leq m$. Thus Δ_O is of type $\mathbf{BC}_m$ or $\mathbf{C}_{n/2}$.

If θ is outer, then $\overline{\theta}(\varepsilon_i) = -\varepsilon_{n+1-i}$ and $T^{\overline{\theta}} = \{t = \mathrm{diag}(t_1,t_2,\dots,t_2^{-1},t_1^{-1})\}$. Restricting the roots to this subtorus, we see that $\overline{\Delta}$ consists of $\pm\varepsilon_i' \pm \varepsilon_j'$, $\pm 2\varepsilon_i'$, and $\pm\varepsilon_i'$ for odd n, where ε_i' are the restrictions of ε_i, $1 \leq i \leq n/2$. The Dynkin diagram of $\widetilde{\Pi}$ has one of the following forms:



depending on whether n is odd or even. Therefore the Kac diagram has a unique white node, namely an extreme one.

The involution ι is trivial, whence either all nodes of the Satake diagram are white or the black nodes are isolated from each other and alternate with the white ones, the extreme nodes being black. (Otherwise, there would exist an inadmissible fragment ○—● .) Thus we obtain Nos. 2–3 of Table 26.3.

Any outer involution has the form $\theta(g) = (g^*)^{-1}$, where $*$ denotes the conjugation with respect to a non-degenerate (skew-)symmetric bilinear form on $\Bbbk^n$. In the symmetric case, choosing an orthonormal basis yields $\theta(g) = (g^\top)^{-1}$, whence $T_1 = T$ is a maximal θ-split torus and $\Delta_O = \Delta$. In a hyperbolic basis, $\theta(g) = (g^\dagger)^{-1}$, where $\dagger$ denotes the transposition with respect to the secondary diagonal. Then $T_0 = T^{\overline{\theta}}$ is a maximal torus in $H = \mathrm{SO}_n(\Bbbk)$. The roots of H are $\pm\varepsilon_i' \pm \varepsilon_j'$, $1 \leq i \neq j \leq n/2$, and $\pm\varepsilon_i'$ for odd n. The space $\mathfrak{m}$ consists of traceless symmetric matrices, and the lowest weight is $-2\varepsilon_1'$.

In the skew-symmetric case, choosing an appropriately ordered symplectic basis yields $\theta(g) = I_{n,n/2}(g^\dagger)^{-1}I_{n,n/2}$. Here T_0 is a maximal torus in $H = \mathrm{Sp}_n(\Bbbk)$ and $T_1 = \{t = \mathrm{diag}(t_1,t_2,\dots,t_2,t_1) \mid t_1 \cdots t_{n/2} = 1\}$ is a maximal θ-split torus. The roots of H are $\pm\varepsilon_i' \pm \varepsilon_j'$, $\pm 2\varepsilon_i'$, $1 \leq i \neq j \leq n/2$, and the lowest weight of $\mathfrak{m}$ is $-\varepsilon_1' - \varepsilon_2'$. The compact roots are $\varepsilon_i - \varepsilon_{n+1-i}$ $(1 \leq i \leq n)$, and Δ_O consists of $\overline{\varepsilon}_i - \overline{\varepsilon}_j$, $1 \leq i \neq j \leq n/2$, thus having the type $\mathbf{A}_{n/2-1}$.

From now on we assume that T_1 is a maximal θ-split torus.

26.6 Weyl Group. Consider the Weyl group W_O of the little root system Δ_O.

Proposition 26.19. $W_O \simeq N_{H^0}(T_1)/Z_{H^0}(T_1) \simeq N_G(T_1)/Z_G(T_1)$.

Proof. First we prove that each element of W_O is induced by an element of $N_{H^0}(T_1)$. It suffices to consider a root reflection $r_{\overline{\alpha}}$. Let $T_1^{\overline{\alpha}} \subseteq T_1$ be the connected kernel of $\overline{\alpha}$. Replacing G by $Z_G(T_1^{\overline{\alpha}})$ we may assume that $W_O = \{e, r_{\overline{\alpha}}\}$. The same argument as in Lemma 26.15 shows that $P^- = \theta(P) = hPh^{-1}$ for some $h \in H^0$. It follows that $h \in N_{H^0}(L) = N_{H^0}(T_1)$ acts on $\mathfrak{X}(T_1)$ as $r_{\overline{\alpha}}$.

On the other hand, $N_G(T_1)$ acts on T_1 as a subgroup of the "big" Weyl group $W = N_G(T)/T$. Indeed, any $g \in N_G(T_1)$ normalizes $L = Z_G(T_1)$ and may be replaced by another element in gL normalizing T. Since the Weyl chambers of W_O in $\mathfrak{X}(T_1) \otimes \mathbb{Q}$ are the intersections of Weyl chambers of W with $\mathfrak{X}(T_1) \otimes \mathbb{Q}$, the orbits of $N_G(T_1)/Z_G(T_1)$ intersect them in single points. Thus $N_G(T_1)/Z_G(T_1)$ cannot be bigger than W_O. This concludes the proof. □

26.7 *B*-orbits. Since O is spherical, there are finitely many B-orbits in O (Corollary 6.5). Their structure plays an important rôle in some geometric problems and, for $\Bbbk = \mathbb{C}$, in the representation theory of the real reductive Lie group $G(\mathbb{R})$ acting on the Riemannian symmetric space $O(\mathbb{R})$, the non-compact real form of O [Vog]. The classification and the adherence relation for B-orbits were described in [Sp1], [RS1], [RS2] (cf. Example 6.7). We explain the basic classification result under the assumption $H = G^\theta$. This is not an essential restriction [RS2, 1.1(b)].

By Proposition 26.3, O is identified with $\tau(G)$, where G (and B) acts by twisted conjugation.

Proposition 26.20. *The (twisted) B-orbits in $\tau(G) \simeq O$ intersect $N_G(T)$ in T-orbits. Thus $\mathfrak{B}(O)$ is in bijective correspondence with the set of twisted T-orbits in $N_G(T) \cap \tau(G)$.*

Proof. Consider a B-orbit $Bgo \subseteq O$. By Lemma 26.5, replacing g by bg, $b \in B$, one may assume that $g^{-1}Tg$ is a θ-stable maximal torus in $g^{-1}Bg$. This holds if and only if $\tau(g) \in N_G(T)$. One the other hand, taking another point $g'o \in Bgo$, $g' = bgh, b \in B$, $h \in H$, we have $\tau(g') = \theta(b)\tau(g)b^{-1} \in N_G(T)$ if and only if $\tau(g') = \theta(t)\tau(g)t^{-1}$, where $b = tu$, $t \in T$, $u \in U$, by standard properties of the Bruhat decomposition [Hum, 28.4]. □

There is a natural map $\mathfrak{B}(O) \to W$, $Bgo \mapsto w$, where $\theta(B)wB$ is the unique Bruhat double coset containing the respective B-orbit $\tau(BgH)$. By Proposition 26.20, $\tau(BgH) \cap N_G(T) \subseteq wT$. This map plays an important rôle in the study of B-orbits [RS1], [RS2]. Its image is contained in the set of *twisted involutions* $\{w \in W \mid \theta(w) = w^{-1}\}$, but in general is neither injective nor surjective onto this set.

Example 26.21. Let $G = \mathrm{GL}_n(\Bbbk)$, $\theta(g) = (g^\top)^{-1}$, $H = \mathrm{O}_n(\Bbbk)$. Then $\tau(G)$ is the set of non-degenerate symmetric matrices, viewed as quadratic forms on $\Bbbk^n$. The group B of upper-triangular matrices acts on $\tau(G)$ by base changes preserving the standard flag in $\Bbbk^n$. It is an easy exercise in linear algebra that, for any inner product on $\Bbbk^n$, one can choose a basis $e_1, \dots, e_n$ compatible with a given flag and having the property that for any i there is a unique j such that $(e_i, e_j) = 1$ and $(e_i, e_k) = 0$, $\forall k \neq j$. The matrix of the quadratic form in this basis is the permutation matrix of the involution transposing i and j. It lies in $N_G(T)$ (where T is the diagonal torus) and is uniquely determined by the B-orbit of the quadratic form. Thus $\mathfrak{B}(O)$ is in bijection with the set of involutions in $W = \mathrm{S}_n$.

26.8 Colored Equipment. Now we describe the colored equipment of a symmetric space, according to [Vu2].

The weight lattice of a symmetric space is read off the Satake diagram, at least up to a finite extension. Let $Z = Z(G)^0$ and ω_i be the fundamental weights corresponding to the simple roots $\alpha_i \in \Pi$.

Proposition 26.22. *If G is of simply connected type, then*

$$\Lambda(O) = \mathfrak{X}(Z/Z \cap H) \oplus \left\langle \widehat{\omega}_j,\ \omega_k + \omega_{\iota(k)} \,\middle|\, j,k \right\rangle, \tag{26.2}$$

where j,k run over all ι-fixed, resp. ι-unstable, white nodes of the Satake diagram, and $\widehat{\omega}_j = \omega_j$ or $2\omega_j$, depending on whether the j-th node is adjacent to a black one or not. In the general case, $\Lambda(O)$ is a sublattice of finite index in the r.h.s. of (26.2).

Remark 26.23. The weight lattice $\Lambda(O) = \mathfrak{X}(T/T \cap H) = \mathfrak{X}(T_1/T_1 \cap H)$ injects into $\mathfrak{X}(T_1)$ via restriction of characters from T to T_1. The space $\mathscr{E} = \mathrm{Hom}(\Lambda(O), \mathbb{Q})$ is then identified with $\mathfrak{X}^*(T_1) \otimes \mathbb{Q}$. The second direct summand in the r.h.s. of (26.2) is nothing else but the doubled weight lattice $2(\mathbb{Z}\Delta_O^\vee)^*$ of the little root system Δ_O. Indeed, $\widehat{\omega}_j/2$ and $(\omega_k + \omega_{\iota(k)})/2$ restrict to the fundamental weights dual to the simple coroots $\overline{\alpha}_j^\vee = \alpha_j^\vee - \theta(\alpha_j^\vee)$ or $\alpha_j^\vee$ and $\overline{\alpha}_k^\vee = \alpha_k^\vee - \theta(\alpha_k^\vee)$.

Proof. Without loss of generality we may assume that G is semisimple simply connected, whence $H = G^\theta$. The sublattice $\Lambda(O) \subseteq \mathfrak{X}(T)$ consisting of the weights vanishing on T^θ, i.e., of $\mu - \theta(\mu)$, $\mu \in \mathfrak{X}(T)$, is contained in $\mathfrak{X}(T/T_0) = \{\lambda \in \mathfrak{X}(T) \mid \theta(\lambda) = -\lambda\} = \langle \omega_j,\ \omega_k + \omega_{\iota(k)} \mid j,k \rangle$. The latter lattice injects into $\mathfrak{X}(T_1)$ so that $\Lambda(O)$ is identified with $2\mathfrak{X}(T_1)$. It remains to prove that $\mathfrak{X}(T_1) = (\mathbb{Z}\Delta_O^\vee)^*$ or, equivalently, that $\mathfrak{X}^*(T_1) = \mathbb{Z}\Delta_O^\vee$ is the coroot lattice of the little root system.

We have $\mathfrak{X}^*(T) = \mathbb{Z}\Delta^\vee$ and $\mathfrak{X}^*(T_1) = \mathbb{Z}\Delta^\vee \cap \mathscr{E} \supseteq \mathbb{Z}\Delta_O^\vee$. The *alcoves* (= fundamental polyhedra, see [Bou1, Ch. VI, §2, n°1]) of the affine Weyl group $W_{\mathrm{aff}}(\Delta_O)$ are the intersections of $\mathscr{E}$ with alcoves of $W_{\mathrm{aff}}(\Delta_O)$. Hence each alcove of $W_{\mathrm{aff}}(\Delta_O)$ contains a unique point from $\mathfrak{X}^*(T_1)$. It follows that $\mathfrak{X}^*(T_1)$ coincides with $\mathbb{Z}\Delta_O^\vee$. □

Let $\mathbf{C} = \mathbf{C}(\Delta_+)$ denote the dominant Weyl chamber of a root system Δ (with respect to a chosen subset of positive roots Δ_+). The weight semigroup $\Lambda_+(O)$ is contained both in $\Lambda(O)$ and in $\mathbf{C}$. Note that $\mathbf{C} \cap \mathscr{E} = \mathbf{C}(\Delta_O^+)$.

Proposition 26.24. $\Lambda_+(O) = \Lambda(O) \cap \mathbf{C}(\Delta_O^+)$.

Proof. Since $\Lambda_+(O)$ is the semigroup of all lattice points in a cone (see 15.1), it suffices to prove that $\mathbb{Q}_+\Lambda_+(O) = \mathbf{C}(\Delta_O^+)$. Take any dominant $\lambda \in \Lambda(O)$. We prove that $2\lambda \in \Lambda_+(O)$.

First note that $\lambda = -\theta(\lambda)$ is orthogonal to compact roots, whence λ is extended to P and $V^*(\lambda) = \operatorname{Ind}_P^G \Bbbk_{-\lambda}$. Consider another dual Weyl module obtained by twisting the G-action by θ: $V^*(\lambda)^\theta = \operatorname{Ind}_{\theta(P)}^G \Bbbk_{-\theta(\lambda)} \simeq V^*(\lambda^*)$. We have the canonical H-equivariant linear isomorphism $\omega : V^*(\lambda)^\theta \xrightarrow{\sim} V^*(\lambda)$. (If the dual Weyl modules are realized in $\Bbbk[G]$ as in Example 2.10, then ω is just the restriction of θ acting on $\Bbbk[G]$.) In other words, $\omega \in (V^*(\lambda) \otimes V(\lambda^*))^H$. Note that ω maps a T-eigenvector of weight μ to an eigenvector of weight $\theta(\mu)$. Hence

$$\omega = v_{-\lambda} \otimes v'_{-\lambda} + \sum_{\mu \neq \lambda} v_{\theta(\mu)} \otimes v'_{-\mu}, \tag{26.3}$$

where v_χ, v'_χ denote basic eigenvectors of weight χ in $V^*(\lambda)$ and $V(\lambda^*)$, respectively. Applying the homomorphisms $V(\lambda^*) \to V^*(\lambda)$, $v'_{-\lambda} \mapsto v_{-\lambda}$, and $V^*(\lambda) \otimes V^*(\lambda) \to V^*(2\lambda)$ (induced by multiplication in $\Bbbk[G]$), we obtain a nonzero element $\overline{\omega} \in V^*(2\lambda)^H$, whence $2\lambda \in \Lambda_+(O)$ by (2.2). □

Now we are ready to describe the colors and G-valuations of a symmetric space.

Theorem 26.25. *The colors of a symmetric space O are represented by the vectors from $\frac{1}{2}\Pi_O^\vee \subset \mathscr{E}$ (where $\Pi_O^\vee$ is the base of $\Delta_O^\vee \subset \mathfrak{X}^*(T_1)$). The valuation cone $\mathscr{V}$ is the antidominant Weyl chamber of $\Delta_O^\vee$ in $\mathscr{E}$.*

Corollary 26.26. *W_O is the little Weyl group of O in the sense of* 22.3.

Proof. Without loss of generality G is assumed to be of simply connected type. In the notation of Remarks 13.4 and 15.1, each $f \in \Bbbk[O]_\lambda^{(B)}$ is represented as $f = \eta_1^{d_1} \cdots \eta_s^{d_s}$, where the η_i are equations of the colors $D_i \in \mathscr{D}^B$, $d_i \in \mathbb{Z}_+$, and $\lambda = \sum d_i \lambda_i$, $\sum d_i \chi_i = 0$, where (λ_i, χ_i) are the biweights of η_i.

In the notation of Proposition 26.22, if $\lambda = \widehat{\omega}_j$ or $\omega_k + \omega_{\iota(k)}$, then $f = \eta_j$, or $\eta'_j\eta''_j$, or η_k, or $\eta'_k\eta''_k$, where the biweights of $\eta_j, \eta'_j, \eta''_j, \eta_k, \eta'_k, \eta''_k$ are $(\widehat{\omega}_j, 0)$, (ω_j, χ_j), $(\omega_j, -\chi_j)$, $(\omega_k + \omega_{\iota(k)}, 0)$, (ω_k, χ_k), $(\omega_{\iota(k)}, -\chi_k)$, respectively, for some nonzero $\chi_j, \chi_k \in \mathfrak{X}(H)$. In particular, the respective colors $D_j, D'_j, D''_j, D_k, D'_k, D''_k$ are pairwise distinct, and all colors occur among them since these f's span the multiplicative semigroup $\Bbbk[O]^{(B)}/\Bbbk[O]^\times$ by Proposition 26.24 and Remark 26.23. The assertion on colors stems now from Remarks 15.1 and 26.23.

Now we treat G-valuations. Take any $v = v_D \in \mathscr{V}$, where D is a G-stable prime divisor on a G-model X of $\Bbbk(O)$. It follows from the local structure theorem that $F = \overline{T_1 o}$ is an $N_H(T_1)$-stable subvariety of X intersecting D in the union of T_1-stable prime divisors D_{wv}, $w \in W_O$, that correspond to wv regarded as T_1-valuations of $\Bbbk(T_1 o)$ (cf. Proposition 23.13). By Theorem 21.1 $\mathscr{V}$ contains the antidominant Weyl chamber. It remains to show as in the proof of Theorem 22.13 that different vectors from $\mathscr{V}$ cannot be W_O-equivalent. □

The proof of Theorem 26.25 shows that the map $\varkappa : \mathscr{D}^B \to \mathscr{E}$ may be non-injective if H is not semisimple. There is a more precise description of colors in the spirit of Proposition 26.20 [Sp1, 5.4], [CS, §4].

It suffices to consider simple G. Assume first that H is connected. For any $\overline{\alpha}^\vee \in \Pi_O^\vee$ there exist either a unique or exactly two colors mapping to $\overline{\alpha}^\vee/2$. They correspond to the twisted T-orbits in $\tau(G) \cap r_\alpha T$ (for real α) or in $\tau(G) \cap \big(r_{\theta(\alpha)} r_\alpha T \cup r_{\theta(\iota(\alpha))} r_{\iota(\alpha)} T\big)$ (for complex α).

If H is semisimple, then such an orbit (and the respective color D_α) is always unique. In particular, $\iota(\alpha) = \alpha$ or $\iota(\alpha) = -\theta(\alpha) \perp \alpha$.

If H is not semisimple (*Hermitian case*), then inspection of Table 26.3 shows that $\dim Z(H) = 1$ and Δ_O is of type $\mathbf{BC}_n$ or $\mathbf{C}_n$. The color mapped to $\overline{\alpha}^\vee/2$ is unique except for the case where $\overline{\alpha}^\vee$ is the short simple coroot.

In the latter case, if α is complex, then $\tau(G) \cap r_{\theta(\alpha)} r_\alpha T$ and $\tau(G) \cap r_{\theta(\iota(\alpha))} r_{\iota(\alpha)} T$ are the twisted T-orbits corresponding to the two colors $D_\alpha, D_{\iota(\alpha)}$ mapped to $\overline{\alpha}^\vee/2$. Here $\Delta_O = \mathbf{BC}_n$ and $c(G/H') = 0$.

If α is real, then $\tau(G) \cap r_\alpha T$ consists of two twisted T-orbits corresponding to the two colors $D_\alpha^\pm$ mapped to $\alpha^\vee/2$ and swapped by $\operatorname{Aut}_G O \simeq \mathbf{Z}_2$. Here $\Delta_O = \mathbf{C}_n$ and $c(G/H') = 1$.

For disconnected H the divisors $D_\alpha^\pm \in \mathscr{D}^B(G/H^0)$ patch together into a single divisor $D_\alpha \in \mathscr{D}^B(G/H)$.

26.9 Coisotropy Representation. The (co)isotropy representation $H : \mathfrak{m}$ has nice invariant-theoretic properties in characteristic zero. They were examined by Kostant and Rallis [KoR]. From now on assume that $\operatorname{char} \Bbbk = 0$.

Semisimple elements in $\mathfrak{m}$ are exactly those having closed H-orbits, and the unique closed H-orbit in $\overline{H\xi}$ ($\xi \in \mathfrak{m}$) is $H\xi_s$. General elements of $\mathfrak{m}$ are semisimple. One may deduce it from the fact that T^*O is symplectically stable (Proposition 8.14) or prove directly: $\mathfrak{g} = \mathfrak{l} \oplus [\mathfrak{g}, \mathfrak{t}_1] \implies \mathfrak{m} = \mathfrak{t}_1 \oplus [\mathfrak{h}, \mathfrak{t}_1] \implies \mathfrak{m} = \overline{H\mathfrak{t}_1}$. This argument also shows that H-invariant functions on $\mathfrak{m}$ are uniquely determined by their restrictions to $\mathfrak{t}_1$. A more precise result was obtained by Kostant and Rallis.

Proposition 26.27 ([KoR]). *Every semisimple H-orbit in $\mathfrak{m}$ intersects $\mathfrak{t}_1$ in a W_O-orbit. Restriction of functions yields an isomorphism $\Bbbk[\mathfrak{m}]^H \simeq \Bbbk[\mathfrak{t}_1]^{W_O}$.*

Proof. Every semisimple element $\xi \in \mathfrak{m}$ is contained in the Lie algebra of a maximal θ-split torus. Hence by Lemma 26.15, $\xi' = (\operatorname{Ad} h)\xi \in \mathfrak{t}_1$ for some $h \in H$. If $\xi \in \mathfrak{t}_1$, then $T_1, h^{-1}T_1h$ are two maximal θ-split tori in $Z_G(\xi)$. Again by Lemma 26.15, $zT_1z^{-1} = h^{-1}T_1h$ for some $z \in Z_G(\xi) \cap H$, whence $h' = hz \in N_H(\mathfrak{t}_1)$, $\xi' = (\operatorname{Ad} h')\xi \in W_O\xi$.

The second assertion is a particular case of Proposition 25.7. It suffices to observe that the surjective birational morphism $\mathfrak{m}/\!/H \to \mathfrak{t}_1/W_O$ of two normal affine varieties has to be an isomorphism. □

Global analogues of these results for the H-action on O (in any characteristic) were obtained by Richardson [Ri2].

26.10 Flats. It is not incidental that the description of the valuation cone of a symmetric space was obtained by the same reasoning as in 23.5.

Proposition 26.28 ([Kn5, §6]). *Flats in O are exactly the G-translates of $T_1 \cdot o$.*

Proof. It suffices to consider flats F_α, $\alpha \in T_o^{\mathrm{pr}} O$. We have $T^*O = G *_H \mathfrak{m}$, $\alpha = e * \xi$, $\xi = \Phi(\alpha) \in \mathfrak{m}^{\mathrm{pr}}$. By Proposition 26.27, $\xi \in (\mathrm{Ad}\, h)\mathfrak{t}_1^{\mathrm{pr}}$, $h \in H^0$. It follows that $G_\xi = hLh^{-1}$, whence $F_\alpha = hLo = hT_1 o$. □

The W_O-action on the flat $T_1 \cdot o$ comes from $N_H(T_1)$.

In the case $\Bbbk = \mathbb{C}$, flats in O are (G-translates of) the complexifications of maximal totally geodesic flat submanifolds in a Riemannian symmetric space $O(\mathbb{R})$ which is a real form of O [Hel1].

27 Algebraic Monoids and Group Embeddings

27.1 Algebraic Monoids. Similarly to algebraic groups, defined by superposing the concepts of an abstract group and an algebraic variety, it is quite natural to consider *algebraic semigroups*, i.e., algebraic varieties equipped with an associative multiplication law which is a regular map.

Example 27.1. All linear operators on a finite-dimensional vector space V form an algebraic semigroup $\mathrm{L}(V) \simeq \mathrm{L}_n(\Bbbk)$ ($n = \dim V$). The operators (matrices) of rank $\leq r$ form a closed subsemigroup $\mathrm{L}^{(r)}(V)$ ($\mathrm{L}_n^{(r)}(\Bbbk)$), a particular example of a determinantal variety.

However the category of *all* algebraic semigroups is immense. (For instance, every algebraic variety X turns into an algebraic semigroup being equipped with the "zero" multiplication $X \times X \to \{0\}$, where $0 \in X$ is a fixed element.) In order to make the theory really substantive, one has to restrict the attention to algebraic semigroups not too far from algebraic groups.

Definition 27.2. An *algebraic monoid* is an algebraic semigroup with unit, i.e., an algebraic variety X equipped with a morphism $\mu : X \times X \to X$, $\mu(x,y) =: x \cdot y$ (the *multiplication law*), and with a distinguished *unity element* $e \in X$ such that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $e \cdot x = x \cdot e = x$, $\forall x,y,z \in X$.

Let $G = G(X)$ denote the group of invertible elements in X. The following elementary result can be found, e.g., in [Rit1, §2].

Proposition 27.3. *G is open in X.*

Proof. Since the left translation $x \mapsto g \cdot x$ by an element $g \in G$ is an automorphism of X, it suffices to prove that G contains an open subset of an irreducible component of X. Without loss of generality we may assume that X is irreducible. Let p_1, p_2 be the two projections of $\mu^{-1}(e) \subset X \times X$ to X. By the fiber dimension theorem, every component of $\mu^{-1}(e)$ has dimension $\geq \dim X$, and $p_i^{-1}(e) = (e,e)$. Hence p_i are dominant maps and $G = p_1(\mu^{-1}(e)) \cap p_2(\mu^{-1}(e))$ is a dense constructible set containing an open subset of X. □

Corollary 27.4. *G is an algebraic group.*

Those irreducible components of X which do not intersect G do not "feel the presence" of G and their behavior is beyond control. Therefore it is reasonable to restrict oneself to algebraic monoids X such that $G = G(X)$ is dense in X. In this case, left translations by G permute the components of X transitively, and many questions are reduced to the case where X is irreducible.

Monoids of this kind form an interesting category of algebraic structures closely related to algebraic groups (e.g., they arise as the closures of linear algebraic groups in the spaces of linear operators). The theory of algebraic monoids was created in major part during the last 30 years by M. S. Putcha, L. E. Renner, E. B. Vinberg, A. Rittatore, et al. The interested reader may consult a detailed survey [Ren3] of the theory from the origin up to the latest developments. In this section, we discuss algebraic monoids from the viewpoint of equivariant embeddings. A link between these two theories is provided by the following result.

Theorem 27.5 ([Rit1, §2]).

(1) *Any algebraic monoid X is a $(G \times G)$-equivariant embedding of $G = G(X)$, where the factors of $G \times G$ act by left/right multiplication, having a unique closed $(G \times G)$-orbit.*
(2) *Conversely, any* affine *$(G \times G)$-equivariant embedding $X \hookleftarrow G$ carries a structure of algebraic monoid with $G(X) = G$.*

Proof. (1) One has only to prove the uniqueness of a closed orbit $Y \subseteq X$. Note that $X \cdot Y \cdot X = \overline{G \cdot Y \cdot G} = Y$, i.e., Y is a (two-sided) *ideal* in X. For any other ideal $Y' \subseteq X$ we have $Y \cdot Y' \subseteq Y \implies Y = Y \cdot Y' \subseteq Y'$. Thus Y is the smallest ideal, called the *kernel* of X.
(2) The actions of the left and right copy of $G \times G$ on X define coactions $\Bbbk[X] \to \Bbbk[G] \otimes \Bbbk[X]$ and $\Bbbk[X] \to \Bbbk[X] \otimes \Bbbk[G]$, which are the restrictions to $\Bbbk[X] \subseteq \Bbbk[G]$ of the comultiplication $\Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$. Hence the image of $\Bbbk[X]$ lies in $(\Bbbk[G] \otimes \Bbbk[X]) \cap (\Bbbk[X] \otimes \Bbbk[G]) = \Bbbk[X] \otimes \Bbbk[X]$, and we have a comultiplication in $\Bbbk[X]$. Now G is open in $X = \overline{G}$ and consists of invertibles. For any invertible $x \in X$, we have $xG \cap G \neq \emptyset$, and hence $x \in G$. □

Remark 27.6. Assertion (2) was first proved for reductive G by Vinberg [Vin2] in a different way.

Among general algebraic groups, affine (= linear) ones occupy a privileged position due to their most rich and interesting structure. The same holds for algebraic monoids. We provide two results confirming this observation.

Theorem 27.7 ([Mum, §4]). *Complete irreducible algebraic monoids are just Abelian varieties.*

Theorem 27.8 ([Rit3]). *An algebraic monoid X is affine provided that $G(X)$ is affine.*

This theorem was proved by Renner [Ren1] for quasiaffine X using some structure theory, and Rittatore [Rit3] reduced the general case to the quasiaffine one by considering total spaces of certain line bundles over X.

A theorem of Barsotti [Bar] and Rosenlicht [Ros] says that every connected algebraic group G has a unique affine normal connected subgroup G_{aff} such that the quotient group $G_{\mathrm{ab}} = G/G_{\mathrm{aff}}$ is an Abelian variety. An analogous structure result for algebraic monoids was obtained by Brion and Rittatore [BR]: for any normal irreducible algebraic monoid X the quotient map $G = G(X) \to G_{\mathrm{ab}}$ extends to a homomorphism of algebraic monoids $X \to G_{\mathrm{ab}}$, whose fiber at unity $X_{\mathrm{aff}} = \overline{G_{\mathrm{aff}}}$ is an affine algebraic monoid.

Theorem 27.9. *Any affine algebraic monoid X admits a closed homomorphic embedding $X \hookrightarrow \mathrm{L}(V)$. Furthermore, $G(X) = X \cap \mathrm{GL}(V)$.*

The proof is essentially the same as that of a similar result for algebraic groups [Hum, 8.6]. Thus the adjectives "affine" and "linear" are synonyms for algebraic monoids, in the same way as for algebraic groups.

In the notation of Theorem 27.9, the space of matrix entries $\mathrm{M}(V)$ generates $\Bbbk[X] \subseteq \Bbbk[G]$. Generally, $\Bbbk[X] \supset \mathrm{M}(V)$ if and only if the representation $G : V$ is extendible to X. It follows from Theorem 27.9 and (2.1) that

$$\Bbbk[X] = \bigcup \mathrm{M}(V) \tag{27.1}$$

over all G-modules V that are X-modules (cf. Proposition 2.14).

Example 27.10. By Theorem 27.5(2), every affine toric variety X carries a natural structure of algebraic monoid extending the multiplication in the open torus T. By Theorem 27.9, X is the closure of T in $\mathrm{L}(V)$ for some faithful representation $T : V$, i.e., a closed submonoid in the monoid of all diagonal matrices in some $\mathrm{L}_n(\Bbbk)$. The coordinate algebra $\Bbbk[X]$ is the semigroup algebra of the semigroup $\Sigma \subseteq \mathfrak{X}(T)$ consisting of all characters $T \to \Bbbk^\times$ extendible to X. Conversely, every finitely generated semigroup $\Sigma \ni 0$ such that $\mathbb{Z}\Sigma = \mathfrak{X}(T)$ defines a toric monoid $X \supseteq T$.

27.2 Reductive Monoids. The classification and structure theory for algebraic monoids is most well developed in the case where the group of invertibles is reductive.

Definition 27.11. An irreducible algebraic monoid X is called *reductive* if $G = G(X)$ is a reductive group.

In the sequel we consider only reductive monoids, thus returning to the general convention of our survey that G is a connected reductive group. By Theorems 27.5 and 27.8, reductive monoids are nothing else but $(G \times G)$-equivariant affine embeddings of G. They were classified by Vinberg [Vin2] in characteristic zero. Rittatore [Rit1] extended this classification to arbitrary characteristic using the embedding theory of spherical homogeneous spaces.

Considered as a homogeneous space under $G \times G$ acting by left/right multiplication, G is a symmetric space (Example 26.10). All θ-stable maximal tori of $G \times G$ are of the form $T \times T$, where T is a maximal torus in G. The maximal θ-split tori are $(T \times T)_1 = \{(t^{-1}, t) \mid t \in T\}$. Choose a Borel subgroup $B \supseteq T$ of G. Then $B^- \times B$ is a

Borel subgroup in $G\times G$ containing $T\times T$ and $\theta(B^-\times B)=B\times B^-$ is the opposite Borel subgroup.

The weight lattice $\Lambda=\mathfrak{X}(T\times T/\operatorname{diag}T)=\{(-\lambda,\lambda)\mid\lambda\in\mathfrak{X}(T)\}$ is identified with $\mathfrak{X}(T)$ and the little root system with $\frac{1}{2}\Delta$. The eigenfunctions $\mathbf{f}_\lambda\in\Bbbk(G)^{(B^-\times B)}$ ($\lambda\in\mathfrak{X}(T)$) are defined on the "big" open cell $U^-\times T\times U\subseteq G$ by the formula $\mathbf{f}_\lambda(u^-tu)=\lambda(t)$. For $\lambda\in\mathfrak{X}_+$ they are matrix entries: $\mathbf{f}_\lambda(g)=\langle v_{-\lambda},gv_\lambda\rangle$, where $v_\lambda\in V$, $v_{-\lambda}\in V^*$ are $B^\pm$-eigenvectors of weights $\pm\lambda$.

By Theorem 26.25, the valuation cone $\mathscr{V}$ is identified with the antidominant Weyl chamber in $\mathfrak{X}^*(T)\otimes\mathbb{Q}$ (cf. Example 24.9) and the colors are represented by the simple coroots $\alpha_1^\vee,\dots,\alpha_l^\vee\in\Pi^\vee$. In fact, the respective colors are $D_i=\overline{B^-r_{\alpha_i}B}$. Indeed, the equation of D_i in $\Bbbk[\widetilde{G}]$ is $\mathbf{f}_{\omega_i}$, where ω_i denote the fundamental weights.

By Example 24.9, every $(G\times G)$-valuation is proportional to $v=v_\gamma$, where $\gamma\in\mathfrak{X}^*(T)$ is antidominant, and $v(\mathbf{f}_\lambda)=\langle\gamma,\lambda\rangle$, $\forall\lambda\in\mathfrak{X}_+$, whence v is identified with γ (as a vector in the valuation cone).

Now Corollary 15.5 yields

Theorem 27.12. *Normal reductive monoids X are in bijection with strictly convex cones $\mathscr{C}=\mathscr{C}(X)\subset\mathfrak{X}^*(T)\otimes\mathbb{Q}$ generated by all simple coroots and finitely many antidominant vectors.*

Remark 27.13. The normality assumption is not so restrictive, because the multiplication on X lifts to its normalization $\widetilde{X}$ turning it into a monoid with the same group of invertibles.

Corollary 27.14. *There are no non-trivial monoids with semisimple group of invertibles.*

Corollary 27.15 ([Put], [Rit1, Pr. 9]). *Every normal reductive monoid has the structure $X=(X_0\times G_1)/Z$, where X_0 is a monoid with zero, and Z is a finite central subgroup in $G(X_0)\times G_1$ not intersecting the factors.*

Proof. Identify $\mathfrak{X}(T)\otimes\mathbb{Q}$ with $\mathscr{E}=\mathfrak{X}^*(T)\otimes\mathbb{Q}$ via a W-invariant inner product. Consider an orthogonal decomposition $\mathscr{E}=\mathscr{E}_0\oplus\mathscr{E}_1$, where $\mathscr{E}_0=\langle\mathscr{C}\cap\mathscr{V}\rangle$, $\mathscr{E}_1=(\mathscr{C}\cap\mathscr{V})^\perp$. It is easy to see that each root is contained in one of the $\mathscr{E}_i$. Then $G=G_0\cdot G_1=(G_0\times G_1)/Z$, where G_i are the connected normal subgroups with $\mathfrak{X}^*(T\cap G_i)=\mathscr{E}_i\cap\mathfrak{X}^*(T)$. Take a reductive monoid $X_0\supseteq G_0$ defined by $\mathscr{C}_0=\mathscr{C}\cap\mathscr{E}_0$. Since $\operatorname{int}\mathscr{C}_0$ intersects $\mathscr{V}(G_0)=\mathscr{V}\cap\mathscr{E}_0$, the kernel of X_0 is a complete variety, and hence a single point 0, the zero element with respect to the multiplication on X_0. Now X coincides with $(X_0\times G_1)/Z$, because both monoids have the same colored data. □

This classification can be made more transparent via coordinate algebras and representations. Recall from 15.1 that $\Bbbk[X]^{U^-\times U}=\Bbbk[\mathscr{C}^\vee\cap\mathfrak{X}(T)]$. The algebra $\Bbbk[X]$ itself is given by (27.1). It remains to determine which representations of G extend to X.

Proposition 27.16. *The following conditions are equivalent:*

(1) *The representation $G:V$ is extendible to X.*

(2) *The highest weights of the simple factors of* V *are in* $\mathscr{C}^\vee$.
(3) *All dominant* T*-weights of* V *are in* $\mathscr{C}^\vee$.

Proof. (1) $\Longrightarrow$ (2) Choose a G-stable filtration of V with simple factors and consider the associated graded G-module $\operatorname{gr} V$. If $G : V$ extends to X, then $\operatorname{gr} V$ is an X-module. Hence $\mathbf{f}_\lambda \in \Bbbk[X]$ whenever λ is a highest weight of a simple factor of $\operatorname{gr} V$.
(2) $\Longleftrightarrow$ (3) All T-weights of V are obtained from the highest weights of simple factors by subtracting positive roots. The structure of $\mathscr{C}$ implies that all dominant vectors obtained this way from $\lambda \in \mathscr{C}^\vee$ belong to $\mathscr{C}^\vee$.
(3) $\Longrightarrow$ (1) It follows from Example 24.9 that $\overline{T} \subseteq X$ intersects all $(G\times G)$-orbits, cf. Proposition 27.18 below. Thus it suffices to prove that $G : V$ extends to $\overline{T}$. Choose a closed embedding $X \hookrightarrow \mathrm{L}(V_0)$ such that the dominant T-weights of V_0 generate $\mathscr{C}^\vee \cap \mathfrak{X}(T)$. Then, clearly, $\Bbbk[\overline{T}] = \Bbbk[W\mathscr{C}^\vee \cap \mathfrak{X}(T)]$. Since all T-weights of V are in $W\mathscr{C}^\vee$, they are well defined on $\overline{T}$. □

Corollary 27.17. *If* $X \subseteq \mathrm{L}(V)$ *is a closed submonoid, then* $\mathscr{C}^\vee = \mathscr{K}(V) \cap \mathbf{C}$*, where* $\mathscr{K}(V)$ *denotes the convex cone spanned by the* T*-weights of* V.

Proof. The proposition implies that $\mathscr{C}^\vee \supseteq \mathscr{K}(V) \cap \mathbf{C}$. On the other hand, all $(T\times T)$-weights of $\Bbbk[X]$ are of the form $(-\lambda,\mu)$, $\lambda,\mu \in \mathscr{K}(V)$, whence $\mathscr{C}^\vee \subseteq \mathscr{K}(V)$. □

In characteristic zero, Proposition 27.16 together with (27.1) yields

$$\Bbbk[X] = \bigoplus_{\lambda \in \mathscr{C}^\vee \cap \mathfrak{X}(T)} \mathrm{M}(V(\lambda)) \tag{27.2}$$

(cf. Theorem 2.15 and (2.3)). In positive characteristic, $\Bbbk[X]$ has a "good" filtration with factors $V^*(\lambda)\otimes V^*(\lambda^*)$ [Do], [Rit2, §4], [Ren3, Cor. 9.9].

27.3 Orbits. The embedding theory provides a combinatorial encoding for $(G\times G)$-orbits in X, which reflects the adherence relation. This description can be made more explicit using the following

Proposition 27.18. *Suppose that* $X \hookleftarrow G$ *is an equivariant normal embedding. Then* $F = \overline{T}$ *intersects each* $(G\times G)$*-orbit* $Y \subset X$ *in finitely many* T*-orbits permuted transitively by* W*. Exactly one of these orbits* $F_Y \subseteq F \cap Y$ *satisfies* $\operatorname{int}\mathscr{C}_{F_Y} \cap \mathscr{V} \neq \emptyset$*; then* $\mathscr{C}_{F_Y} = \bigcup w(\mathscr{C}_Y \cap \mathscr{V})$ *over all* $w \in W$ *such that* $w(F_Y) = F_Y$.

Remark 27.19. Since T is a flat of G (Proposition 26.28), some of the assertions stem from the results of §23. However, the proposition here is more precise. In particular, it completely determines the fan of F.

Proof. Take any $v \in \mathscr{S}_Y$; then $v = v_\gamma$, $\gamma \in \mathfrak{X}^*(T) \cap \mathscr{V}$, and $\exists \lim_{t\to 0}\gamma(t) = \gamma(0) \in Y$. The associated parabolic subgroup $P = P(\gamma)$ contains B^-. Consider the Levi decomposition $P = P_\mathrm{u} \leftthreetimes L$, $L \supseteq T$. One verifies that $(G\times G)_{\gamma(0)} \supseteq (P_\mathrm{u}^- \times P_\mathrm{u})\cdot \operatorname{diag} L$. It easily follows that $(B^- \times B)\gamma(0) = \mathring{Y}$ is the open $(B^-\times B)$-orbit in Y and $F_Y := T\gamma(0) = \mathring{Y}^{\operatorname{diag} T}$ is the unique T-orbit in F intersecting $\mathring{Y}$.

In view of Example 24.8, this implies $\operatorname{int}\mathscr{C}_{F_Y} \supseteq (\operatorname{int}\mathscr{C}_Y) \cap \mathscr{V}$. On the other hand, each T-orbit in $F \cap Y$ is accessed by a one-parameter subgroup $\gamma \in \mathfrak{X}^*(T)$, $\gamma(0) \in Y$. Taking $w \in W$ such that $w\gamma \in \mathscr{V}$ yields $w(T\gamma(0)) = F_Y$. All assertions of the proposition are deduced from these observations. □

Now suppose that $X \subseteq \mathrm{L}(V)$ is a closed normal submonoid and denote $\mathscr{K} = \mathscr{K}(V)$.

Theorem 27.20. *The $(G \times G)$-orbits in X are in bijection with the faces of $\mathscr{K}$ whose interiors intersect $\mathbf{C}$. The orbit Y corresponding to a face $\mathscr{F}$ is represented by the T-equivariant projector $e_{\mathscr{F}}$ of V onto the sum of T-eigenspaces of weights in $\mathscr{F}$. The cone $\mathscr{C}_Y$ is dual to the barrier cone $\mathscr{K} \cap \mathbf{C} - \mathscr{F} \cap \mathbf{C}$ of $\mathscr{K} \cap \mathbf{C}$ at the face $\mathscr{F} \cap \mathbf{C}$, and $\mathscr{D}_Y^B$ consists of the simple coroots orthogonal to $\mathscr{F}$.*

Proof. A complete set of T-orbit representatives in $F = \overline{T}$ is formed by the limits of one-parameter subgroups, i.e., by the $e_{\mathscr{F}}$ over *all* faces $\mathscr{F}$ of $\mathscr{K}$. The respective cones in the fan of F are the dual faces $\mathscr{F}^* = \mathscr{K}^\vee \cap \mathscr{F}^\perp$ of $\mathscr{K}^\vee = W(\mathscr{C} \cap \mathscr{V})$. By Proposition 27.18, the orbits Y are bijectively represented by those $e_{\mathscr{F}}$ which satisfy $\operatorname{int}\mathscr{F}^* \cap \mathscr{V} \neq \emptyset$. This happens if and only if $\mathscr{F}^*$ lies on a face of $\mathscr{C}$ of the same dimension (namely on $\mathscr{C}_Y$) or, equivalently, $\mathscr{F}$ contains a face of $\mathscr{C}^\vee = \mathscr{K} \cap \mathbf{C}$ of the same dimension (namely $\mathscr{C}_Y^* = \mathscr{F} \cap \mathbf{C}$), i.e., $\operatorname{int}\mathscr{F} \cap \mathbf{C} \neq \emptyset$. The assertion on $(\mathscr{C}_Y, \mathscr{D}_Y)$ stems from the description of a dual face. □

Example 27.21. Let $G = \mathrm{GL}_n(\Bbbk)$ and $X = \mathrm{L}_n(\Bbbk)$. For B and T take the standard Borel subgroup of upper-triangular matrices and diagonal torus, respectively. We have $\mathfrak{X}(T) = \langle \varepsilon_1, \dots, \varepsilon_n \rangle$, where the ε_i are the diagonal matrix entries of T. We identify $\mathfrak{X}(T)$ with $\mathfrak{X}^*(T)$ via the inner product such that the ε_i form an orthonormal basis. Let $(k_1, \dots, k_n)$ denote the coordinates on $\mathfrak{X}(T) \otimes \mathbb{Q}$ with respect to this basis. The Weyl group $W = S_n$ permutes them.

The weights $\lambda_i = \varepsilon_1 + \dots + \varepsilon_i$ span $\mathfrak{X}(T)$ and $\mathbf{f}_{\lambda_i} \in \Bbbk[X]$ are the upper-left corner i-minors of a matrix. Put $D_i = \{x \in X \mid \mathbf{f}_{\lambda_i}(x) = 0\}$. Then $\mathscr{D}^B = \{D_1, \dots, D_{n-1}\}$, D_i are represented by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\forall i < n$, and D_n is the unique G-stable prime divisor, $v_{D_n} = \varepsilon_n$.

Therefore $\mathscr{C} = \{k_1 + \dots + k_i \geq 0,\ i = 1, \dots, n\}$ is the cone spanned by $\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n$, and $\mathscr{C}^\vee = \{k_1 \geq \dots \geq k_n \geq 0\}$ is spanned by λ_i. The lattice vectors of $\mathscr{C}^\vee$ are exactly the dominant weights of polynomial representations (cf. Proposition 27.16). The lattice vectors of $\mathscr{K} = W\mathscr{C}^\vee = \{k_1, \dots, k_n \geq 0\}$ are all polynomial weights of T.

The $(G \times G)$-orbits in X are $Y_r = \{x \in X \mid \operatorname{rk} x = r\}$. Clearly, $\mathscr{D}_{Y_r}^B = \{D_i \mid r < i < n\}$ and $\mathscr{C}_{Y_r}$ is a face of $\mathscr{C}$ cut off by the equations $k_1 = \dots = k_r = 0$. The dual face $\mathscr{C}_{Y_r}^*$ of $\mathscr{C}^\vee$ is the dominant part of the face $\mathscr{F}_r = \{k_i \geq 0 = k_j \mid i \leq r < j\} \subseteq \mathscr{K}$, and all faces of $\mathscr{K}$ whose interiors intersect $\mathbf{C} = \{k_1 \geq \dots \geq k_n\}$ are obtained this way. Clearly, the respective projectors $e_{\mathscr{F}_r} = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)$ are the $(G \times G)$-orbit representatives, and the representatives of all T-orbits in $\overline{T}$ are obtained from $e_{\mathscr{F}_r}$ by the W-action.

27.4 Normality and Smoothness. In characteristic zero, it is possible to classify (to a certain extent) arbitrary (not necessarily normal) reductive monoids [Vin2] via their coordinate algebras similarly to (27.2). The question is to describe finitely generated $(G \times G)$-stable subalgebras of $\Bbbk[G]$ with the quotient field $\Bbbk(G)$. They are of the form

$$\Bbbk[X] = \bigoplus_{\lambda \in \Sigma} \mathrm{M}(V(\lambda)), \tag{27.3}$$

where Σ is a finitely generated subsemigroup of $\mathfrak{X}_+$ such that $\mathbb{Z}\Sigma = \mathfrak{X}(T)$ and the r.h.s. of (27.3) remains closed under multiplication, i.e., all highest weights of $V(\lambda) \otimes V(\mu)$ belong to Σ whenever $\lambda, \mu \in \Sigma$. Such a semigroup Σ is called *perfect*.

Definition 27.22. We say that $\lambda_1, \dots, \lambda_m$ *G-generate* Σ if Σ consists of all highest weights of G-modules $V(\lambda_1)^{\otimes k_1} \otimes \dots \otimes V(\lambda_m)^{\otimes k_m}$, $k_1, \dots, k_m \in \mathbb{Z}_+$. (In particular any generating set G-generates Σ.) All weights in Σ are of the form $\sum k_i \lambda_i - \sum l_j \alpha_j$, $k_i, l_j \in \mathbb{Z}_+$.

Example 27.23. In Example 27.21, $\Sigma = \mathscr{C}^\vee \cap \mathfrak{X}(T)$ is generated by $\lambda_1, \dots, \lambda_n$ and G-generated by λ_1.

It is easy to see that $X \hookrightarrow \mathrm{L}(V)$ if and only if the highest weights $\lambda_1, \dots, \lambda_m$ of the simple summands of V G-generate Σ. The highest weight theory implies that $\mathscr{K} = \mathscr{K}(V)$ is the W-span of

$$\mathscr{K} \cap \mathbf{C} = (\mathbb{Q}_+\{\lambda_1, \dots, \lambda_m, -\alpha_1, \dots, -\alpha_l\}) \cap \mathbf{C}. \tag{27.4}$$

Theorem 27.20 generalizes to this context.

By Theorem D.5(3), X is normal if and only if $\Bbbk[X]^{U^- \times U} = \Bbbk[\Sigma]$ is integrally closed, i.e., if and only if Σ is the semigroup of all lattice vectors in a polyhedral cone. In general, taking the integral closure yields

$$\mathbb{Q}_+\Sigma = \mathscr{C}^\vee = \mathscr{K} \cap \mathbf{C},$$

where $\mathscr{C}$ is the cone associated with the normalization of X. Indeed, the inclusion $\mathbb{Q}_+\Sigma \subseteq \mathscr{K} \cap \mathbf{C}$ stems from the structure of Σ, $\mathscr{K} \cap \mathbf{C} \subseteq \mathscr{C}^\vee$ stems from the structure of $\mathscr{C}$, and $\mathbb{Q}_+\Sigma = \mathscr{C}^\vee$ is due to Lemma D.6. Here is a representation-theoretic interpretation: a multiple of each dominant vector in $\mathscr{K}$ eventually occurs as a highest weight in a tensor power of V, see [Tim4, §2] for a direct proof.

Given $G : V$, the above normality condition for $X \subseteq \mathrm{L}(V)$ is generally not easy to verify, because the reconstruction of Σ from $\{\lambda_1, \dots, \lambda_m\}$ requires decomposing tensor products of arbitrary G-modules. Of course, there is no problem if λ_i already generate $\mathscr{K} \cap \mathfrak{X}_+$—a sufficient condition for normality. Here is an effective necessary condition:

Proposition 27.24 ([Ren2], [Ren3, Th. 5.4(b)]). *If X is normal, then $F = \overline{T}$ is normal, i.e., the T-weights of V generate $\mathscr{K} \cap \mathfrak{X}(T)$.*

Proof. We can increase V by adding new highest weights λ_i so that $\lambda_1, \dots, \lambda_m$ will generate $\Sigma = \mathscr{K} \cap \mathfrak{X}_+$. (This operation does not change X and F.) Then

$W\{\lambda_1,\dots,\lambda_m\}$ generates $\mathscr{K}\cap\mathfrak{X}(T)$, i.e., $\Bbbk[F]=\Bbbk[\mathscr{K}\cap\mathfrak{X}(T)]$ is integrally closed. □

If $V=V(\lambda)$ is irreducible, then the center of G acts by homotheties, whence $G=\Bbbk^\times\cdot G_0$, where G_0 is semisimple, $\mathfrak{X}(T)\subseteq\mathbb{Z}\oplus\mathfrak{X}(T\cap G_0)$ is a cofinite sublattice, and $\lambda=(1,\lambda_0)$. De Concini showed that $\mathscr{K}(V(\lambda))\cap\mathfrak{X}_+$ is G-generated by the T-dominant weights of $V(\lambda)$ [Con]. However Σ contains no T-weights of $V(\lambda)$ except λ. It follows that X is normal if and only if λ_0 is a minuscule weight for G_0 [Con], [Tim4, §12].

It turns out that Example 27.21 is essentially the unique non-trivial example of a smooth reductive monoid.

Theorem 27.25 (cf. [Ren2], [Tim4, §11]). *Smooth reductive monoids are of the form $X=(G_0\times\mathrm{L}_{n_1}(\Bbbk)\times\cdots\times\mathrm{L}_{n_s}(\Bbbk))/Z$, where $Z\subset G_0\times\mathrm{GL}_{n_1}(\Bbbk)\times\cdots\times\mathrm{GL}_{n_s}(\Bbbk)$ is a finite central subgroup not intersecting $\mathrm{GL}_{n_1}(\Bbbk)\times\cdots\times\mathrm{GL}_{n_s}(\Bbbk)$.*

Proof. By Corollary 27.15, $X=G_0*_Z X_0$, where X_0 has the zero element. Thus it suffices to consider monoids with zero. We explain how to handle this case in characteristic zero.

Assume that $X\subseteq\mathrm{L}(V)$. There exists a coweight $\gamma\in\operatorname{int}\mathscr{C}\cap\mathscr{V}$, $\gamma\perp\Delta$. It defines a one-parameter subgroup $\gamma(t)\in Z(G)$ contracting V to 0 (as $t\to 0$). The algebra $\mathscr{A}=\mathscr{A}(V)$ spanned by X in $\mathrm{L}(V)$ is semisimple, i.e., a product of matrix algebras, and T_0X is an ideal in $\mathscr{A}$. As X is smooth and the multiplication by $\gamma(t)$ contracts X to 0, the equivariant projection $X\to T_0X$ is an isomorphism. □

27.5 Group Embeddings. We conclude this section with a discussion of arbitrary (not necessarily affine) equivariant embeddings of G. For simplicity, we assume that $\operatorname{char}\Bbbk=0$.

In the same way as a faithful linear representation $G:V$ defines a reductive monoid $\overline{G}\subseteq\mathrm{L}(V)$, a faithful projective representation $G:\mathbb{P}(V)$ (arising from a linear representation of a finite cover of G in V) defines a projective completion $X=\overline{G}\subseteq\mathbb{P}(\mathrm{L}(V))$. These group completions are studied in [Tim4]. There are two main tools to reduce their study to reductive monoids.

First, the cone $\widehat{X}\subseteq\mathrm{L}(V)$ over X is a reductive monoid whose group of invertibles $\widehat{G}$ is the extension of G by homotheties. Conversely, any such monoid gives rise to a projective completion. This allows the transfer of some of the above results to projective group completions. For instance, Theorem 27.20 transfers verbatim if we only replace the weight cone $\mathscr{K}(V)$ by the weight polytope $\mathscr{P}=\mathscr{P}(V)$ (= the convex hull of the T-weights of V), see [Tim4, §9] for details.

Another approach, suitable for local study, is to use the local structure theorem. By the above, closed $(G\times G)$-orbits $Y\subset X$ correspond to the dominant vertices $\lambda\in\mathscr{P}$, and the representatives are $y=[v_\lambda\otimes v_{-\lambda}]$, where $v_\lambda\in V$, $v_{-\lambda}\in V^*$ are $B^\pm$-eigenvectors of weights $\pm\lambda$, $\langle v_\lambda,v_{-\lambda}\rangle\neq 0$. Consider the parabolic $P=P(\lambda)$ and its Levi decomposition $P=P_{\mathrm{u}}\rtimes L$, $L\supseteq T$. Then $V_0=\langle v_{-\lambda}\rangle^\perp$ is an L-stable complement to $\langle v_\lambda\rangle$ in V. Put $\mathring{X}=X_{\mathbf{f}_\lambda}$.

Lemma 27.26. $\mathring{X}\simeq P_{\mathrm{u}}^-\times Z\times P_{\mathrm{u}}$, *where $Z\simeq\overline{L}\subseteq\mathrm{L}(V_0\otimes\Bbbk_{-\lambda})$ is a reductive monoid with the zero element y.*

Proof. Applying Corollary 4.5 to the projectivization of $G\times G : \mathrm{L}(V) = V\otimes V^*$ and intersecting with X, we obtain a neighborhood of the desired structure with $Z = X\cap\mathbb{P}\big(\Bbbk^\times(v_\lambda\otimes v_{-\lambda})+E_0\big)$, where

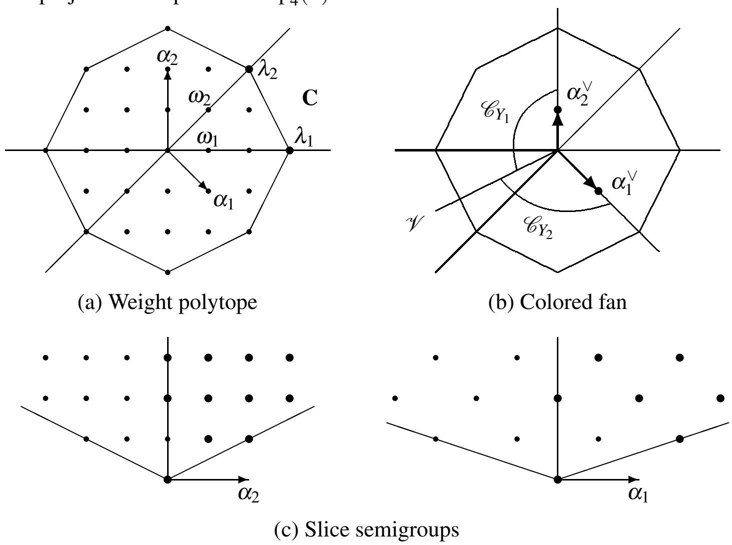
$$E_0 = \big((\mathfrak{g}\times\mathfrak{g})(v_{-\lambda}\otimes v_\lambda)\big)^\perp = (\mathfrak{g}v_{-\lambda}\otimes v_\lambda + v_{-\lambda}\otimes\mathfrak{g}v_\lambda)^\perp \supseteq V_0\otimes V_0^* = \mathrm{L}(V_0).$$

Hence $Z=\overline{L}\subseteq\mathbb{P}\big(\Bbbk^\times(v_\lambda\otimes v_{-\lambda})\oplus\mathrm{L}(V_0)\big)\simeq\mathrm{L}(V_0\otimes\Bbbk_{-\lambda})$. □

The monoids Z are transversal slices to the closed orbits in X. They can be used to study the local geometry of X. For instance, one can derive criteria for normality and smoothness [Tim4, §§10,11].

Example 27.27. Take $G=\mathrm{Sp}_4(\Bbbk)$, with the simple roots $\alpha_1=\varepsilon_1-\varepsilon_2$, $\alpha_2=2\varepsilon_2$, and the fundamental weights $\omega_1=\varepsilon_1$, $\omega_2=\varepsilon_1+\varepsilon_2$, $\pm\varepsilon_i$ being the weights of the tautological representation $\mathrm{Sp}_4(\Bbbk):\Bbbk^4$. Let $\lambda_1=3\omega_1$, $\lambda_2=2\omega_2$ be the highest weights of the simple summands of V. The weight polytope $\mathscr{P}$ is depicted in Fig. 27.1(a), the highest weights are indicated by bold dots. There are two closed orbits $Y_1,Y_2\subset X$.

Fig. 27.1 A projective completion of $\mathrm{Sp}_4(\Bbbk)$



The respective Levi subgroups are $L_1=\mathrm{SL}_2(\Bbbk)\times\Bbbk^\times$ and $L_2=\mathrm{GL}_2(\Bbbk)$, with the simple roots α_2 and α_1, respectively.

Consider the slice monoids Z_i for Y_i. The weight semigroups of $F_i=\overline{T}$ (the closure in Z_i) are plotted by dots in Fig. 27.1(c), the bold dots corresponding to the weight semigroups Σ_i of Z_i. (They are easily computed using the Clebsch–Gordan formula.) We can now see that F_i are normal, but Z_i are not, i.e., X is non-normal along Y_1, Y_2. However, if we increase V by adding two highest weights $\lambda_3 = 2\omega_1$, $\lambda_4 = \omega_1 + \omega_2$, then X becomes normal. Its colored fan is depicted in Fig. 27.1(b).

The projective completions of adjoint simple groups in projective linear operators on fundamental and adjoint representation spaces were studied in detail in [Tim4, §12]. In particular, the orbital decomposition was described, and normal and smooth completions were identified.

Example 27.28. Suppose that $G = \mathrm{SO}_{2l+1}(\Bbbk)$ and $V = V(\omega_i)$ is a fundamental representation. We have a unique closed orbit $Y \subset X$. If $i < l$, then $L \not\simeq \mathrm{GL}_{n_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{n_s}(\Bbbk)$. Hence Z and X are singular. But for $i = l$ (the spinor representation), $L \simeq \mathrm{GL}_l(\Bbbk)$ and $V(\omega_l) \otimes \Bbbk_{-\omega_l}$ is L-isomorphic to $\bigwedge^\bullet \Bbbk^l$. It follows that $Z \simeq \mathrm{L}_l(\Bbbk)$, whence X is smooth.

Example 27.29. Suppose that all vertices of $\mathscr{P}$ are regular weights. Then the slice monoids Z are toric and their weight semigroups Σ are generated by the weights $\mu - \lambda$, where μ runs over all T-weights of V. The variety X is toroidal, and normal (smooth) if and only if each Σ consists of all lattice vectors in the barrier cone of $\mathscr{P}$ at λ (resp. Σ is generated by linearly independent weights).

In particular, if $V = V(\lambda)$ is a simple module of regular highest weight, then $\Sigma = \mathbb{Z}_+(-\Pi)$, whence X is smooth. This is a particular case of a wonderful completion, see §30.

A interesting model for the wonderful completion of G in terms of Hilbert schemes was proposed by Brion [Bri17]. Namely, given a generalized flag variety $M = G/Q$, he proves that the closure $X = \overline{(G \times G)[\operatorname{diag} M]}$ in the Hilbert scheme (or the Chow variety) of $M \times M$ is isomorphic to the wonderful completion. If $G = (\operatorname{Aut} M)^0$ (e.g., if $Q = B$), then X is an irreducible component of the Hilbert scheme (the Chow variety). All fibers of the universal family over X are reduced and Cohen–Macaulay (even Gorenstein if $Q = B$).

Toroidal and wonderful group completions were studied intensively in the framework of the general theory of toroidal and wonderful varieties (see §29–§30) and on their own. De Concini and Procesi [CP3] and Strickland [Str2] computed ordinary and equivariant rational cohomology of smooth toroidal completions over $\Bbbk = \mathbb{C}$ (see also [BCP], [LP]). Brion [Bri14] carried out a purely algebraic treatment of these results replacing cohomology by (equivariant) Chow rings.

The basis of the Chow ring $A(X)$ of a smooth toroidal completion $X = \overline{G}$ is given by the closures of the Białynicki-Birula cells [B-B1], which are isomorphic to affine spaces and intersect $(G \times G)$-orbits in $(B^- \times B)$-orbits [BL, 2.3]. The latter were described in [Bri14, 2.1]. The $(B^- \times B)$-orbit closures in X are smooth in codimension 1, but singular in codimension 2 (apart from trivial exceptions arising from $G = \mathrm{PSL}_2(\Bbbk)$) [Bri14, §2]. For wonderful X, the Białynicki-Birula cells are described in [Bri14, 3.3], and the closures of the cells intersecting G (= the closures in X of $(B^- \times B)$-orbits in G) are normal and Cohen–Macaulay [BPo]. The geometry of $(B^- \times B)$-orbit closures in X was studied in [Sp4], [Ka].

The class of reductive group embeddings is not closed under degenerations. Alexeev and Brion [AB1], [AB2] introduced a more general class of (*stable*) *reductive varieties* closed under flat degenerations with irreducible (resp. reduced) fibers. Affine (stable) reductive varieties may be defined as normal affine spherical $(G\times G)$-varieties X such that $\Lambda(X)=\Lambda(G)\cap\mathscr{S}$ for some subspace $\mathscr{S}\subseteq\Lambda(G)\otimes\mathbb{Q}$ (resp. as seminormal connected unions of reductive varieties); projective (stable) reductive varieties are the projectivizations of affine ones. Affine reductive varieties provide examples of algebraic semigroups without unit.

Alexeev and Brion gave a combinatorial classification and described the orbital decomposition for stable reductive varieties in the spirit of Theorems 27.12, 27.20. They constructed moduli spaces for affine stable reductive varieties embedded in a $(G\times G)$-module and for stable reductive pairs, i.e., projective stable reductive varieties with a distinguished effective ample divisor containing no $(G\times G)$-orbit.

27.6 Enveloping and Asymptotic Semigroups. An interesting family of reductive varieties was introduced by Vinberg [Vin2]. Consider the group $\widehat{G}=(G\times T)/Z$, where $Z=\{(t^{-1},t)\mid t\in Z(G)\}$. The cone $\mathscr{C}\subset\mathscr{E}(\widehat{G})$ spanned by (the projections to $\mathscr{E}$ of) $(\alpha_i^\vee,0)$ and $(-\omega_j^\vee,\omega_j^\vee)$, where $\omega_j^\vee$ are the fundamental coweights, defines a normal reductive monoid $\operatorname{Env}G$, called the *enveloping semigroup* of G, with group of invertibles $\widehat{G}$. The projection $\mathscr{E}(\widehat{G})\to\mathscr{E}(T/Z(G))$ maps $\mathscr{C}$ onto $\mathbf{C}$. Hence by Theorem 15.10 we have an equivariant map $\pi_G:\operatorname{Env}G\to\mathbb{A}^l$, where $G\times G$ acts on $\mathbb{A}^l$ trivially and T acts with the weights $-\alpha_1,\dots,-\alpha_l$.

The algebra $\Bbbk[\operatorname{Env}G]=\bigoplus_{\chi\in\lambda+\mathbb{Z}_+\Pi}\mathrm{M}(V(\lambda))\otimes\Bbbk\chi$ is a free module over $\Bbbk[\mathbb{A}^l]=\Bbbk[\mathbb{Z}_+\Pi]$ and $\Bbbk[\operatorname{Env}G]^{U^-\times U}=\Bbbk[\mathfrak{X}_+]\otimes\Bbbk[\mathbb{Z}_+\Pi]$, i.e., all schematic fibers of π_G have the same algebra of $(U^-\times U)$-invariants $\Bbbk[\mathfrak{X}_+]$. Hence π_G is flat and all its fibers are reduced and irreducible by Theorem D.5(1), i.e., $\operatorname{Env}G$ is the total space of a family of reductive varieties, in the sense of Appendix E.3. (In fact, $\mathbb{A}^l=(\operatorname{Env}G)/\!\!/(G\times G)$ and π_G is the categorical quotient map.)

It is easy to see that the fibers of π_G over points with nonzero coordinates are isomorphic to G. Degenerate fibers are obtained from G by a deformation of the multiplication law in $\Bbbk[G]$. In particular, the "most degenerate" fiber $\operatorname{As}G:=\pi_G^{-1}(0)$, called the *asymptotic semigroup* of G, is just the horospherical contraction of G (see 7.3). In a sense, the asymptotic semigroup reflects the behavior of G at infinity.

The enveloping semigroup is used in [AB1, 7.5] to construct families of affine reductive varieties with given general fiber X: $\operatorname{Env}X=(\operatorname{Env}G\times X)/\!\!/G$, where G acts as $\{e\}\times\operatorname{diag}G\times\{e\}\subset G\times G\times G\times G$, so that $\Bbbk[\operatorname{Env}X]=\bigoplus_{\chi\in\lambda+\mathbb{Z}_+\Pi}\Bbbk[X]_{(\lambda)}\otimes\Bbbk\chi\subseteq\Bbbk[X\times T]$. The map π_G induces a flat morphism $\pi_X:\operatorname{Env}X\to\mathbb{A}^l$ with reduced and irreducible fibers.

It was proved in [AB1, 7.6] that π_X is a locally universal family of reductive varieties with general fiber X, i.e., every flat family of affine reductive varieties with reduced fibers over an irreducible base is locally a pullback of π_X. The universal property for enveloping semigroups was already noticed in [Vin2].

Example 27.30. Let us describe the enveloping and asymptotic semigroups of $G=\mathrm{SL}_n(\Bbbk)$ for small n, using the notation of Example 27.21. Here $\Lambda_+(\operatorname{Env}G)$ is generated by (ω_i,ω_i), $(0,\alpha_i)$, $i=1,\dots,n-1$. Recall that $\omega_i=\varepsilon_1+\dots+\varepsilon_i$ is the highest weight of $\bigwedge^i\Bbbk^n$. Thus $\operatorname{Env}\mathrm{SL}_n(\Bbbk)$ is the closure in $\mathrm{L}(V)$ of the image of $\mathrm{SL}_n(\Bbbk)\times T$ acting on $V=\bigwedge^\bullet\Bbbk^n\oplus\Bbbk^{n-1}$, where $\mathrm{SL}_n(\Bbbk)$ acts on $\bigwedge^\bullet\Bbbk^n$ in a natural way, and T

acts on $\bigwedge^k \Bbbk^n$ by the weight $\varepsilon_1 + \cdots + \varepsilon_k$ and on $\Bbbk^{n-1}$ by the weights $\varepsilon_i - \varepsilon_{i+1}$. In other words, the image of $\mathrm{SL}_n(\Bbbk) \times T$ consists of tuples of the form

$$(t_1 g, \dots, t_1 \cdots t_k \bigwedge^k g, \dots, t_1/t_2, \dots, t_{n-1}/t_n),$$

where $g \in \mathrm{SL}_n(\Bbbk)$, $t = \mathrm{diag}(t_1, \dots, t_n) \in T$ $(t_1 \cdots t_n = 1)$. It follows that $\mathrm{Env}\,\mathrm{SL}_2(\Bbbk) = \{(a,z) \mid a \in \mathrm{L}_2(\Bbbk),\ z \in \Bbbk,\ \det a = z\} \simeq \mathrm{L}_2(\Bbbk)$ and $\mathrm{As}\,\mathrm{SL}_2(\Bbbk)$ is the subsemigroup of degenerate matrices. Under the identification $\bigwedge^2 \Bbbk^3 \simeq (\Bbbk^3)^* \simeq \Bbbk^3$,

$$\mathrm{Env}\,\mathrm{SL}_3(\Bbbk) = \big\{(a_1,a_2,z_1,z_2) \bigm| a_i \in \mathrm{L}_3(\Bbbk),\ z_i \in \Bbbk,\\ \textstyle\bigwedge^2 a_i = z_i a_j\ (i \neq j),\ a_1^\top a_2 = a_1 a_2^\top = z_1 z_2 e\big\}$$

and $\mathrm{As}\,\mathrm{SL}_3(\Bbbk) = \big\{(a_1,a_2) \bigm| a_i \in \mathrm{L}_3(\Bbbk),\ \mathrm{rk}\,a_i \leq 1,\ a_1^\top a_2 = a_1 a_2^\top = 0\big\}$.

28 S-varieties

28.1 General S-varieties. Horospherical varieties of complexity 0 form another class of spherical varieties whose structure and embedding theory is understood better than in the general case.

Definition 28.1. An *S-variety* is an equivariant embedding of a horospherical homogeneous space $O = G/S$.

This terminology is due to Popov and Vinberg [VP], though they considered only the affine case. General S-varieties were studied by Pauer [Pau1], [Pau2] in the case where S is a maximal unipotent subgroup of G.

S-varieties are spherical. We shall examine them from the viewpoint of the Luna–Vust theory. In order to apply it, we have to describe the colored space $\mathscr{E} = \mathscr{E}(O)$.

It is convenient to assume that $S \supseteq U^-$; then $S = P_\mathrm{u}^- \rtimes L_0$ for a certain parabolic $P \supseteq B$ with the Levi subgroup $L \supseteq L_0 \supseteq L'$ and the unipotent radical P_u (Lemma 7.4). We may assume that $L \supseteq T$. Put $T_0 = T \cap L_0$.

We have $\Lambda(O) = \mathfrak{X}(A)$, where $A = P^-/S \simeq L/L_0 \simeq T/T_0$. By Theorem 21.10, $\mathscr{V}(O) = \mathscr{E}$. The space $\mathscr{E} = \mathfrak{X}^*(A) \otimes \mathbb{Q}$ may be identified with the orthocomplement of $\mathfrak{X}^*(T_0) \otimes \mathbb{Q}$ in $\mathfrak{X}^*(T) \otimes \mathbb{Q}$. It follows from the Bruhat decomposition that the colors on O are of the form $D_\alpha = \overline{B r_\alpha o}$, $\alpha \in \Pi \setminus \Pi_0$, where $\Pi_0 \subseteq \Pi$ is the simple root set of L. An argument similar to that in 27.2 shows that D_α maps to $\overline{\alpha^\vee}$, the image of $\alpha^\vee$ under the projection $\mathfrak{X}^*(T) \to \mathfrak{X}^*(A)$.

Theorem 15.4(3) says that normal S-varieties are classified by colored fans in $\mathscr{E}$, each fan consisting of finitely many colored cones $(\mathscr{C}_i, \mathscr{R}_i)$, so that the cones $\mathscr{C}_i$ form a polyhedral fan in $\mathscr{E}$, $\mathscr{R}_i \subseteq \mathscr{D}^B$, and each $\mathscr{C}_i \setminus \{0\}$ contains all $\overline{\alpha^\vee}$ such that $D_\alpha \in \mathscr{R}_i$. The colored cones in a fan correspond to the G-orbits Y_i in the respective S-variety X, and X is covered by simple open S-subvarieties $X_i = \{x \in X \mid \overline{Gx} \supseteq Y_i\}$.

The following result "globalizing" Theorem 15.17 is a nice example of how the combinatorial embedding theory of §15 helps to clarify the geometric structure of

S-varieties. For any G-orbit $Y \subseteq X$ let $P(Y) = P[\mathscr{D}^B \setminus \mathscr{D}^B_Y]$ be the normalizer of the open B-orbit in Y and let $S(Y) \subseteq P(Y)$ be the normalizer of general U-orbits, so that $S(Y)^-$ is the stabilizer of $G : Y$ (see 7.2). The Levi subgroup $L(Y) \subseteq P(Y)$ containing T has the simple root set $\Pi_0 \cup \{\alpha \in \Pi \mid D_\alpha \in \mathscr{D}^B_Y\}$, and $S(Y) = P(Y)_{\mathrm{u}} \rightthreetimes L(Y)_0$, where the Levi subgroup $L(Y)_0$ is intermediate between $L(Y)$ and $L(Y)'$ and is in fact the common kernel of all characters in $\Lambda(Y) = \mathfrak{X}(A) \cap \mathscr{C}_Y^\perp$.

Theorem 28.2 (cf. [Pau1, 5.4]). *Let X be a simple normal S-variety with the unique closed G-orbit $Y \subseteq X$.*

(1) *There exists a $P(Y)^-$-stable affine closed subvariety $Z \subseteq X$ such that $P(Y)^-_{\mathrm{u}}$ acts on Z trivially and $X \simeq G *_{P(Y)^-} Z$.*
(2) *There exists an $S(Y)^-$-stable closed subvariety $Z_0 \subseteq Z$ with a fixed point such that $Z \simeq P(Y)^- *_{S(Y)^-} Z_0 \simeq L(Y) *_{L(Y)_0} Z_0$ and $X \simeq G *_{S(Y)^-} Z_0$.*
(3) *The varieties Z and Z_0 are equivariant affine embeddings of $L(Y)/L(Y) \cap S$ and $L(Y)_0/L(Y)_0 \cap S$ whose weight lattices are $\mathfrak{X}(A)$ and $\mathfrak{X}(A)/\mathfrak{X}(A) \cap \mathscr{C}_Y^\perp$, colored spaces are $\mathscr{E}$ and $\mathscr{E}_0 := \langle \mathscr{C}_Y \rangle$, and colored cones coincide with $(\mathscr{C}_Y, \mathscr{D}^B_Y)$.*

Proof. The idea of the proof is to construct normal affine S-varieties Z and Z_0 with the colored data as in (3) and then to verify that the colored data of $L(Y) *_{L(Y)_0} Z_0$ coincide with those of Z and the colored data of $G *_{P(Y)^-} Z$ with those of X. In each case both varieties under consideration are simple normal embeddings of one and the same homogeneous space. The restriction of B-eigenfunctions to the fiber of each homogeneous bundle above preserves the orders along B-stable divisors. It follows that the colored cones of both varieties coincide with the colored cone of the fiber, whence the varieties are isomorphic. Note that Z_0 contains a fixed point since it is determined by a colored cone of full dimension. □

28.2 Affine Case. The theorem shows that the local geometry of (normal) S-varieties is completely reduced to the affine case (even to affine S-varieties with a fixed point). Affine S-varieties were studied in [VP] in characteristic 0 and in [Gr2, §17] in arbitrary characteristic.

First note that O is quasiaffine if and only if all $\overline{\alpha^\vee}$ are nonzero (whenever $\alpha \in \Pi \setminus \Pi_0$) and generate a strictly convex cone in $\mathscr{E}$ (Corollary 15.6). This holds if and only if there exists a dominant weight λ such that $\langle \lambda, \Pi^\vee \setminus \Pi_0^\vee \rangle > 0$ and $\lambda|_{T_0} = 1$, i.e, if and only if S is regularly embedded in the stabilizer of a lowest vector of weight $-\lambda$ (cf. Theorem 3.13).

Theorem 28.3. *Let X be the normal affine S-variety determined by a colored cone $(\mathscr{C}, \mathscr{D}^B)$. Then*

$$\Bbbk[X] \simeq \bigoplus_{\lambda \in \mathfrak{X}(A) \cap \mathscr{C}^\vee} V^*(\lambda^*) \subseteq \Bbbk[G/S] = \bigoplus_{\lambda \in \mathfrak{X}(A) \cap \mathbf{C}} V^*(\lambda^*).$$

Here $V^*(\lambda^*)$ is identified with $\Bbbk[G]^{(B^-)}_{-\lambda}$ via right translation by w_G on G. If the semigroup $\mathfrak{X}(A) \cap \mathscr{C}^\vee$ is generated by dominant weights $\lambda_1, \dots, \lambda_m$, then $X \simeq \overline{Gv} \subseteq V(\lambda_1^*) \oplus \cdots \oplus V(\lambda_m^*)$, where $v = v_{-\lambda_1} + \cdots + v_{-\lambda_m}$ is the sum of respective lowest vectors.

Proof. Observe that $R=\bigoplus_{\lambda\in\mathfrak{X}(A)\cap\mathscr{C}^\vee}V^*(\lambda^*)$ is the largest subalgebra of $\Bbbk[G/S]$ with the given algebra of U-invariants $R^U=\Bbbk[X]^U\simeq\Bbbk[\mathfrak{X}(A)\cap\mathscr{C}^\vee]$. Hence $R\supseteq\Bbbk[X]\supseteq\langle G\cdot R^U\rangle$, and the extension is integral by Lemma D.4. Now $R=\Bbbk[X]$ since $\Bbbk[X]$ is integrally closed.

It is easy to see that $\overline{Gv}$ is an affine embedding of O such that $\Bbbk[\overline{Gv}]$ is generated by $V^*(\lambda_1^*)\oplus\cdots\oplus V^*(\lambda_m^*)\subset\Bbbk[O]$. By Lemma 2.23, $\Bbbk[\overline{Gv}]=R$. □

Every (even non-normal) affine S-variety with the open orbit O is realized in a G-module V as $X=\overline{Gv}$, $v\in V^S$. We may assume that $V=\langle Gv\rangle$ and decompose $v=v_{-\lambda_1}+\cdots+v_{-\lambda_m}$, where $v_{-\lambda_i}$ are B^--eigenvectors of certain antidominant weights $-\lambda_i$.

In characteristic zero, $V\simeq V(\lambda_1^*)\oplus\cdots\oplus V(\lambda_m^*)$ and the same arguments as in the proof of Theorem 28.3 show that $\Bbbk[X]=\bigoplus_{\lambda\in\Sigma}V^*(\lambda^*)$, where Σ is the semigroup generated by $\lambda_1,\dots,\lambda_m$, and the dual Weyl modules $V^*(\lambda^*)\simeq V(\lambda)$ are the (simple) G-isotypic components of $\Bbbk[X]$. It is easy to see that $Gv\simeq O$ if and only if $\lambda_1,\dots,\lambda_m$ span $\mathfrak{X}(A)$. Thus we obtain the following

Proposition 28.4 ([VP, 3.1, 3.4]). *In characteristic zero, affine S-varieties X with the open orbit O bijectively correspond to finitely generated semigroups Σ of dominant weights spanning $\mathfrak{X}(A)$, via $\Sigma=\Lambda_+(X)$. The variety X is normal if and only if the semigroup Σ is saturated, i.e., $\Sigma=\mathbb{Q}_+\Sigma\cap\mathfrak{X}(A)$. Moreover, the saturation $\widetilde{\Sigma}=\mathbb{Q}_+\Sigma\cap\mathfrak{X}(A)$ of Σ corresponds to the normalization $\widetilde{X}$ of X.*

G-orbits in an affine S-variety $X=\overline{Gv}\subseteq V$ have a transparent description "dual" to that in Theorem 15.4(4).

Proposition 28.5 ([VP, Th. 8]). *The orbits in X are in bijection with the faces of $\mathscr{C}^\vee=\mathbb{Q}_+\lambda_1+\cdots+\mathbb{Q}_+\lambda_m$. The orbit corresponding to a face $\mathscr{F}$ is represented by $v_{\mathscr{F}}=\sum_{\lambda_i\in\mathscr{F}}v_{-\lambda_i}$. The adherence of orbits agrees with the inclusion of faces.*

Proof. We have $X=G\overline{Tv}$ since $\overline{Tv}$ is B^--stable (Proposition 2.7). The T-orbits in $\overline{Tv}$ are represented by $v_{\mathscr{F}}$ over all faces $\mathscr{F}\subseteq\mathscr{C}^\vee$, and the adherence of orbits agrees with the inclusion of faces. On the other hand, it is easy to see that the U^--fixed point set in each G-orbit of X is a T-orbit. Hence distinct $v_{\mathscr{F}}$ represent distinct G-orbits. □

In characteristic zero, one can describe the defining equations of X in V. Let $c=\sum\xi_i\xi_i^*\in\mathrm{U}\mathfrak{g}$ be the Casimir element with respect to a G-invariant inner product on $\mathfrak{g}$, ξ_i and ξ_i^* being mutually dual bases. It is well known that c acts on $V(\lambda^*)$ by a scalar $c(\lambda)=(\lambda+2\rho,\lambda)$. Note that $c(\lambda)$ depends on λ monotonously with respect to the partial order induced by positive roots: if $\lambda=\mu+\sum k_i\alpha_i$, $k_i\geq 0$, then $c(\lambda)=c(\mu)+\sum k_i\big((\lambda+2\rho,\alpha_i)+(\alpha_i,\mu)\big)\geq c(\mu)$, and the inequality is strict, except for $\lambda=\mu$. The following result is due to Kostant:

Proposition 28.6 ([LT]). *If* $\operatorname{char}\Bbbk=0$ *and $\lambda_1,\dots,\lambda_m$ are linearly independent, then $\mathscr{I}(X)\lhd\Bbbk[V]$ is generated by the relations*

$$c(x_i\otimes x_j)=(\lambda_i+\lambda_j+2\rho,\lambda_i+\lambda_j)(x_i\otimes x_j),\qquad i,j=1,\dots,m,$$

where x_k denotes the projection of $x \in V$ to $V(\lambda_k^*)$.

Remark 28.7. There is a characteristic-free version of this result, due to Kempf and Ramanathan [KeR], asserting that $\mathscr{I}(X)$ is generated by quadratic relations, see also [BKu, Ex. 3.5.E(1)].

Proof. The algebra $\Bbbk[V] = \bigoplus_{k_1,\dots,k_m} \mathrm{S}^{k_1}V(\lambda_1) \otimes \cdots \otimes \mathrm{S}^{k_m}V(\lambda_m)$ is multigraded and $\mathscr{I}(X)$ is a multihomogeneous ideal. The structure of $\Bbbk[X]$ implies that each homogeneous component $\mathscr{I}(X)_{k_1,\dots,k_m}$ is the kernel of the natural map $\mathrm{S}^{k_1}V(\lambda_1) \otimes \cdots \otimes \mathrm{S}^{k_m}V(\lambda_m) \to V(k_1\lambda_1 + \cdots + k_m\lambda_m)$.

Consider a series of linear endomorphisms $\pi = c - c(\sum k_i\lambda_i)\mathbf{1}$ of the subspaces $\mathrm{S}^{k_1,\dots,k_m}V = \mathrm{S}^{k_1}V(\lambda_1^*) \otimes \cdots \otimes \mathrm{S}^{k_m}V(\lambda_m^*) \subset \mathrm{S}^\bullet V$. Note that $\operatorname{Ker}\pi \simeq V(\sum k_i\lambda_i^*)$ is the highest irreducible component of $\mathrm{S}^{k_1,\dots,k_m}V$, annihilated by $\mathscr{I}(X)_{k_1,\dots,k_m}$, and $\operatorname{Im}\pi \simeq \mathscr{I}(X)^*_{k_1,\dots,k_m}$ is the complementary G-module.

It follows that $\mathscr{I}(X)$ is spanned by the coordinate functions of all $\pi(x_1^{k_1}\cdots x_m^{k_m})$. An easy calculation shows that

$$\pi(x_1^{k_1}\cdots x_m^{k_m}) = \sum_i \frac{k_i(k_i-1)}{2}\pi(x_i^2)x_1^{k_1}\cdots x_i^{k_i-2}\cdots x_m^{k_m} + \sum_{i<j} k_ik_j\pi(x_ix_j)x_1^{k_1}\cdots x_i^{k_i-1}\cdots x_j^{k_j-1}\cdots x_m^{k_m}.$$

Thus $\mathscr{I}(X)$ is generated by the relations $\pi(x_ix_j) = 0$, $i,j = 1,\dots,m$. □

If the generators of $\Lambda_+(X)$ are not linearly independent, one has to extend the defining equations of X by those arising from the linear dependencies between the λ_i, see [Sm-E].

Proposition 17.1 allows the divisor class group of a normal affine S-variety X to be computed. Every Weil divisor is rationally equivalent to a B-stable one $\delta = \sum m_\alpha D_\alpha + \sum m_i Y_i$, where Y_i are the G-stable prime divisors corresponding to the generators v_i of the rays of $\mathscr{C}$ containing no colors. The divisor δ is principal if and only if $m_\alpha = \langle\lambda, \alpha^\vee\rangle$ and $m_i = \langle\lambda, v_i\rangle$ for a certain $\lambda \in \mathfrak{X}(A)$. This yields a finite presentation for $\operatorname{Cl} X$. In particular, we have

Proposition 28.8. *A normal affine S-variety X is factorial if and only if $\Lambda_+(X)$ is generated by weights $\lambda_1,\dots,\lambda_s, \pm\lambda_{s+1},\dots,\pm\lambda_r$ ($s \le r$), where the λ_i are linearly independent and the projection $\mathfrak{X}(T) \to \mathfrak{X}(T \cap G')$ maps them to distinct fundamental weights or to* 0.

For semisimple G, we conclude that factorial S-varieties are those corresponding to weight semigroups Σ generated by some of the fundamental weights [VP, Th. 11].

The simplest class of affine S-varieties is formed by *HV-varieties*, i.e., cones of highest (or lowest) vectors $X = \overline{Gv_{-\lambda}}$, $v_{-\lambda} \in V(\lambda^*)^{(B^-)}$, see 11.1. Particular examples are quadratic cones or Grassmann cones of decomposable polyvectors. The above results on affine S-varieties imply Proposition 11.2, which describes basic properties of HV-varieties. It follows from Proposition 28.6 that an HV-cone is defined by quadratic equations in the ambient simple G-module. For a Grassmann cone we recover the Plücker relations between the coordinates of a polyvector.

28.3 Smoothness. Now we describe smooth S-varieties in characteristic zero. By Theorem 28.2, the problem is reduced to affine S-varieties with a fixed point, which are nothing else but G-modules with a dense orbit of a U-fixed vector.

Lemma 28.9. *If a G-module V is an S-variety, then $V = V_0 \oplus V_1 \oplus \cdots \oplus V_s$ so that $Z = Z(G)^0$ acts on V_0 with linearly independent weights of multiplicity 1 and each V_i ($i > 0$) is a simple submodule acted on non-trivially by a unique simple factor $G_i \subseteq G$, $G_i \simeq \mathrm{SL}(V_i)$ or $\mathrm{Sp}(V_i)$.*

Proof. Since Z has a dense orbit in $V_0 = V^{G'}$, it acts with linearly independent weights of multiplicity 1. If G_i acts non-trivially on two simple submodules V_i, V_j, and $v_i \in V_i^{(B)}$, $v_j \in V_j^{(B^-)}$, then the stabilizer of $v_i + v_j$ is not horospherical, i.e., V is not an S-variety. Therefore we may assume that V is irreducible and each simple factor of G acts non-trivially.

Then G acts transitively on $\mathbb{P}(V)$, which implies that $G' \simeq \mathrm{SL}(V)$ or $\mathrm{Sp}(V)$ [Oni2]. Indeed, we have $V = \mathfrak{b}v_{-\lambda}$, where $v_{-\lambda} \in V$ is a lowest vector. Hence there exists a unique root δ such that $e_\delta v_{-\lambda} = v_{\lambda^*}$ is a highest vector. One easily deduces that the root system of G is indecomposable and δ is the highest root, so that $\delta = \lambda + \lambda^*$ is the sum of two dominant weights, whence the assertion. □

The colored data of such a G-module V are easy to write down. Namely $\Pi \setminus \Pi_0 = \{\alpha_1, \dots, \alpha_s\}$, where α_i are the first simple roots in some components of Π having the type $\mathbf{A}_l$ or $\mathbf{C}_l$. The weight lattice $\mathfrak{X}(A)$ is spanned by linearly independent weights $\lambda_1, \dots, \lambda_r$, where $\lambda_1, \dots, \lambda_s$ are the highest weights of V_i^*, which project to the fundamental weights ω_i corresponding to α_i, and $\lambda_{s+1}, \dots, \lambda_r$ are the weights of V_0^*, which are orthogonal to Π. The cone $\mathscr{C}$ is spanned by the basis $\overline{\alpha_1^\vee}, \dots, \overline{\alpha_s^\vee}, v_{s+1}, \dots, v_r$ of $\mathfrak{X}^*(A)$ dual to $\lambda_1, \dots, \lambda_r$. Using Theorem 28.2 we derive the description of colored data of arbitrary smooth S-varieties:

Theorem 28.10 (cf. [Pau2, 3.5]). *An S-variety X is smooth if and only if all colored cones $(\mathscr{C}_Y, \mathscr{D}_Y^B)$ in the colored fan of X satisfy the following properties:*

(1) *$\mathscr{C}_Y$ is generated by a part of a basis of $\mathfrak{X}^*(A)$, and all $\overline{\alpha^\vee}$ such that $D_\alpha \in \mathscr{D}_Y^B$ are among the generators.*
(2) *The simple roots α such that $D_\alpha \in \mathscr{D}_Y^B$ are isolated from each other in the Dynkin diagram of G, and each α is connected with at most one component Π_α of Π_0; moreover, $\{\alpha\} \cup \Pi_\alpha$ has the type $\mathbf{A}_l$ or $\mathbf{C}_l$, α being the first simple root therein.*

The condition (1) is equivalent to the local factoriality of X.

29 Toroidal Embeddings

In this section we assume that $\operatorname{char} \Bbbk = 0$.

29.1 Toroidal Versus Toric Varieties. Recall that a G-equivariant normal embedding X of a spherical homogeneous space $O = G/H$ is said to be *toroidal* if $\mathscr{D}_Y^B = \emptyset$ for each G-orbit $Y \subseteq X$. Toroidal embeddings are defined by fans in $\mathscr{V}$, and G-morphisms between them correspond to subdivisions of these fans in the same way as in toric geometry [Ful2]. There is a more direct relation between toroidal and toric varieties. Put $P = P(O)$, with the Levi decomposition $P = P_{\mathrm{u}} \rightthreetimes L$ and other notation from 4.2 and 7.2.

Theorem 29.1 ([BPa, 3.4], [Bri13, 2.4]). *A toroidal embedding $X \hookleftarrow O$ is covered by G-translates of an open P-stable subset*

$$\mathring{X} = X \setminus \bigcup_{D \in \mathscr{D}^B} D \simeq P *_L Z \simeq P_{\mathrm{u}} \times Z,$$

where Z is a locally closed L-stable subvariety pointwise fixed by L_0. The variety Z is a toric embedding of $A = L/L_0$ defined by the same fan as X, and the G-orbits in X intersect Z in A-orbits.

Proof. The problem is easily reduced to the case where X contains a unique closed orbit Y with $\mathscr{C}_Y = \mathscr{V}$, $\mathscr{D}_Y^B = \emptyset$. Such toroidal embeddings, called standard, are discussed in 30.1. Indeed, consider another spherical homogeneous space $\overline{O} = G/N_G(H)$. Then $\overline{\mathscr{V}} = \mathscr{V}(\overline{O}) = \mathscr{V}/(\mathscr{V} \cap -\mathscr{V})$ is strictly convex, whence there exists a standard embedding $\overline{X} \hookleftarrow \overline{O}$. The canonical map $\varphi : O \to \overline{O}$ extends to $X \to \overline{X}$ by Theorem 15.10. We have $P(\overline{O}) = P$, and $\mathring{X}, Z$ are the preimages of the respective subvarieties defined for $\overline{X}$.

For standard X one applies the local structure theorem in a neighborhood of Y: by Theorem 15.17, $\mathring{X} = \mathring{X}_Y \simeq P *_L Z$, where Z is toric since $Z \cap O$ is a single A-orbit.

The G-stable divisors in X intersect $\mathring{X}$ in the P-stable divisors and Z in the A-stable divisors. Each G-(A-)orbit in X (in Z) meets $\mathring{X}$ and is an intersection of G-(A-)stable divisors, whence the assertion on orbits. The assertion on fans is easy, cf. Remark 15.19. □

It follows that toroidal varieties are locally toric. They inherit many nice geometric properties from toric varieties. On the other hand, each spherical variety is the image of a toroidal one by a proper birational equivariant map: to obtain this toroidal covering variety, just remove all colors from the fan. This universality of toroidal varieties can be used to derive some properties of spherical varieties from the toroidal case.

29.2 Smooth Toroidal Varieties. A toroidal variety is smooth if and only if all cones of its fan are simplicial and generated by a part of a basis of $\Lambda(O)^*$: for toric varieties this is deduced from the description of the coordinate algebra [Ful2, 2.1] (cf. Example 15.8) and the general case follows by Theorem 29.1. For a singular toroidal variety one may construct an equivariant desingularization by subdividing its fan, cf. [Ful2, 2.6].

Every (smooth) toroidal variety admits an equivariant (smooth) completion, which is defined by adding new cones to the fan in order to cover all of $\mathscr{V}(O)$. Smooth complete toroidal varieties have other interesting characterizations.

Theorem 29.2 ([BiB]). *For a smooth G-variety X consider the following conditions:*

(1) *X is toroidal.*
(2) *There is a dense open orbit $O \subseteq X$ such that $\partial X = X \setminus O$ is a divisor with normal crossings, each orbit $Gx \subset X$ is locally the intersection of several components of ∂X, and G_x has a dense orbit in $T_xX/\mathfrak{g}x$.*
(3) *There is a G-stable divisor $D \subset X$ with normal crossings such that $\mathscr{G}_X = \mathscr{T}_X(-\log D)$.*
(4) *X is spherical and pseudo-free.*

Then (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). If X is complete or spherical, then all conditions are equivalent.

G-varieties satisfying the condition (2), resp. (3), are known as *regular* in the sense of Bifet–de Concini–Procesi [BCP], resp. of Ginzburg [Gin].

Proof. (1) $\Longrightarrow$ (2)&(3) Theorem 29.1 reduces the problem to smooth toric varieties. The latter are covered by invariant affine open charts of the form $X = \mathbb{A}^m \times (\mathbb{A}^1 \setminus 0)^{n-m}$, where $(\mathbb{k}^\times)^n$ acts in the natural way, so that $D = \partial X$ is the union of coordinate hyperplanes $\{x_i = 0\}$, X is isomorphic to the normal bundle of the closed orbit, and $\mathscr{T}_X(-\log D)$ is a free sheaf spanned by velocity fields $x_1\partial_1, \dots, x_n\partial_n$ ($\partial_i := \partial/\partial x_i$).
(2) $\Longleftrightarrow$ (3) First observe that $O = X \setminus D$ is a single G-orbit if and only if $\mathscr{G}_{X\setminus D} = \mathscr{T}_{X\setminus D}$. Now consider a neighborhood of any $x \in D$. Due to local nature of the conditions (2) and (3), we may assume that all components $D_1, \dots, D_k$ of D contain x. Choose local parameters $x_1, \dots, x_n$ at x such that D_i are locally defined by the equations $x_i = 0$. Let $\partial_1, \dots, \partial_n$ denote the vector fields dual to $\mathrm{d}x_1, \dots, \mathrm{d}x_n$. Then $\mathscr{T}_X(-\log D)$ is locally generated by $x_1\partial_1, \dots, x_k\partial_k, \partial_{k+1}, \dots, \partial_n$.

Let $Y = D_1 \cap \dots \cap D_k$ and $\pi : N = \operatorname{Spec} \mathrm{S}^\bullet(\mathscr{I}_Y/\mathscr{I}_Y^2) \to Y$ be the normal bundle. There is a natural embedding $\pi^*\mathscr{T}_X(-\log D)|_Y \hookrightarrow \mathscr{T}_N$: each vector field in $\mathscr{T}_X(-\log D)$ preserves $\mathscr{I}_Y$, whence induces a derivation of $\mathrm{S}^\bullet(\mathscr{I}_Y/\mathscr{I}_Y^2)$. The image of $\pi^*\mathscr{T}_X(-\log D)|_Y$ is $\mathscr{T}_N(-\log \bigcup N_i)$, where N_i are the normal bundles to Y in D_i. Indeed, $\bar{x}_i = x_i \bmod \mathscr{I}_Y^2$ ($i \le k$), $\bar{x}_j = \pi^* x_j|_Y$ ($j > k$) are local parameters on N and $x_i\partial_i, \partial_j$ induce the derivations $\bar{x}_i\bar{\partial}_i, \bar{\partial}_j$. Note that $N = \bigoplus L_i$, where $L_i = \bigcap_{j\neq i} N_j$ are G-stable line subbundles. Hence the G_x-action on $T_xX/T_xY = N(x) = \bigoplus L_i(x)$ is diagonalizable.

Condition (2) implies that Gx is open in Y and the weights of $G_x : L_i(x)$ are linearly independent. This yields velocity fields $\bar{x}_i\bar{\partial}_i$ on $N(x)$ and in transversal directions, which locally generate $\mathscr{T}_N(-\log \bigcup N_i)$. Therefore $\mathscr{T}_X(-\log D)|_Y$ is generated by velocity fields. By Nakayama's lemma, $\mathscr{T}_X(-\log D) = \mathscr{G}_X$ in a neighborhood of x.

Conversely, (3) implies that $\mathscr{G}_Y = \mathscr{T}_Y$ and $\mathscr{G}_N = \mathscr{T}_N(-\log \bigcup N_i)$. Hence Gx is open in Y and $N|_{Gx} = G *_{G_x} T_xX/\mathfrak{g}x$ has an open G-orbit. Thus $T_xX/\mathfrak{g}x$ contains an open G_x-orbit.
(2)&(3) $\Longrightarrow$ (4) Since $\mathscr{T}_X(-\log \partial X)$ is locally free, the implication is trivial provided that X is spherical. It remains to prove that X is spherical if it is complete.

A closed orbit $Y \subseteq X$ intersects a B-chart $\mathring{X} \simeq P *_L Z$, where $L \subseteq P = P(Y)$ is the Levi subgroup and Z is an L-stable affine subvariety intersecting Y in a single point z. Since the maximal torus $T \subseteq L \subseteq G_z = P^-$ acts on $T_z Z \simeq T_z X/\mathfrak{g}z$ with linearly independent weights, $Z \simeq T_z Z$ contains an open T-orbit, whence $\mathring{X}$ has an open B-orbit.
(4) $\Longrightarrow$ (1) There is a morphism $X \to \mathrm{Gr}_k(\mathfrak{g})$, $x \mapsto [\mathfrak{h}_x]$, extending the map $x \mapsto [\mathfrak{g}_x]$ on O. If X is not toroidal, then there exists a G-orbit $Y \subset X$ contained in a color $D \subset X$. Then we have $\mathfrak{b} + \mathfrak{h}_{gy} = \mathfrak{b} + (\mathrm{Ad}\, g)\mathfrak{h}_y \neq \mathfrak{g}$, $\forall y \in Y$, $g \in G$, i.e., $\mathfrak{h}_y$ is not spherical.

To obtain a contradiction, it suffices to prove that all $\mathfrak{h}_x$ are spherical subalgebras. Passing to a toroidal variety mapping onto X, one may assume that X itself is toroidal. Consider the normal bundle N to $Y = Gx$. Since $\mathscr{G}_N = \pi^* \mathscr{G}_X|_Y$, $\mathfrak{h}_x$ is the stabilizer subalgebra of general position for $G : N$. But N is spherical, because the minimal B-chart $\mathring{X}$ of Y is P-isomorphic to $N|_{Y \cap \mathring{X}}$. □

29.3 Cohomology Vanishing. Toric varieties and generalized flag varieties form two "extreme" classes of toroidal varieties. A number of geometric and cohomological results generalize from these particular cases to general toroidal varieties. A powerful vanishing theorem was proved by Bien and Brion (1991) under some restrictions and refined by Knop (1992).

Theorem 29.3. *If X is a smooth complete toroidal variety, then*

$$\mathrm{H}^i\left(X, \mathrm{S}^\bullet \mathscr{T}_X(-\log \partial X)\right) = 0, \quad \forall i > 0.$$

For flag varieties, this result is due to Elkik (vanishing of higher cohomology of the tangent sheaf was proved already by Bott in 1957). In fact, Bien and Brion proved a twisted version of Theorem 29.3 [BiB, 3.2]: $\mathrm{H}^i(X, \mathscr{L} \otimes \mathrm{S}^\bullet \mathscr{T}_X(-\log \partial X)) = 0$ for all $i > 0$ and any globally generated line bundle $\mathscr{L}$ on X, under a technical condition that the stabilizer H of O is parabolic in a reductive subgroup of G. (Generally, higher cohomology of globally generated line bundles vanishes on every complete spherical variety, see Corollary 31.7.)

In view of Theorem 29.2, Theorem 29.3 stems from a more general vanishing result of Knop:

Theorem 29.4 ([Kn6, 4.1]). *If X is a pseudo-free smooth equivariant completion of a homogeneous space O, then* $\mathrm{H}^i(X, \mathscr{U}_X^{(m)}) = \mathrm{H}^i(X, \mathrm{S}^m \mathscr{G}_X) = 0$, $\forall i > 0$, $m \geq 0$.

Synopsis of a proof. The assertions on $\mathscr{U}_X$ are reduced to those on $\mathrm{S}^\bullet \mathscr{G}_X = \mathrm{gr}\, \mathscr{U}_X$. Since $\pi_X : T^{\mathfrak{g}}X \to X$ is an affine morphism, the Leray spectral sequence reduces the question to proving $\mathrm{H}^i(T^{\mathfrak{g}}X, \mathscr{O}_{T^{\mathfrak{g}}X}) = 0$. The localized moment map $\overline{\Phi} : T^{\mathfrak{g}}X \to M_X$ factors through $\widetilde{\Phi} : T^{\mathfrak{g}}X \to \widetilde{M}_X$. As $\widetilde{M}_X$ is affine, $\mathrm{H}^i(T^{\mathfrak{g}}X, \mathscr{O}_{T^{\mathfrak{g}}X}) = \mathrm{H}^0\left(\widetilde{M}_X, \mathrm{R}^i \widetilde{\Phi}_* \mathscr{O}_{T^{\mathfrak{g}}X}\right)$, and it remains to prove that $\mathrm{R}^i \widetilde{\Phi}_* \mathscr{O}_{T^{\mathfrak{g}}X} = 0$. Here one applies to $\widetilde{\Phi}$ a version of Kollár's vanishing theorem [Kn6, 4.2]:

> If Y is smooth, Z has rational singularities, and $\varphi : Y \to Z$ is a proper morphism with connected general fibers F, which satisfy $\mathrm{H}^i(F, \mathscr{O}_F) = 0$, $\forall i > 0$, then $\mathrm{R}^i \varphi_* \mathscr{O}_Y = 0$ for all $i > 0$.

It remains to verify the conditions. The morphism $\widetilde{\Phi}$ is proper by Example 8.2. The variety $\widetilde{M}_X$ has rational singularities by [Kn6, 4.3]. To show vanishing of the higher cohomology of $\mathscr{O}_F$, it suffices to prove that F is unirational [Se2]. Here one may assume that $X = O$, $\widetilde{\Phi} : T^*O \to \widetilde{M}_O$. Unirationality of the fibers of the moment map is the heart of the proof [Kn6, §5]. □

There is a relative version of Theorem 29.4 asserting that $\mathrm{R}^i\psi_*\mathscr{U}_X^{(m)} = \mathrm{R}^i\psi_*\mathrm{S}^m\mathscr{G}_X = 0$, $\forall i > 0$, for a proper G-invariant morphism $\psi : X \to Y$ separating general orbits, where X is smooth pseudo-free and Y has rational singularities.

29.4 Rigidity. The vanishing theorems of Bien–Brion and Knop have a number of important consequences. For instance, on a pseudo-free smooth completion X of O the symbol map $\operatorname{gr}\mathrm{H}^0(X,\mathscr{U}_X) \to \Bbbk[T^{\mathfrak{g}}X]$ is surjective. In the toroidal case, $\mathrm{H}^1(X,\mathscr{T}_X(-\log\partial X)) = 0$ implies that the pair $(X,\partial X)$ is locally rigid, by the deformation theory of Kodaira–Spencer [Ser]. Using this observation, Alexeev and Brion proved Luna's conjecture on rigidity of spherical subgroups.

Theorem 29.5 ([AB3, §3]). *For any (irreducible) G-variety with spherical (general) orbits, the stabilizers of points in general position are conjugate.*

Proof. Let $\mathscr{X}$ be a G-variety with spherical orbits. Passing to an open subset, we may assume that $\mathscr{X}$ is smooth quasiprojective and there exists a smooth G-invariant morphism $\pi : \mathscr{X} \to Z$ whose fibers contain dense orbits. Regarding $\mathscr{X}$ as a family of spherical G-orbit closures, we may replace $\mathscr{X}$ by a birationally isomorphic family of smooth projective toroidal varieties.

Indeed, there is a locally closed G-embedding of $\mathscr{X}$ into $\mathbb{P}(V)$, and therefore into $\mathbb{P}(V)\times Z$, for some G-module V. Replacing $\mathscr{X}$ by its closure and taking a pseudo-free desingularization, we may assume that $\mathscr{X}$ is pseudo-free and π is a projective morphism. By Theorem 29.2, the fibers of π are smooth projective toroidal varieties. Shrinking Z if necessary, we obtain that the G-orbits of non-maximal dimension in $\mathscr{X}$ form a divisor with normal crossings $\partial\mathscr{X} = \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_k$ whose components $\mathscr{D}_i$ are smooth over Z.

Morally, an equivariant version of Kodaira–Spencer theory should imply that all fibers of π are G-isomorphic, which should complete the proof. An alternative argument uses nested Hilbert schemes (see Appendix E.2).

Let X be any fiber of π, with $\partial X = D_1 \cup \cdots \cup D_k$, $D_i = \mathscr{D}_i \cap X$. Applying a suitable Veronese map, we satisfy a technical condition that the restriction map $V^* \to \mathrm{H}^0(X,\mathscr{O}(1))$ is surjective.

The nested Hilbert scheme Hilb parameterizes tuples $(Y,Y_1,\dots,Y_k)$ of projective subvarieties $Y_i \subseteq Y \subseteq \mathbb{P}(V)$ having the same Hilbert polynomials as $X,D_1,\dots,D_k$. The varieties $\mathscr{X},\mathscr{D}_1,\dots,\mathscr{D}_k$ are obtained as the pullbacks under $Z \to \mathrm{Hilb}$ of the universal families $\mathscr{Y},\mathscr{Y}_1,\dots,\mathscr{Y}_k \to \mathrm{Hilb}$. The groups $\mathrm{GL}(V)$ and G act on Hilb in a natural way, so that Hilb^G parameterizes tuples of G-subvarieties. Since the centralizer $\mathrm{GL}(V)^G$ of G maps G-subvarieties to G-isomorphic ones, it suffices to prove that the $\mathrm{GL}(V)^G$-orbit of $(X,D_1,\dots,D_k)$ is open in Hilb^G.

This is done by considering tangent spaces. Let $\mathscr{N}_Z,\mathscr{N}_{Z_i/Z}$ denote the normal bundles to Z in $\mathbb{P}(V)$, resp. to Z_i in Z. By Proposition E.7, $T_{(X,D_1,\dots,D_k)}\mathrm{Hilb} =$

$\mathrm{H}^0(X,\mathscr{N})$, where $\mathscr{N} \subset \mathscr{N}_X \oplus \mathscr{N}_{D_1} \oplus \cdots \oplus \mathscr{N}_{D_k}$ is formed by tuples $(\xi,\xi_1,\dots,\xi_k)$ of normal vector fields such that $\xi|_{D_i} = \xi_i \bmod \mathscr{N}_{D_i/X}$, $i=1,\dots,k$. (These vector fields define infinitesimal deformations of $X,D_1,\dots,D_k$, so that the deformation of D_i is determined by the deformation of X modulo a deformation inside X.) There are exact sequences

$$0 \longrightarrow \mathscr{T}_X(-\log\partial X) \longrightarrow \mathscr{T}_{\mathbb{P}(V)}|_X \longrightarrow \mathscr{N} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{O}_X \longrightarrow V \otimes \mathscr{O}_X(1) \longrightarrow \mathscr{T}_{\mathbb{P}(V)}|_X \longrightarrow 0.$$

Taking cohomology yields

$$\mathrm{H}^0\left(X,\mathscr{T}_{\mathbb{P}(V)}\right) \longrightarrow T_{(X,D_1,\dots,D_k)}\mathrm{Hilb} \longrightarrow \mathrm{H}^1(X,\mathscr{T}_X(-\log\partial X)) = 0,$$
$$V \otimes \mathrm{H}^0(X,\mathscr{O}(1)) \longrightarrow \mathrm{H}^0\left(X,\mathscr{T}_{\mathbb{P}(V)}\right) \longrightarrow \mathrm{H}^1(X,\mathscr{O}_X) = 0.$$

(The first cohomologies vanish by Theorem 29.3 and [Se2], since X is a smooth projective rational variety.) Hence the differential of the orbit map

$$\mathfrak{gl}(V) \simeq V \otimes V^* \longrightarrow V \otimes \mathrm{H}^0(X,\mathscr{O}(1)) \longrightarrow \mathrm{H}^0\left(X,\mathscr{T}_{\mathbb{P}(V)}\right) \longrightarrow T_{(X,D_1,\dots,D_k)}\mathrm{Hilb}$$

is surjective. By linear reductivity of G, the composite map

$$\mathfrak{gl}(V)^G \longrightarrow T_{(X,D_1,\dots,D_k)}(\mathrm{Hilb}^G) \subseteq \left(T_{(X,D_1,\dots,D_k)}\mathrm{Hilb}\right)^G$$

is surjective as well. Hence $(X,D_1,\dots,D_k)$ is a smooth point of Hilb^G and the orbit $\mathrm{GL}(V)^G(X,D_1,\dots,D_k)$ is open. □

29.5 Chow Rings. Cohomology rings of smooth complete toroidal varieties (over $\Bbbk = \mathbb{C}$) were computed by Bifet–de Concini–Procesi [BCP], see also [LP] for toroidal completions of symmetric spaces. By Corollary 18.4, cohomology coincides with the Chow ring in this situation. The most powerful approach is through equivariant cohomology or the equivariant intersection theory of Edidin–Graham, see [Bri15]. In particular, Chow (or cohomology) rings of smooth (complete) toric varieties and flag varieties are easily computed in this way [BCP, I.4], [Bri11, 2, 3], [Bri15], cf. 18.4.

29.6 Closures of Flats. The local structure of toroidal varieties can be refined in order to obtain a full description for the closures of generic flats.

Proposition 29.6 ([Kn5, 8.3]). *The closure of a generic twisted flat in a toroidal variety X is a normal toric variety whose fan is the W_X-span of the fan of X.*

Proof. It suffices to choose the toric slice Z in Theorem 29.1 in such a way that the open A-orbit in Z is a generic (twisted) flat F_α. Then $\overline{Z} = \overline{F}_\alpha$ (the closure in X), so that Theorem 29.1 and Proposition 23.19 imply the claim.

If X is smooth and T^*X is symplectically stable, then the conormal bundle to general U-orbits extends to a trivial subbundle $\mathring{X} \times \mathfrak{a}^* \hookrightarrow T^*\mathring{X}(\log\partial X)$, the trivializing sections being $\mathrm{d}\mathbf{f}_\lambda/\mathbf{f}_\lambda$, $\lambda \in \Lambda$. The logarithmic moment map restricts to $\Phi : \mathring{X} \times \mathfrak{a}^* \to \mathfrak{a} \oplus \mathfrak{p}_u$, cf. Lemma 23.14. It follows that $\mathring{X} \simeq P *_L Z$, where $Z = \pi_X \Phi^{-1}(\lambda)$, $\lambda \in \mathfrak{a}^{\mathrm{pr}}$, and F_α is the open L-orbit in Z for any $\alpha \in \Phi^{-1}(\lambda) \cap T^*O$.

If X is singular, then it admits a toroidal resolution of singularities $\nu : X' \to X$. Then $\mathring{X}' := \nu^{-1}(\mathring{X}) = X' \setminus \bigcup_{D \in \mathscr{D}^B} D \simeq P *_L Z'$ and $Z' \supseteq F_\alpha$. The map $\Phi : \mathring{X}' \times \mathfrak{a}^* \to \mathfrak{a} \oplus \mathfrak{p}_u$ descends to $\mathring{X}$, because $\Bbbk[\mathring{X}'] = \Bbbk[\mathring{X}]$. Thus one may put $Z = \nu(Z')$.

If T^*X is not symplectically stable, then passing to affine cones and back to projectivizations yields Z such that the open L-orbit in Z is a twisted flat. □

Example 29.7. If X is a toroidal $(G \times G)$-embedding of G, then T is a flat and $F = \overline{T}$ is a toric variety whose fan is the W-span of the fan of X (in the antidominant Weyl chamber), cf. Proposition 27.18. For instance, if $X = \overline{G} \subseteq \mathbb{P}(\mathrm{L}(V))$ for a faithful projective representation $G : \mathbb{P}(V)$ with regular highest weights, then the fan of F is formed by the duals to the barrier cones of the weight polytope $\mathscr{P}(V)$, and the fan of X is its antidominant part (see 27.3).

Example 29.8. Consider the variety of complete conics $X \subset \mathbb{P}(\mathrm{S}^2(\Bbbk^3)^*) \times \mathbb{P}(\mathrm{S}^2\Bbbk^3)$ from Example 17.12. The set $F = \{([q],[q^\vee]) \mid q \text{ diagonal}, \det q \neq 0\}$ is a flat. Using the Segre embedding $\mathbb{P}(\mathrm{S}^2(\Bbbk^3)^*) \times \mathbb{P}(\mathrm{S}^2\Bbbk^3) \hookrightarrow \mathbb{P}(\mathrm{S}^2(\Bbbk^3)^* \otimes \mathrm{S}^2\Bbbk^3)$ and observing that the T-weights occurring in the weight decomposition of $q \otimes q^\vee$ are $2(\varepsilon_i - \varepsilon_j)$, we conclude that the fan of $\overline{F}$ is the set of all Weyl chambers of $G = \mathrm{SL}_3(\Bbbk)$ together with their faces, while the fan of X consists of the antidominant Weyl chamber and its faces.

30 Wonderful Varieties

30.1 Standard Completions. In the study of a homogeneous space O it is useful to consider its equivariant completions. The reason is that properties of O and of related objects (subvarieties and their intersection, functions, line bundles and their sections, etc) often become apparent "at infinity", and equivariant completions of O take into account the points at infinity. Also, complete varieties behave better than non-complete ones from various points of view (e.g., in intersection theory).

Among all equivariant completions of a spherical homogeneous space O one distinguishes two opposite classes. Toroidal completions have nice geometry (see §29) and a universal property: each equivariant completion of O is dominated by a toroidal one. On the other hand, simple completions of O (i.e., those having a unique closed orbit) are the most "economical" ones: their boundaries are "small". Simple completions exist if and only if the valuation cone $\mathscr{V}$ is strictly convex.

These two classes intersect in a unique element, called the standard completion.

Definition 30.1. A spherical subgroup $H \subseteq G$ is called *sober* if $N_G(H)/H$ is finite or, equivalently, if $\mathscr{V}(G/H)$ is strictly convex.

The *standard embedding* of $O = G/H$ is the unique toroidal simple complete G-embedding $X \hookleftarrow O$, defined by the colored cone $(\mathscr{V}, \emptyset)$, provided that H is sober. A smooth standard embedding is called *wonderful*.

The standard embedding has a universal property: for any toroidal completion $X' \hookleftarrow O$ and any simple completion $X'' \hookleftarrow O$, there exist unique proper birational G-morphisms $X' \to X \to X''$ extending the identity map on O.

Wonderful embeddings were first introduced by de Concini and Procesi [CP1] for symmetric spaces. Their remarkable properties were studied by many researchers (see below) mainly in characteristic zero, though some results in special cases, e.g., for symmetric spaces [CS], are obtained in arbitrary characteristic. For simplicity, we assume that $\operatorname{char} \Bbbk = 0$ from now on.

Every spherical subgroup $H \subseteq G$ is contained in the smallest sober overgroup $H \cdot N_G(H)^0$ normalizing H. This stems, e.g., from the following useful lemma.

Lemma 30.2. *If $H \subseteq G$ is a spherical subgroup, then $N_G(H) = N_G(\overline{H})$ for any intermediate subgroup $\overline{H}$ between H and $N_G(H)$.*

Proof. As $N_G(H)/H$ is Abelian, we have $N_G(H) \subseteq N_G(\overline{H})$. In particular, $N_G(H) = N_G(H^0)$. To prove the converse inclusion, we may assume without loss of generality that H is connected and $\mathfrak{b} + \mathfrak{h} = \mathfrak{g}$. Then the right multiplication by $N_G(\overline{H})$ preserves $BH = B\overline{H}$, the unique open $(B \times H)$-orbit in G. Hence the $N_G(\overline{H})$-action on $\Bbbk(G)$ by right translations of an argument preserves $\Bbbk[G]^{(B\times H)}$ (= the set of regular functions on G invertible on BH). Since this action commutes with the G-action by left translations, it preserves $\Bbbk[G]^{(H)}$, whence $\Bbbk(G/H)$, too. Hence $N_G(\overline{H})$ acts on G/H by G-automorphisms, i.e., is contained in $N_G(H)$. □

Now let $H \subseteq G$ be a sober subgroup and X be the standard embedding of $O = G/H$. The local structure theorem reveals the orbit structure and local geometry of X: by Theorem 29.1 there are an affine open chart $\mathring{X} = X \setminus \bigcup_{D \in \mathscr{D}^B} D$ and a closed subvariety $Z \subset \mathring{X}$ such that $\mathring{X}$ is stable under $P = P(O)$, the Levi subgroup $L \subset P$ leaves Z stable and acts on it via the quotient torus $A = L/L_0$, $\mathring{X} \simeq P *_L Z \simeq P_{\mathrm{u}} \times Z$, and each G-orbit of X intersects Z in an A-orbit. Actually $\mathring{X}$ is the unique B-chart of X intersecting all G-orbits.

The affine toric variety Z is defined by the cone $\mathscr{V}$, so that $\Bbbk[Z] = \Bbbk[\mathscr{V}^\vee \cap \Lambda]$, where $\Lambda = \Lambda(O) = \mathfrak{X}(A)$. The orbits (of $A : Z$ or of $G : X$) are in an order-reversing bijection with the faces of $\mathscr{V}$, and each orbit closure is the intersection of invariant divisors containing the orbit. If $\mathscr{V}$ is generated by a basis of Λ^*, then $Z \simeq \mathbb{A}^r$ with the natural action of $A \simeq (\Bbbk^\times)^r$; the eigenweight set for $A : Z$ is $\Pi_O^{\min}$. Generally, since $\mathscr{V}$ is simplicial (Theorem 22.13), one deduces that $Z \simeq \mathbb{A}^r/\Gamma$ with the natural action of $A \simeq (\Bbbk^\times)^r/\Gamma$, where $\Gamma \simeq \Lambda^*/N$ is the common kernel of all $\lambda \in \Lambda$ in $(\Bbbk^\times)^r = N \otimes \Bbbk^\times$, the sublattice $N \subseteq \Lambda^*$ being spanned by the indivisible generators of the rays of $\mathscr{V}$.

In particular, X is smooth if and only if $\mathscr{V}$ is generated by a basis of Λ^*, i.e., if and only if $\Lambda = \mathbb{Z}\Delta_O^{\min}$. It is a delicate problem to characterize the (sober) spherical subgroups $H \subseteq G$ such that the standard embedding $X \hookleftarrow O = G/H$ is smooth.

Note that $N_G(H)/H = \operatorname{Aut}_G O$ acts on a finite set $\mathscr{D}^B$.

Definition 30.3. A spherical subgroup $H \subseteq G$ is called *very sober* if $N_G(H)/H$ acts on $\mathscr{D}^B$ effectively. (In particular, H is sober, because $(N_G(H)/H)^0$ leaves $\mathscr{D}^B$ pointwise fixed.) The *very sober hull* of H is the kernel $\overline{H}$ of $N_G(H) : \mathscr{D}^B$. An alternative terminology is: *spherically closed* subgroup, *spherical closure*.

Remark 30.4. It is easy to deduce from Lemma 30.2 that $\overline{H}$ is the smallest very sober subgroup of G containing H as a normal subgroup. The colored space $\overline{\mathscr{E}} = \mathscr{E}(G/\overline{H})$ is identified with $\mathscr{E}/(\mathscr{V}\cap -\mathscr{V})$, the valuation cone is $\overline{\mathscr{V}} = \mathscr{V}/(\mathscr{V}\cap -\mathscr{V})$, and the set of colors $\overline{\mathscr{D}}^B$ is identified with $\mathscr{D}^B$ via pullback.

Observe that $\overline{H}$ is the kernel of $N_G(H) : \mathfrak{X}(H)$ [Kn8, 7.4]. Indeed, (a multiple of) each B-stable divisor δ on O is defined by an equation $\eta \in \Bbbk(G)^{(B\times H)}_{(\lambda,\chi)}$, and each $\chi \in \mathfrak{X}(H)$ arises in this way (because every G-line bundle $\mathscr{L}_{G/H}(\chi)$ has a rational B-eigensection). The right multiplication by $n \in N_G(H)$ maps η to $\eta' \in \Bbbk(G)^{(B\times H)}_{(\lambda,\chi')}$, the equation of $\delta' = n(\delta)$, where $\chi'(h) = \chi(n^{-1}hn)$. Since $\Bbbk(O)^B = \Bbbk$, we have $\chi' = \chi \iff \eta'/\eta = \text{const} \iff \delta' = \delta$.

In particular, $\overline{H} \supseteq Z_G(H)$.

Theorem 30.5 ([Kn8, 7.6, 7.2]). *If H is very sober, then the standard embedding $X \hookleftarrow G/H$ is smooth. In particular, X is smooth if $N_G(H) = H$.*

Remark 30.6. If all simple factors of G are isomorphic to PSL_{n_i}, then very soberness is also a necessary condition for X be smooth [Lu6, 7.1]. This is not true in general: $S^{n-1} = \mathrm{SO}_n/\mathrm{SO}_{n-1}$ and $\mathrm{SL}_4/\mathrm{Sp}_4$ are symmetric spaces of rank 1, and hence their standard embeddings are smooth (Proposition 30.18), while $\overline{\mathrm{SO}_{n-1}} = \mathrm{S}(\mathrm{O}_1\times \mathrm{O}_{n-1})$, $\overline{\mathrm{Sp}_4} = \mathrm{Sp}_4\cdot Z(\mathrm{SL}_4)$.

Proof. By Theorem 23.25, $S_O = \bigcap_{\alpha\in\Delta_O^{\min}} \operatorname{Ker}\alpha \hookrightarrow \operatorname{Aut}_G O = N_G(H)/H$. It suffices to show that S_O fixes all colors; then $S_O = \{e\}$, i.e., $\Delta_O^{\min}$ spans Λ.

Take any $D \in \mathscr{D}^B$. Replacing D by a multiple, we may assume that $\mathscr{O}(D)$ is G-linearized. Consider the total space $\widehat{O} = \widehat{G}/\widehat{H}$ of $\mathscr{O}(-D)^\times$, where $\widehat{G} = G\times\Bbbk^\times$, cf. Remark 20.8. Using the notation of Remark 20.8, we have

$$0 \longrightarrow \Lambda \longrightarrow \widehat{\Lambda} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

$\mathscr{V} = \widehat{\mathscr{V}}/(\widehat{\mathscr{V}}\cap -\widehat{\mathscr{V}})$, and $\Delta_{\widehat{O}}^{\min} = \Delta_O^{\min}$. Therefore $S_O = S_{\widehat{O}}/\Bbbk^\times$.

However the pullback $\widehat{D} \subset \widehat{O}$ of D is principal. Since $S_{\widehat{O}}$ multiplies the equation of $\widehat{D}$ by scalars, it leaves $\widehat{D}$ stable, whence S_O leaves D stable. □

30.2 Demazure Embedding. If $N_G(H) = H$, then $O \simeq G[\mathfrak{h}]$, the orbit of $\mathfrak{h}$ in $\mathrm{Gr}_k(\mathfrak{g})$, $k = \dim\mathfrak{h}$. The closure $X(\mathfrak{h}) = \overline{G[\mathfrak{h}]} \subseteq \mathrm{Gr}_k(\mathfrak{g})$ is called the *Demazure embedding*.

Proposition 30.7 ([Los1]). *If $N_G(H) = H$, then $X(\mathfrak{h})$ is the wonderful embedding of O.*

Proof. The standard embedding $X \hookleftarrow O$ is wonderful by Theorem 30.5. Brion proved that X is the normalization of $X(\mathfrak{h})$ [Bri8, 1.4]. Finally, Losev proved that the normalization map $X \to X(\mathfrak{h})$ is an isomorphism [Los1]. We give a proof of Brion's result referring to [Los1] for the rest.

The decomposition $\mathfrak{g} = \mathfrak{p}_u \oplus \mathfrak{a} \oplus \mathfrak{h}$ yields $\mathfrak{h} = \mathfrak{l}_0 \oplus \langle e_{-\alpha} + \xi_\alpha \mid \alpha \in \Delta^+ \setminus \Delta_L^+\rangle$, where $\xi_\alpha \in \mathfrak{p}_u \oplus \mathfrak{a}$ is the projection of $-e_{-\alpha}$ along $\mathfrak{h}$. Hence

$$\widehat{\mathfrak{h}} = \widehat{\mathfrak{l}_0} \wedge \bigwedge_{\alpha \in \Delta^+ \setminus \Delta_L^+} (e_{-\alpha} + \xi_\alpha) = \widehat{\mathfrak{s}} + \text{terms of higher } T\text{-weights},$$

where $\widehat{\mathfrak{q}} \in \bigwedge^\bullet \mathfrak{g}$ denotes a generator of $[\mathfrak{q}] \in \mathrm{Gr}(\mathfrak{g})$, $\mathfrak{s} = \mathfrak{l}_0 \oplus \mathfrak{p}_{\mathrm{u}}^-$, and the weights of other terms differ from that of $\widehat{\mathfrak{s}}$ by $\sum(\alpha_i + \beta_i)$, $\alpha_i, \beta_i \in \Delta^+ \setminus \Delta_L^+$ or $\beta_i = 0$.

Let $Z(\mathfrak{h})$ be the closure of $T[\mathfrak{h}]$ in the affine chart defined by non-vanishing of the covector dual to $\widehat{\mathfrak{s}}$. It is an affine toric variety with the fixed point $[\mathfrak{s}]$. Thus $Y = G[\mathfrak{s}] \subset X(\mathfrak{h})$ is a closed orbit. The local structure theorem in a neighborhood of $[\mathfrak{s}]$ provides a B-chart $\mathring{X}(\mathfrak{h}) \subset X(\mathfrak{h})$, $\mathring{X}(\mathfrak{h}) \simeq P_{\mathrm{u}} \times Z(\mathfrak{h})$. Note that for any $[\mathfrak{q}] \in Z(\mathfrak{h}) \setminus T[\mathfrak{h}]$ the subalgebra $\mathfrak{q}$ is transversal to $\mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{a}$ while $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{q}) \cap \mathfrak{a} \neq 0$, whence $\dim G_{[\mathfrak{q}]} > \dim H$. It follows that $\mathring{X}(\mathfrak{h})$ intersects no colors, i.e., $X(\mathfrak{h})$ is toroidal in a neighborhood of Y.

On the other hand, every smooth toroidal embedding of O maps to $X(\mathfrak{h})$ by Theorem 29.2. It follows that the normalization of $X(\mathfrak{h})$ is simple, and hence wonderful. □

If $N_G(H) \neq H$, then $X(\mathfrak{h})$ is the wonderful embedding of $G/N_G(H)$.

30.3 Case of a Symmetric Space. Let G be an adjoint semisimple group and let $H = G^\theta$ be a symmetric subgroup. Here $N_G(H) = H$. We have $\Lambda(O) = \mathfrak{X}(T/T^\theta) = \{\mu - \theta(\mu) \mid \mu \in \mathfrak{X}(T)\}$, where T is a θ-stable maximal torus such that T_1 is a maximal θ-split torus. Hence $\Lambda(O)$ is the root lattice of $2\Delta_O$. Since $\mathscr{V}(O)$ is the antidominant Weyl chamber of $\Delta_O^\vee$ in $\Lambda(O)^* \otimes \mathbb{Q}$ (by Theorem 26.25), $\Delta_O^{\min}$ is the reduced root system associated with $2\Delta_O$. It follows that the standard completion X is smooth in this case.

Wonderful completions of symmetric spaces were studied in [CP1], [CS]. In particular, a geometric realization for a wonderful completion as an embedded projective variety was constructed. Take any $\lambda \in \Lambda(\widetilde{G}/\widetilde{G}^\theta) \cap \operatorname{int} \mathbf{C}(\Delta_O^+)$. There exists a unique (up to proportionality) nonzero $\widetilde{G}^\theta$-fixed vector $v' \in V^*(\lambda)$. Then $X' = \overline{G[v']} \subseteq \mathbb{P}(V^*(\lambda))$ is the wonderful embedding of $G[v'] \simeq O$.

Indeed, a natural closed embedding $\mathbb{P}(V^*(\lambda)) \hookrightarrow \mathbb{P}(V^*(2\lambda))$ (given by the multiplication $V^*(\lambda) \otimes V^*(\lambda) \to V^*(2\lambda)$ in $\Bbbk[\widetilde{G}]$) identifies X' with $X'' = \overline{G[v'']}$, where $v'' \in V^*(2\lambda)$ is a unique $\widetilde{G}^\theta$-fixed vector. As X'' is a simple projective embedding of $G[v'']$, the natural map $O \to G[v'']$ extends to $X \to X''$. On the other hand, the homomorphism $V^*(\lambda) \otimes V^*(\lambda) \to V^*(2\lambda)$ maps ω to v'', where ω is defined by (26.3). Let Z'' be the closure of $T[v'']$ in the affine chart of $\mathbb{P}(V^*(2\lambda))$ defined by non-vanishing of the highest covector of weight 2λ. From (26.3) it is easy to deduce that $Z'' \simeq \mathbb{A}^r$ is acted on by T via the eigenweight set $\Pi_O^{\min}$ and the closed orbit $G[v_{-2\lambda}]$ is transversal to Z'' at $[v_{-2\lambda}]$. Hence $Z \xrightarrow{\sim} Z''$, $\mathring{X} \xrightarrow{\sim} PZ'' \simeq P_{\mathrm{u}} \times Z''$, and finally $X \xrightarrow{\sim} X'' \simeq X'$. (A similar reasoning shows $X \simeq \overline{G[\omega]}$. A slight refinement carries over the construction to positive characteristic [CS].)

Another model for the wonderful completion is the Demazure embedding. First note that $\mathfrak{h} = \mathfrak{l}_0 \oplus \langle e_\alpha + e_{\theta(\alpha)} \mid \alpha \in \Delta^+ \setminus \Delta_L^+ \rangle$. Arguing as in the proof of Proposition 30.7, we see that $Z(\mathfrak{h}) = \overline{T[\mathfrak{h}]} \simeq \mathbb{A}^r$ is acted on by T with the eigenweights $\alpha - \theta(\alpha)$, $\alpha \in \Pi \setminus \Delta_L$, and $Y = G[\mathfrak{s}]$ is transversal to $Z(\mathfrak{h})$ at $[\mathfrak{s}]$. This yields $\mathscr{C}_Y = \mathscr{V}$.

Now the Luna–Vust theory together with the description of the colored data for symmetric spaces implies that $X(\mathfrak{h})$ is wonderful [CP1]. The varieties $X(\mathfrak{h})$ were first considered by Demazure in the case, where $G = \mathrm{PSL}_n(\Bbbk)$ and H is the projective orthogonal or symplectic group [Dem4].

30.4 Canonical Class. Using the Demazure embedding, Brion computed the canonical class of any spherical variety.

Proposition 30.8 ([Bri8, 1.6]). *Suppose that X is a spherical variety with the open orbit $O \simeq G/H$. Consider the G-morphism $\varphi : O \to \mathrm{Gr}_k(\mathfrak{g})$, $\varphi(o) = [\mathfrak{h}]$, $k = \dim H$. Then a canonical divisor of X is*

$$K_X = -\sum_i D_i - \overline{\varphi^*\mathscr{H}} = -\sum_i D_i - \sum_{D \in \mathscr{D}^B} m_D D,$$

where D_i runs over all G-stable prime divisors in X, $\mathscr{H}$ is a hyperplane section of $X(\mathfrak{h})$ in $\mathbb{P}(\bigwedge^k \mathfrak{g})$, and $m_D \in \mathbb{N}$.

Explicit formulæ for m_D are given in [Bri12, 4.2], [Lu5, 3.6]. In the notation of 30.10, $m_D = 1$ unless $D = D_\alpha \in \mathscr{D}^b$, in which case $m_D = 2\langle \rho_G - \rho_L, \alpha^\vee \rangle \geq 2$.

Proof. Removing all G-orbits of codimension > 1, we may assume that X is smooth and toroidal. Then by Theorem 29.2, φ extends to X, and we have an exact sequence

$$0 \longrightarrow \varphi^*\mathscr{E} \longrightarrow \mathscr{O}_X \otimes \mathfrak{g} \longrightarrow \mathscr{T}_X(-\log \partial X) \longrightarrow 0,$$

where $\mathscr{E}$ is the tautological vector bundle on $\mathrm{Gr}_k(\mathfrak{g})$. Taking the top exterior powers yields $\omega_X \otimes \mathscr{O}_X(\partial X) \simeq \bigwedge^k \varphi^*\mathscr{E} = \mathscr{O}_X(-\varphi^*\mathscr{H})$, whence the first expression for K_X. If $\mathscr{H}$ is defined by a covector in $(\bigwedge^k \mathfrak{g}^*)^{(B)}$ dual to $\widehat{\mathfrak{s}}$, then $\mathring{X}(\mathfrak{h}) = X(\mathfrak{h}) \setminus \mathscr{H}$ intersects all G-orbits in open B-orbits, and hence $\varphi^*\mathscr{H} = \sum m_D D$ with $m_D > 0$ for all $D \in \mathscr{D}^B$. □

Using the characterization of ample divisors on complete spherical varieties (Corollary 17.24), one deduces that certain wonderful embeddings (e.g., flag varieties, most wonderful completions of symmetric spaces, primitive wonderful varieties of rank 1, see 30.8) are Fano varieties (i.e., the anticanonical divisor is ample).

30.5 Cox Ring. In the study of projective varieties it is very helpful to use homogeneous coordinates. Polynomials in homogeneous coordinates are not functions, but sections of a very ample line bundle and its powers. Instead of taking one line bundle, one may consider all line bundles and their sections. Thus one arrives at the notion of a total coordinate ring or a Cox ring of an algebraic variety [BH]. For simplicity, we define it under some technical restrictions.

Definition 30.9. Suppose that X is a locally factorial irreducible algebraic variety such that $\mathrm{Pic}\,X$ is free and finitely generated. The *Cox sheaf* of X is a sheaf of graded $\mathscr{O}_X$-algebras $\mathscr{R}_X = \bigoplus_\delta \mathscr{O}_X(\delta)$, where δ runs over a subgroup of $\mathrm{CaDiv}\,X$ mapped isomorphically onto $\mathrm{Pic}\,X$. The multiplication in $\mathscr{R}_X$ is given by choosing compatible identifications $\mathscr{O}(\delta) \otimes \mathscr{O}(\delta') \simeq \mathscr{O}(\delta + \delta')$, which is possible since $\mathrm{Pic}\,X$ is free. The *Cox ring* of X is $\mathscr{R}(X) = \mathrm{H}^0(X, \mathscr{R}_X) = \bigoplus_\delta \mathrm{H}^0(X, \mathscr{O}(\delta))$.

$\mathscr{R}_X$ and $\mathscr{R}(X)$ do not depend (up to isomorphism) on the lifting $\operatorname{Pic} X \hookrightarrow \operatorname{CaDiv} X$. The ring $\mathscr{R}(X)$ was first introduced by Cox for smooth complete toric X, in which case $\mathscr{R}(X)$ is a polynomial algebra [Cox].

Example 30.10. For $X = \mathbb{P}(V)$ one has $\mathscr{R}(X) = \Bbbk[V]$.

Put $\mathring{\mathscr{X}} = \operatorname{Spec}_{\mathscr{O}_X} \mathscr{R}_X$. It is a quasiaffine variety [BH, Pr. 3.10] and a principal T_X-bundle over X, where $T_X = \operatorname{Hom}(\operatorname{Pic} X, \Bbbk^\times)$ is the *Néron–Severi torus*. If $\mathscr{R}(X)$ is finitely generated, then $\mathscr{X} = \operatorname{Spec} \mathscr{R}(X)$ is a factorial affine variety containing $\mathring{\mathscr{X}}$ as an open subset with the complement of codimension > 1 [BH]. Indeed, $\mathring{\mathscr{X}}$ is covered by affine open subsets $\mathring{\mathscr{X}}_\eta \simeq \mathscr{X}_\eta$, where $\eta \in \mathrm{H}^0(X, \mathscr{O}(\delta))$, $\Bbbk[\mathring{\mathscr{X}}] = \Bbbk[\mathscr{X}]$, and each T_X-stable divisor on $\mathscr{X}$ is pulled back from a divisor on X, and hence is principal.

The Cox ring of a wonderful variety was investigated by Brion in [Bri18]. Here we describe his results.

Let $X \hookleftarrow O = G/H$ be a wonderful embedding, where G may and will be assumed semisimple and simply connected. By Corollary 17.10, $\operatorname{Pic} X$ is freely generated by the classes of colors $D \in \mathscr{D}^B$. The canonical B-eigensection η_D of $\mathscr{O}(D)$ with $\operatorname{div} \eta_D = D$ may be regarded as a function in $\Bbbk[G]^{(B\times H)}$ (= the equation of the preimage of D in G) of eigenweight $\widehat{\lambda}_D = (\lambda_D, \chi_D)$, so that λ_D is the eigenweight of the section and $\mathscr{O}_{G/H}(D) \simeq \mathscr{L}(-\chi_D)$, cf. Remark 13.4. The biweights $\widehat{\lambda}_D$ are linearly independent by Remark 15.1. Consider also the canonical G-invariant sections $\eta_1, \dots, \eta_r$ corresponding to the components $D_1, \dots, D_r$ of $X \setminus O$.

Proposition 30.11. (1) *$\mathscr{R}(X)^U = \Bbbk[\eta_1, \dots, \eta_r, \eta_D \mid D \in \mathscr{D}^B]$ and $\mathscr{R}(X)^G = \Bbbk[\eta_1, \dots, \eta_r]$ are polynomial algebras.*
(2) *$\mathscr{R}(X)$ is a free module over $\mathscr{R}(X)^G$ and a finitely generated algebra.*
(3) *The categorical quotient map $\pi_G : \mathscr{X} \to \mathscr{X}/\!\!/G \simeq \mathbb{A}^r$ is flat with reduced and normal fibers.*

Proof. (1) stems from the fact that the divisor of each $(B \times T_X)$-eigensection in $\mathscr{R}(X)^U$ is uniquely expressed as a non-negative integral linear combination of $D_1, \dots, D_r, D \in \mathscr{D}^B$. Since $\mathscr{R}(X)^U$ is free over $\mathscr{R}(X)^G$, this implies (2) in view of Theorem D.5(1). By (2), π_G is flat, and all fibers have one and the same algebra of U-invariants isomorphic to $\Bbbk[\eta_D \mid D \in \mathscr{D}^B]$, which yields (3) by Theorem D.5. $\square$

Let H_0 denote the common kernel of all characters in $\mathfrak{X}(H)$. Then $T_O = H/H_0$ is a diagonalizable group with $\mathfrak{X}(T_O) = \mathfrak{X}(H)$. There is a commutative diagram

$$\begin{array}{ccccc} \operatorname{Pic} O & \longleftarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Pic} Y \\ \| & & \| & & \| \\ \mathfrak{X}(T_O) & \longleftarrow & \mathbb{Z}\mathscr{D}^B & \longrightarrow & \mathfrak{X}(P) \\ \chi_D & \longleftarrow & D & \longrightarrow & \lambda_D, \end{array} \tag{30.1}$$

where $Y \subseteq X$ is the closed G-orbit and $P = P(O)$. The left arrows are surjective, with kernels consisting of $\sum m_D D$ such that $\sum m_D D = \operatorname{div} \mathbf{f}_\lambda$ on O for some $\lambda \in \Lambda(O) =$

$\mathbb{Z}\Pi_O^{\min}$, i.e., $m_D = \langle D, \lambda \rangle$. In particular, $T_O \hookrightarrow T_X \simeq (\Bbbk^\times)^{\mathscr{D}^B}$, $h \mapsto (\chi_D(h))_{D\in\mathscr{D}^B}$. A homomorphism $T \to T_X$, $t \mapsto (\lambda_D(t))_{D\in\mathscr{D}^B}$, induces an isomorphism $A = T/T_0 \xrightarrow{\sim} T_X/T_O$. Indeed, $\mathfrak{X}(T_X/T_O)$ consists of the elements $\sum\langle D,\lambda\rangle D$, which are mapped bijectively to $\sum\langle D,\lambda\rangle\lambda_D = \lambda \in \Lambda(O) = \mathfrak{X}(A)$.

The group $G\times T$ acts on $\mathscr{X}$ via the above homomorphism $T \to T_X$. The preimage $\widehat{O} \subseteq \mathring{\mathscr{X}}$ of $O \subseteq X$ is a single $(G\times T_X)$- or $(G\times T)$-orbit consisting of G-orbits isomorphic to G/H_0 and transitively permuted by T_X, so that $\widehat{O} \simeq T_X *_{T_O} G/H_0 \simeq T *_{T_0} G/H_0$. It follows that

$$\Bbbk[\widehat{O}] \simeq \bigoplus_{\substack{\chi=\sum m_D\chi_D\\ \lambda=\sum m_D\lambda_D\\ m_D\in\mathbb{Z}}} \Bbbk[G/H_0]^{(H)}_{\chi} \otimes \Bbbk\lambda^{-1} =$$

$$= \bigoplus_{\substack{\lambda\in\Lambda_+(G/H_0)=\sum\mathbb{Z}_+\lambda_D\\ \mu\in\lambda+\Lambda(G/H)}} \Bbbk[G/H_0]_{(\lambda)} \otimes \Bbbk\mu^{-1} \subseteq \Bbbk[G/H_0\times T].$$

Note that $D_i \sim \sum\langle D,\lambda_i\rangle D$, where $\lambda_i \in \Pi_X^{\min}$, $\langle v_{D_i},\lambda_i\rangle = -1$, whence $\eta_D,\eta_i \in \Bbbk[\mathscr{X}] \subseteq \Bbbk[\widehat{O}]$ are $(B\times T)$-eigenfunctions of biweights (λ_D,λ_D) and $(0,\lambda_i)$, respectively. Since they generate $\Bbbk[\mathscr{X}]^U$, we deduce the following

Theorem 30.12. (1) $\mathscr{R}(X) \simeq \bigoplus_{\substack{\lambda\in\Lambda_+(G/H_0)\\ \mu\in\lambda+\mathbb{Z}_+\Pi_X^{\min}}} \Bbbk[G/H_0]_{(\lambda)}\otimes\Bbbk\mu^{-1}$.

(2) π_G *is also the categorical quotient map for the action* $\widehat{G} = G\times T_O : \mathscr{X}$.
(3) $\pi_G : \mathscr{X} \to \mathbb{A}^r$ *is a* T_X*-equivariant flat family of affine spherical* $\widehat{G}$*-varieties with categorical quotient by* U *isomorphic to* $\mathbb{A}^{|\mathscr{D}^B|}$, *where* $\widehat{B} = B\times T_O$ *and* $\widehat{T} = T\times T_O$ *act by weights* $-\widehat{\lambda}_D$.
(4) *General fibers of* π_G *are isomorphic to* $G/\!/H_0$ *and* $\pi_G^{-1}(0)$ *is the horospherical contraction of* $G/\!/H_0$.

Proof. (1) follows by choosing in $\Bbbk[\widehat{O}]$ those isotypic components which correspond to $(B\times T)$-eigenweights of $\mathscr{R}(X)^U$. Assertions (2) and (3) are easily derived from the structure of $\mathscr{R}(X)^U$. To deduce (4) from (1), observe that $\mathscr{V}(G/H_0)$ is the preimage of $\mathscr{V}(X)$, whence $\Pi_{G/H_0}^{\min}$ is proportional to $\Pi_X^{\min}$ and, by (T_0), $-\Pi_{G/H_0}^{\min}$ generates the cone spanned by tails of $G/\!/H_0$. □

Example 30.13. If $O = G/Z(G)$ is the adjoint group of G considered as a symmetric space (Example 26.10), then $\mathscr{X} = \operatorname{Env} G$ is the enveloping semigroup.

Generators and relations for $\mathscr{R}(X)$ are described in [Bri18, 3.3].

The Cox sheaf and ring may be defined in a more general setup than above [EKW], [Hau]. Namely let X be any normal variety such that $\operatorname{Cl} X$ is finitely generated. The Cox sheaf $\mathscr{R}_X$ and the Cox ring $\mathscr{R}(X)$ are defined by the formulæ of Definition 30.9, where δ runs over a set of representatives for the divisor classes in $\operatorname{Cl} X$, and $\mathscr{O}_X(\delta)$ is the corresponding reflexive sheaf. If $\operatorname{Cl} X$ is free or $\Bbbk[X]^\times = \Bbbk^\times$, then one may choose multiplication maps $\mathscr{O}(\delta)\otimes\mathscr{O}(\delta')\to\mathscr{O}(\delta+\delta')$ (which are isomorphisms over X^{reg}) making $\mathscr{R}_X$ into a sheaf of graded commutative associative $\mathscr{O}_X$-algebras in a canonical way independent on the choice of representatives of divisor classes, so that $\mathscr{R}(X)$ is indeed a ring. Note that $\mathscr{R}_X=\iota_*\mathscr{R}_{X^{\mathrm{reg}}}$, where $\iota: X^{\mathrm{reg}}\hookrightarrow X$ and $\mathscr{R}(X)=\mathscr{R}(X^{\mathrm{reg}})$. Here X^{reg} may be replaced by any other open subset of X with the complement of codimension >1. So (at least for a free divisor class group) the concepts of Cox sheaf and Cox ring in general can be reduced to Definition 30.9. An equivariant version of these concepts is obtained by replacing $\mathrm{Cl}(X)$ with the group $\mathrm{Cl}_G(X)$ of G-linearized divisor classes (i.e., isomorphism classes of G-linearized reflexive sheaves of rank 1) on a G-variety X. (If $\mathrm{Cl}_G(X)$ has torsion, then one has to require that there are no non-constant G-invariant invertible functions on X.)

If $\mathscr{R}_X$ is a sheaf of finitely generated $\mathscr{O}_X$-algebras (which holds, e.g., whenever X is $\mathbb{Q}$-factorial or $\mathscr{R}(X)$ is finitely generated), then $\mathring{\mathscr{X}}=\operatorname{Spec}_{\mathscr{O}_X}\mathscr{R}_X$ is a quasi-affine normal variety equipped with a natural action of a (possibly disconnected) diagonalizable group T_X such that $\mathfrak{X}(T_X)=\mathrm{Cl}(X)$ (or $\mathrm{Cl}_G(X)$), and the natural map $\mathring{\mathscr{X}}\to X$ is a good quotient for the T_X-action. If $\mathscr{R}(X)$ is finitely generated, then $\mathscr{X}=\operatorname{Spec}\mathscr{R}(X)$ is a normal affine variety containing $\mathring{\mathscr{X}}$ as an open subset with complement of codimension >1.

Cox rings of arbitrary spherical varieties were computed by Brion in [Bri18, §4]. Let X be a spherical variety with the open orbit $O\simeq G/H$, where G may and will be assumed to be of simply connected type. For simplicity, we impose a (not very restrictive) condition $\Bbbk[X]^\times=\Bbbk^\times$.

Note that $\mathscr{R}(X)=\mathscr{R}(X')$, where $X'\subseteq X$ is a smooth open subset obtained by removing all G-orbits of codimension >1. Therefore X may be assumed smooth and toroidal whenever necessary.

As in the wonderful case, we use the notation η_D and $\eta_1,\dots,\eta_k$ for the canonical sections corresponding to the colors D and the G-stable prime divisors $D_1,\dots,D_k$, respectively. Since G' is simply connected semisimple, $\mathscr{R}_X$ admits a unique G'-linearization, so that η_D are $(B\cap G')$-semiinvariant of eigenweights λ_D and η_j are G'-invariant.

Proposition 30.11 extends to the general spherical case, except that G is replaced by G'.

Let $\overline{X}$ denote the wonderful embedding of $\overline{O}=G/\overline{H}$. Various objects related to $\overline{X}$ (divisors, sections, ...) will be denoted in the same way as for X, but equipped with a bar. The natural rational map $\varphi: X \dashrightarrow \overline{X}$ (which is regular if and only if X is toroidal) gives rise to a homomorphism $\varphi^*:\mathscr{R}(\overline{X})\to\mathscr{R}(X)$. It is easy to see that $\varphi^*\eta_{\overline{D}}=\eta_D$, where $D=\varphi^{-1}(\overline{D})$, and $\varphi^*\overline{\eta}_i=\prod_j\eta_j^{\langle v_{D_j},-\lambda_i\rangle}$.

Theorem 30.14 ([Bri18, 4.3]). *$\mathscr{R}(X)\simeq\mathscr{R}(\overline{X})\otimes_{\mathscr{R}(\overline{X})^{G'}}\mathscr{R}(X)^{G'}$. In geometric terms, $\mathscr{X}\simeq\overline{\mathscr{X}}\times_{\overline{\mathscr{X}}/\!/G'}\mathscr{X}/\!/G'$.*

A G-linearization of $\mathscr{R}_X$ may not exist and even if it exists, it may be not unique. In order to take into account the G-action, it is more convenient to use the G-equivariant version of the Cox sheaf and ring. The above results generalize to this setup: $\mathscr{O}(D)$ and $\mathscr{O}(D_j)$ are equipped with canonical G-linearizations so that η_D and η_j

are $Z(G)^0$-invariant, $\mathscr{R}(X)^U$ is freely generated by η_D, η_j as a $\Bbbk[Z(G)^0]$-algebra, and Theorem 30.14 extends with G' replaced by G, see [Bri18, §4].

30.6 Wonderful Varieties. Wonderful embeddings can be characterized intrinsically by the configuration of G-orbits.

Theorem 30.15 ([Lu4]). *A smooth complete G-variety X is a wonderful embedding of a spherical homogeneous space if and only if it satisfies the following conditions:*

(1) *X contains a dense open orbit O.*
(2) *$X \setminus O$ is a divisor with normal crossings, i.e., its components $D_1, \dots, D_r$ are smooth and intersect transversally.*
(3) *For each tuple $1 \le i_1 < \dots < i_k \le r$, the set $D_{i_1} \cap \dots \cap D_{i_k} \setminus \bigcup_{i \ne i_1,\dots,i_k} D_i$ is a single G-orbit. (In particular, it is non-empty.)*

G-varieties satisfying the conditions of the theorem are called *wonderful varieties*.

Sketch of a proof. Wonderful embeddings obviously satisfy the conditions (1)–(3), as a particular case of Theorem 29.2: the toric slice $Z \simeq \mathbb{A}^r$ is transversal to all orbits and the G-stable prime divisors intersect it in the coordinate hyperplanes.

To prove the converse, consider the local structure of X in a neighborhood of the closed orbit Y which is provided by an embedding of X into a projective space. Let $P = P_u \rtimes L$ be a Levi decomposition of $P = P(Y)$. There is a B-chart $\mathring{X} \simeq P_u \times Z$ such that Z is a smooth L-stable locally closed subvariety intersecting Y transversally at the unique P^--fixed point z. It is easy to see that a general dominant one-parameter subgroup $\gamma \in \mathfrak{X}^*(Z(L))$ contracts $\mathring{X}$ to z. Hence Z is L-isomorphic to T_zZ.

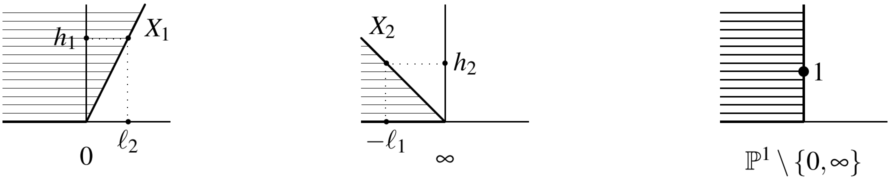
Consider the wonderful subvarieties $X_i = \bigcap_{j \ne i} D_j$ and let λ_i be the T-weights of $T_z(Z \cap X_i)$, $i = 1, \dots, r$. Since $T_zZ = \bigoplus T_z(Z \cap X_i)$, it suffices to prove that $\lambda_1, \dots, \lambda_r$ are linearly independent.

The latter is reduced to the cases $r = 1$ or 2. Indeed, if we already know that X_i and $X_{ij} = \bigcap_{k \ne i,j} D_k$ are wonderful embeddings of spherical spaces, then $\Pi_{X_i}^{\min} = \{\lambda_i\}$ and $\Pi_{X_{ij}}^{\min} = \{\lambda_i, \lambda_j\}$. Thus the λ_i are positive linear combinations of positive roots located at obtuse angles to each other. This implies the linear independence.

The case $r = 1$ stems from Proposition 30.17.

The case $r = 2$ can be reduced to $G = \mathrm{SL}_2$. Indeed, assuming that λ_1, λ_2 are proportional, we see that $c(X) = r(X) = 1$. By Proposition 10.3, O is obtained from a 3-dimensional homogeneous SL_2-space by parabolic induction. Let us describe the colored hypercone $(\mathscr{C}_Y, \mathscr{D}_Y^B)$.

Since T_zZ is contracted to 0 by γ, we have $\Lambda(X) = \mathbb{Z}\lambda$ and $\lambda_i = h_i\lambda$, where $\langle \lambda, \gamma \rangle > 0$ and h_1, h_2 are coprime positive integers. Without loss of generality $\ell_1 h_1 - \ell_2 h_2 = 1$ for some $\ell_1, \ell_2 \in \mathbb{N}$. Consider $T_z(Z \cap X_i)$ as coordinate axes in $T_zZ \simeq Z$ and extend the respective coordinates to $f_1, f_2 \in \Bbbk(X)^{(B)}$. Then we may put $\mathbf{f}_\lambda = f_2^{\ell_2}/f_1^{\ell_1}$, and $\Bbbk(X)^B = \Bbbk(f_2^{h_1}/f_1^{h_2})$. We have the following picture for $(\mathscr{C}_Y, \mathscr{D}_Y^B)$ ($f_2^{h_1}/f_1^{h_2}$ is regarded as an affine coordinate on $\mathbb{P}^1$, colors in $\mathscr{D}_Y^B$ are marked by bold dots, and the rays corresponding to the G-stable divisors X_1, X_2 are marked, too):



By Corollary 12.14, $\mathscr{C}_Y \supseteq \mathscr{V}$. Since $\mathscr{D}_Y^B$ does not contain central colors, Propositions 20.13 and 14.4 imply that X is induced from a wonderful SL_2-variety. However it is easy to see (e.g., from the classification in [Tim2, §5]) that there exist no SL_2-germs with the colored data as above. (Luna uses different arguments in [Lu4].) □

Wonderful varieties play a distinguished rôle in the study of spherical homogeneous spaces, because they are the canonical completions of these spaces having nice geometric properties. To a certain extent this rôle is analogous to that of (generalized) flag varieties in the theory of reductive groups. For symmetric spaces this was already observed by de Concini and Procesi [CP1]. For general spherical spaces this principle was developed by Brion, Knop, Luna, et al [BPa], [Bri8], [Kn8], [Lu3], [Lu5], [Lu6].

30.7 How to Classify Spherical Subgroups. In particular, wonderful varieties are applied to classification of spherical subgroups. The strategy, proposed by Luna, is to reduce the classification to very sober subgroups, which are stabilizers of general position for wonderful varieties, and then to classify the wonderful varieties.

By Theorem 29.5, there are no continuous families of non-conjugate spherical subgroups, and even more:

Proposition 30.16 ([AB3, Cor. 3.2]). *There are finitely many conjugacy classes of sober spherical subgroups $H \subseteq G$.*

Proof. Sober spherical subalgebras of dimension k form a locally closed G-subvariety in $\mathrm{Gr}_k(\mathfrak{g})$. Indeed, the set of spherical subalgebras is open in the variety of k-dimensional Lie subalgebras, and sober subalgebras are those having orbits of maximal dimension. Theorem 29.5 implies that this variety is a finite union of locally closed strata such that all orbits in each stratum have the same stabilizer. But the isotropy subalgebras are nothing else but the points of the strata. Hence each stratum is a single orbit, i.e., there are finitely many sober subalgebras, up to conjugation. As for subgroups, there are finitely many ways to extend H^0 by a (finite) subgroup in $N_G(H^0)/H^0$. □

Note that finiteness fails for non-sober spherical subgroups: H can be extended by countably many diagonalizable subgroups in $N_G(H)/H$.

These results create evidence that spherical subgroups should be classified by some discrete invariants. Such invariants were suggested by Luna, under the names of *spherical systems* and *spherical homogeneous data* (Definition 30.21). They are defined in terms of roots and weights of G and wonderful G-varieties of rank 1.

30.8 Spherical Spaces of Rank 1. For spherical homogeneous spaces of rank 1, standard embeddings are always smooth. Indeed, they are normal G-varieties consisting of two G-orbits—a dense one and another of codimension 1. Furthermore, spherical homogeneous spaces of rank 1 are characterized by existence of a completion by homogeneous divisors.

Proposition 30.17 ([Akh1], [Bri5]). *The following conditions are equivalent:*

(1) *$O = G/H$ is a spherical homogeneous space of rank* 1.
(2) *There exists a smooth complete embedding $X \hookleftarrow O$ such that $X \setminus O$ is a union of G-orbits of codimension* 1.

Moreover, if O is horospherical, then $X \setminus O$ consists of two orbits and $X \simeq G *_Q \mathbb{P}^1$, where $Q \subseteq G$ is a parabolic acting on $\mathbb{P}^1$ via a character. Otherwise $X \setminus O$ is a single orbit and X is a wonderful embedding of O.

Proof. The implication (1) $\Longrightarrow$ (2) and the properties of X easily stem from the Luna–Vust theory: the colored space $\mathscr{E}$ is a line, whence there exists a unique smooth complete toroidal embedding X, which is obtained by adding two homogeneous divisors (corresponding to the two rays of $\mathscr{E}$) if $\mathscr{V} = \mathscr{E}$ and is wonderful if $\mathscr{V}$ is a ray.

To prove (2) $\Longrightarrow$ (1), we consider the local structure of X in a neighborhood of a closed orbit Y. Let $P = P_{\mathrm{u}} \rtimes L$ be a Levi decomposition of $P = P(Y)$. There is a B-chart $\mathring{X} \simeq P_{\mathrm{u}} \times Z$ such that Z is an L-stable affine curve intersecting Y transversally at the unique P^--fixed point. Note that $T \subseteq L$ cannot fix Z pointwise for otherwise O^T would be infinite, which is impossible. Hence $T : Z$ has an open orbit, whence (1). □

Remark 30.18. A similar reasoning proves an embedding characterization of arbitrary rank 1 spaces, due to Panyushev [Pan5]: $r(O) = 1$ if and only if there exists a complete embedding $X \hookleftarrow O$ such that $X \setminus O$ is a divisor consisting of closed G-orbits. Here Z is an affine L-stable subvariety with a pointwise L-fixed divisor $Z \setminus O$ (provided that Y is a general closed orbit), which readily implies that general orbits of $L : Z$ are one-dimensional, whence $r(X) = r(Z) = 1$. On the other hand, it is easy to construct a desired embedding X for a homogeneous space O parabolically induced from SL_2 modulo a finite subgroup, cf. Proposition 10.3.

Spherical homogeneous spaces G/H of rank 1 were classified by Akhiezer [Akh1] and Brion [Bri5]. It is easy to derive the classification from a regular embedding of H into a parabolic $Q \subseteq G$. In the notation of Theorem 9.4 we have an alternative: either $r(M/K) = 1$, $r_{M_*}(Q_{\mathrm{u}}/H_{\mathrm{u}}) = 0$, or vice versa.

In the first case $H_{\mathrm{u}} = Q_{\mathrm{u}}$, i.e., G/H is parabolically induced from an affine spherical homogeneous rank 1 space M/K. Except for the trivial case $M/K \simeq \Bbbk^\times$ (where H is horospherical), K is sober in M and H in G.

In the second case $M = K = M_*$, and $Q_{\mathrm{u}}/H_{\mathrm{u}} \simeq \mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}$ is an M-module such that $(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}) \setminus \{0\}$ is a single M-orbit. Indeed, $\Bbbk[\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}]^{U(M)}$ is generated by one $B(M)$-eigenfunction, namely a highest covector in $(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}})^*$, whence $\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}$ is an HV-cone.

Therefore M acts on $\mathfrak{q}_u/\mathfrak{h}_u \simeq \Bbbk^n$ as $\mathrm{GL}_n(\Bbbk)$ or $\Bbbk^\times \cdot \mathrm{Sp}_n(\Bbbk)$ and the highest weight of $\mathfrak{q}_u/\mathfrak{h}_u$ is a negative simple root.

We deduce that every spherical homogeneous space of rank 1 is either horospherical or parabolically induced from a *primitive* rank 1 space $O = G/H$ with G semisimple and H sober. Primitive spaces are of the two types:

(1) H is reductive.
(2) H is regularly embedded in a maximal parabolic $Q \subseteq G$ which shares a Levi subgroup M with H and $\mathfrak{q}_u/\mathfrak{h}_u$ is a simple M-module of type $\mathrm{GL}_n(\Bbbk) : \Bbbk^n$ or $\Bbbk^\times \cdot \mathrm{Sp}_n(\Bbbk) : \Bbbk^n$ generated by a simple root vector.

Primitive spherical homogeneous spaces of rank 1 are listed in Table 30.1. Those of the first type are easy to classify, e.g., by inspection of Tables 10.1, 10.3, and 26.3. We indicate the embedding $H \hookrightarrow G$ by referring to Table 26.3 (10.1) in the (non-)symmetric case. Primitive spaces of the second type are classified by choosing a Dynkin diagram and its node corresponding to a short simple root α which is adjacent to an extreme node of the remaining diagram, the latter being of type $\mathbf{A}_l$ or $\mathbf{C}_l$. The diagrams are presented in the column "$H \hookrightarrow G$", with the white node corresponding to α.

The wonderful embeddings of spherical homogeneous space of rank 1 are parabolically induced from those of primitive spaces. The latter are easy to describe. For type 1 the construction of 30.3 works whenever $N_G(H) = H$: the wonderful embedding of G/H is realized as $X = \overline{G[v]} \subseteq \mathbb{P}(V(\lambda))$, where $v \in V(\lambda)^{(\widetilde{H})}$, $\lambda \in \Lambda_+(\widetilde{G}/\widetilde{H}^0)$, and $\widetilde{H}$ is the preimage of H in $\widetilde{G}$. If $N_G(H) \neq H$ and λ spans $\Lambda_+(G/H)$, then X is the projective closure of Gv in $\mathbb{P}(V(\lambda) \oplus \Bbbk)$. The simple minimal root of X is the generator of $\Lambda_+(G/H)$.

For type 2 the wonderful embedding is $X = G *_Q \mathbb{P}(Q_u/H_u \oplus \Bbbk)$. Indeed, Q_u acts on the M-module Q_u/H_u by affine translations, whence the projective closure of Q_u/H_u consists of two Q-orbits—the affine part and the hyperplane at infinity. Here $\Pi_X^{\min} = \{w_M \alpha\}$.

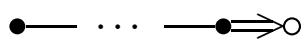
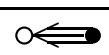
Simple minimal roots of arbitrary wonderful G-varieties of rank 1 are called *spherical roots* of G. They are non-negative linear combinations of Π_G. Let Σ_G denote the set of all spherical roots. It is a finite set, which is easy to find from the classification of wonderful varieties of rank 1.

Spherical roots of reductive groups of simply connected type are listed in Table 30.2. Namely, $\lambda \in \Sigma_G$ if and only if it is a spherical root of a simple factor, or a product of two simple factors, indicated in the first column of the table. For each spherical root λ, we indicate the Dynkin diagram of the simple roots occurring in the decomposition of λ with positive coefficients. The numbering of the simple roots α_i is according to [OV], and α_i, α'_j denote simple roots of different simple factors. For arbitrary G, Σ_G is obtained from $\Sigma_{\widetilde{G}}$ by removing the spherical roots that are not in the weight lattice of G. Note that if $\lambda, \mu \in \Sigma_G$ are proportional, then $\lambda = 2\mu$ or $\mu = 2\lambda$, and also if $\lambda \in \Sigma_G \setminus \mathbb{Z}\Delta_G$, then $2\lambda \in \Sigma_G \cap \mathbb{Z}\Delta_G$.

More generally, two-orbit complete (normal) G-varieties were classified by Cupit-Foutou [C-F1] and Smirnov [Sm-A]. All of them are spherical.

Wonderful varieties of rank 2 were classified by Wasserman [Wa].

Table 30.1 Wonderful varieties of rank 1

No.	G	H	$H \hookrightarrow G$	$\Pi_{G/H}^{\min}$	Wonderful embedding
1	$\mathrm{SL}_2 \times \mathrm{SL}_2$	SL_2	diagonal	$\omega + \omega'$	$X = \{(x:t) \mid \det x = t^2\} \subset \mathbb{P}(\mathrm{L}_2 \oplus \Bbbk)$
2	$\mathrm{PSL}_2 \times \mathrm{PSL}_2$	PSL_2		$2\omega + 2\omega'$	$\mathbb{P}(\mathrm{L}_2)$
3	SL_n	$\mathrm{S}(\mathrm{L}_1 \times \mathrm{L}_{n-1})$	symmetric No. 1	$\omega_1 + \omega_{n-1}$	$\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^*$
4	PSL_2	PO_2	symmetric No. 3	4ω	$\mathbb{P}(\mathfrak{sl}_2)$
5	Sp_{2n}	$\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$	symmetric No. 4	ω_2	$\mathrm{Gr}_2(\Bbbk^{2n})$
6	Sp_{2n}	$B(\mathrm{Sp}_2) \times \mathrm{Sp}_{2n-2}$		ω_2	$\mathrm{Fl}_{1,2}(\Bbbk^{2n})$
7	SO_n	SO_{n-1}	symmetric No. 6	ω_1	$X = \{(x:t) \mid (x,x) = t^2\} \subset \mathbb{P}^n$
8	SO_n	$\mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{n-1})$		$2\omega_1$	$\mathbb{P}^{n-1}$
9	SO_{2n+1}	$\mathrm{GL}_n \rightthreetimes \bigwedge^2 \Bbbk^n$		ω_1	$X = \{(V_1, V_2) \mid V_1 \subset V_1^{\perp}\} \subset \mathrm{Fl}_{n,2n}(\Bbbk^{2n+1})$
10	Spin_7	$\mathbf{G}_2$	non-symmetric No. 10	ω_3	$X = \{(x:t) \mid (x,x) = t^2\} \subset \mathbb{P}(V(\omega_3) \oplus \Bbbk)$
11	SO_7	$\mathbf{G}_2$		$2\omega_3$	$\mathbb{P}(V(\omega_3))$
12	$\mathbf{F}_4$	$\mathbf{B}_4$	symmetric No. 17	ω_1	
13	$\mathbf{G}_2$	SL_3	non-symmetric No. 12	ω_1	$X = \{(x:t) \mid (x,x) = t^2\} \subset \mathbb{P}(V(\omega_1) \oplus \Bbbk)$
14	$\mathbf{G}_2$	$N(\mathrm{SL}_3)$		$2\omega_1$	$\mathbb{P}(V(\omega_1))$
15	$\mathbf{G}_2$	$\mathrm{GL}_2 \rightthreetimes (\Bbbk \oplus \Bbbk^2) \otimes \bigwedge^2 \Bbbk^2$		$\omega_2 - \omega_1$	

30.9 Localization of Wonderful Varieties. For arbitrary wonderful varieties, many questions can be reduced to the case of rank ≤ 2 via the procedure of localization [Lu5], [Lu6, 3.2].

Given a wonderful variety X with the open G-orbit O, there is a bijection $D_i \leftrightarrow \lambda_i$ ($i = 1, \dots, r$) between the component set of ∂X and $\Pi_X^{\min}$. Namely λ_i is orthogonal to the facet of $\mathscr{V}$ complementary to the ray which corresponds to D_i. Also, λ_i is the T-weight of $T_z X / T_z D_i$ at the unique B^--fixed point z.

For any subset $\Sigma \subset \Pi_X^{\min}$, put $X^{\Sigma} = \bigcap_{\lambda_i \notin \Sigma} D_i$, the *localization* of X at Σ. It is a wonderful variety with $\Pi_{X^\Sigma}^{\min} = \Sigma$, and all colors in $\mathscr{D}^B(X^\Sigma)$ are obtained as irreducible components of $\overline{D} \cap X^\Sigma$, $D \in \mathscr{D}^B$. (To see the latter, observe that every color on X^Σ is contained in the zeroes of a B-eigenform in projective coordinates, which extends to X by complete reducibility of G-modules.) In particular, the wonderful subvarieties X_i, X_{ij} of ranks $1,2$ considered in the proof of Theorem 30.15 are the localizations of X at $\{\lambda_i\}$, $\{\lambda_i, \lambda_j\}$, respectively.

Another kind of localization is defined by choosing a subset $I \subset \Pi$. Let P_I be the respective standard parabolic in G, with the standard Levi subgroup L_I, and $T_I = Z(L_I)^0$. Denote by $Z^I, \mathring{X}^I, X^I$ the sets of T_I-fixed points in Z, $\mathring{X}$, and $X_I := P_I \mathring{X} = L_I \mathring{X}$, respectively.

Lemma 30.19. (1) *The contraction by a general dominant one-parameter subgroup $\gamma \in \mathfrak{X}^*(T_I)$ gives a P_I-equivariant retraction $\pi_I : X_I \to X^I$, $\pi_I(x) = \lim_{t \to 0} \gamma(t) x$ (where P_I is assumed to act on X^I via its quotient L_I modulo $(P_I)_{\mathrm{u}}$).*

Table 30.2 Spherical roots

G	Σ_G
$\mathbf{A}_l$	$\alpha_i+\cdots+\alpha_j$ ($\mathbf{A}_{j-i+1}$, $i\le j$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $(\alpha_1+\alpha_3)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=3$), $\alpha_{i-1}+2\alpha_i+\alpha_{i+1}$ ($\mathbf{D}_3$, $1<i<l$), $(\alpha_1+2\alpha_2+\alpha_3)/2$ ($\mathbf{D}_3$, $l=3$)
$\mathbf{B}_l$	$\alpha_i+\cdots+\alpha_j$ ($\mathbf{A}_{j-i+1}$, $i\le j<l$), α_l ($\mathbf{A}_1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $(\alpha_1+\alpha_3)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=3,4$), $\alpha_{i-1}+2\alpha_i+\alpha_{i+1}$ ($\mathbf{D}_3$, $1<i<l-1$), $(\alpha_1+2\alpha_2+\alpha_3)/2$ ($\mathbf{D}_3$, $l=4$), $\alpha_i+\cdots+\alpha_l$ ($\mathbf{B}_{l-i+1}$, $i<l$), $2\alpha_i+\cdots+2\alpha_l$ ($\mathbf{B}_{l-i+1}$, $i<l$), $\alpha_{l-2}+2\alpha_{l-1}+3\alpha_l$ ($\mathbf{B}_3$), $(\alpha_1+2\alpha_2+3\alpha_3)/2$ ($\mathbf{B}_3$, $l=3$)
$\mathbf{C}_l$	$\alpha_i+\cdots+\alpha_j$ ($\mathbf{A}_{j-i+1}$, $i\le j<l$), α_l ($\mathbf{A}_1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $\alpha_{i-1}+2\alpha_i+\alpha_{i+1}$ ($\mathbf{D}_3$, $1<i<l-1$), $\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{l-1}+\alpha_l$ ($\mathbf{C}_{l-i+1}$, $i<l$), $2\alpha_{l-1}+2\alpha_l$ ($\mathbf{C}_2$)
$\mathbf{D}_l$	$\alpha_{i_1}+\cdots+\alpha_{i_k}$ ($\mathbf{A}_k$, $k\ge1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $2\alpha_{i_1}+\alpha_{i_2}+\alpha_{i_3}$ ($\mathbf{D}_3$), $(2\alpha_{i_1}+\alpha_{i_2}+\alpha_{i_3})/2$ ($\mathbf{D}_3$, $l=4$), $2\alpha_i+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$ ($\mathbf{D}_{l-i+1}$, $i<l-1$), $\alpha_i+\cdots+\alpha_{l-2}+(\alpha_{l-1}+\alpha_l)/2$ ($\mathbf{D}_{l-i+1}$, $i<l-1$), $(\alpha_{l-1}+\alpha_l)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$), $(\alpha_1+\alpha_3)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=4$), $(\alpha_1+\alpha_4)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=4$)
$\mathbf{E}_l$	$\alpha_{i_1}+\cdots+\alpha_{i_k}$ ($\mathbf{A}_k$, $k\ge1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $2\alpha_{i_1}+\cdots+2\alpha_{i_{k-2}}+\alpha_{i_{k-1}}+\alpha_{i_k}$ ($\mathbf{D}_k$, $k\ge3$)
$\mathbf{F}_4$	α_i ($\mathbf{A}_1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $\alpha_i+\alpha_{i+1}$ ($\mathbf{A}_2$, $i\ne2$), $\alpha_2+\alpha_3$ ($\mathbf{C}_2$), $2\alpha_2+2\alpha_3$ ($\mathbf{C}_2$), $\alpha_1+2\alpha_2+\alpha_3$ ($\mathbf{C}_3$), $\alpha_2+\alpha_3+\alpha_4$ ($\mathbf{B}_3$), $2\alpha_2+2\alpha_3+2\alpha_4$ ($\mathbf{B}_3$), $3\alpha_2+2\alpha_3+\alpha_4$ ($\mathbf{B}_3$), $2\alpha_1+3\alpha_2+2\alpha_3+\alpha_4$ ($\mathbf{F}_4$)
$\mathbf{G}_2$	α_i ($\mathbf{A}_1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_1+\alpha_2$ ($\mathbf{G}_2$), $2\alpha_1+\alpha_2$ ($\mathbf{G}_2$), $4\alpha_1+2\alpha_2$ ($\mathbf{G}_2$)
$\mathbf{X}_l\times\mathbf{Y}_m$	$\alpha_i+\alpha'_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $(\alpha_l+\alpha'_m)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $\mathbf{X}=\mathbf{Y}=\mathbf{C}$, $l,m\ge1$)

(2) X^I *is a wonderful* L_I*-variety with* $P(X^I)=P\cap L_I$, $\Pi^{\min}_{X^I}=\Pi^{\min}_X\cap\langle I\rangle$, *and the colors of* X^I *are in bijection, given by the pullback along* π_I, *with the* P_I*-unstable colors of* X.
(3) $\mathring{X}^I\simeq(P_\mathrm{u}\cap L_I)\times Z^I$ *is the* $(B\cap L_I)$*-chart of* X^I *intersecting all orbits.*
(4) $(P_I^-)_\mathrm{u}$ *fixes* X^I *pointwise.*

Proof. It is obvious that $\gamma(t)$ contracts $\mathring{X}\simeq P_\mathrm{u}\times Z$ onto $\mathring{X}^I\simeq(P_\mathrm{u}\cap L_I)\times Z^I$, while the conjugation by $\gamma(t)$ contracts P_I to L_I, as $t\to\infty$. Hence π_I extends to a retraction of X_I onto $X^I=L_I\mathring{X}^I$, and $\pi_I^{-1}(\mathring{X}^I)=\mathring{X}$ since $X_I\setminus\mathring{X}$ is closed and γ-stable. Thus the P_I-unstable colors on X intersect X_I and are the pullbacks of the colors on X^I.

Since $(P_I^-)_\mathrm{u}$-orbits are connected, it suffices to prove in (4) that $(P_I^-)_\mathrm{u}x\cap\mathring{X}=\{x\}$, $\forall x\in\mathring{X}^I$. If $gx\in\mathring{X}$ for some $g\in(P_I^-)_\mathrm{u}$, then $\gamma(t)gx=\gamma(t)g\gamma(t)^{-1}x\to x$ as $t\to\infty$, whence $gx=x$, because $\gamma(t)$ contracts $\mathring{X}$ to $\mathring{X}^I$ as $t\to0$.

Now (4) implies that X^I is closed in X, whence complete: otherwise $\overline{X^I}\setminus X^I$ would contain a B^--fixed point distinct from z. The structure of $\mathring{X}^I$ readily implies the remaining assertions on X^I in (2): both G- and B-orbits intersect $\mathring{X}^I$ in the orbits of $(P_{\mathrm{u}} \cap L_I)T$, and $\Pi_X^{\min} \cap \langle I \rangle$ is the set of T-weights of Z^I. □

The wonderful variety X^I is called the *localization* of X at I. It is easy to see that $X^I \subseteq X^\Sigma = GX^I$, where $\Sigma = \Pi_X^{\min} \cap \langle I \rangle$. If $I \supseteq \Pi_L$, then $X^\Sigma = G *_{P_I^-} X^I$.

It is helpful to extend localization to an arbitrary spherical homogeneous space $O = G/H$ using an arbitrary complete toroidal embedding $X \hookleftarrow O$ instead of the wonderful one. For any components $D_1, \dots, D_s$ of ∂X, the intersection $X' = D_1 \cap \dots \cap D_s$ is either empty or a complete toroidal embedding of the G-orbit $O' \subseteq X$ corresponding to the minimal cone $\mathscr{C}'$ in the fan of X containing $v_{D_1}, \dots, v_{D_s}$. If X is smooth, then X' is smooth, too. If X' contains a unique closed G-orbit Y or, equivalently, $\mathscr{C}'$ is a face of a unique maximal cone $\mathscr{C}_Y$ in the fan of X, then X' is the standard embedding of O'. We have $\Pi_{X'}^{\min} = \Sigma := \Pi_X^{\min} \cap (\mathscr{V}')^\perp$, where $\mathscr{V}'$ is the minimal face of $\mathscr{V} = \mathscr{V}(O)$ containing $\mathscr{C}'$. For wonderful X we have $X' = X^\Sigma$. In particular, if $\mathscr{C}'$ is a solid subcone in the facet of $\mathscr{V}$ orthogonal to $\lambda \in \Pi_O^{\min}$, then X' is a wonderful variety of rank 1 with $\Pi_{X'}^{\min} = \{\lambda\}$, whence $\Pi_O^{\min} \subseteq \Sigma_G$.

Also for any $I \subset \Pi$ such that $\Lambda(O) \not\subset \langle I \rangle$ one can find $X \hookleftarrow O$ and a general dominant one-parameter subgroup $\gamma \in \mathfrak{X}^*(T_I)$ such that the image of $-\gamma$ is contained in a unique solid cone $\mathscr{C}_Y$ in the fan of X. (It suffices to take care lest the image of $\langle I \rangle^\perp$ in $\mathscr{E}(O)$ should lie in a hyperplane which separates two neighboring solid cones in the fan.) Starting with $\mathring{X} = \mathring{X}_Y$, one defines X^I as above and generalizes Lemma 30.19, except that in (2) one may only assert that X^I is standard, but it may be no longer wonderful (i.e., be singular if X is so). If $\Lambda(O) \subset \langle I \rangle$, then X^I can be defined for any toroidal embedding $X \hookleftarrow O$ using $\mathring{X} = X \setminus \bigcup_{D \in \mathscr{D}^B} D$. Lemma 30.19 extends to this setup except that X^I is wonderful (resp. standard, complete, smooth) if and only if X is so.

30.10 Types of Simple Roots and Colors. In particular, the localization of a complete toroidal variety X at a single root $\alpha \in \Pi$ yields a smooth complete subvariety X^α of rank ≤ 1 acted on by $S_\alpha = L'_\alpha \simeq \mathrm{SL}_2(\Bbbk)$ or $\mathrm{PSL}_2(\Bbbk)$. The classification of complete varieties of rank ≤ 1, together with Lemma 30.19, allows all simple roots to be subdivided into four types:

(p) $\alpha \in \Pi_L$. Here X^α is a point and P_α leaves all colors stable.
(b) $\alpha \notin \mathbb{Q}_+ \Pi_O^{\min} \cup \Pi_L$. If X^α is wonderful, then $r(X^\alpha) = 0$ whence $X^\alpha = S_\alpha / B \cap S_\alpha \simeq \mathbb{P}^1$; otherwise $r(X^\alpha) = 1$ and $X^\alpha \simeq S_\alpha *_{B \cap S_\alpha} \mathbb{P}^1$, where $B \cap S_\alpha$ acts on $\mathbb{P}^1$ via a character. There is a unique P_α-unstable color $D_\alpha = \overline{\pi_\alpha^{-1}(o)}$ or $\overline{\pi_\alpha^{-1}(e * \mathbb{P}^1)}$.
(a) $\alpha \in \Pi_O^{\min}$. Here $r(X^\alpha) = 1$ and $X^\alpha \simeq \mathbb{P}^1 \times \mathbb{P}^1$. There are two P_α-unstable colors $D_\alpha^+ = \overline{\pi_\alpha^{-1}(\mathbb{P}^1 \times \{o\})}$ and $D_\alpha^- = \overline{\pi_\alpha^{-1}(\{o\} \times \mathbb{P}^1)}$.
(a') $2\alpha \in \Pi_O^{\min}$. Here $r(X^\alpha) = 1$ and $X^\alpha \simeq \mathbb{P}^2 = \mathbb{P}(\mathfrak{s}_\alpha)$. There is a unique P_α-unstable color $D_\alpha = \overline{\pi_\alpha^{-1}(\mathbb{P}(\mathfrak{b} \cap \mathfrak{s}_\alpha))}$.

The type of a color $D \in \mathscr{D}^B$ is defined as the type of $\alpha \in \Pi$ such that P_α moves D. Using Lemma 30.20 below, the localization at $\{\alpha, \beta\} \subseteq \Pi$, and the classification of wonderful varieties of rank ≤ 1, one verifies that, as a rule, each $D \in \mathscr{D}^B$ is moved by a unique P_α, with the following exceptions: $D_\alpha = D_\beta$ if and only if α, β are pairwise orthogonal simple roots of type b such that $\alpha+\beta \in \Pi_O^{\min} \sqcup 2\Pi_O^{\min}$; two sets $\{D_\alpha^\pm\}$ and $\{D_\beta^\pm\}$ may intersect in one color for distinct α,β of type a. (In fact, $D=D'$ for $D\in\{D_\alpha^\pm\}$, $D'\in\{D_\beta^\pm\}$ if and only if $\varkappa(D)=\varkappa(D')$, by axiom (A1) in Definition 30.21.) In particular, each color belongs to exactly one type. We obtain disjoint partitions $\Pi=\Pi^a\sqcup\Pi^{a'}\sqcup\Pi^b\sqcup\Pi^p$, $\mathscr{D}^B=\mathscr{D}^a\sqcup\mathscr{D}^{a'}\sqcup\mathscr{D}^b$ according to the types of simple roots and colors.

Lemma 30.20. *For any $\lambda\in\Lambda(O)$ we have*

$$\begin{aligned} \langle D_\alpha^+,\lambda\rangle+\langle D_\alpha^-,\lambda\rangle &= \langle\alpha^\vee,\lambda\rangle, && \forall\alpha\in\Pi^a,\\ \langle D_\alpha,\lambda\rangle &= \langle\tfrac{\alpha^\vee}{2},\lambda\rangle, && \forall\alpha\in\Pi^{a'},\\ \langle D_\alpha,\lambda\rangle &= \langle\alpha^\vee,\lambda\rangle, && \forall\alpha\in\Pi^b. \end{aligned}$$

Proof. We use the localization at α of a smooth complete toroidal embedding $X \hookleftarrow O$. Let $Y^\alpha\simeq S_\alpha/(B^-\cap S_\alpha)\simeq\mathbb{P}^1$ be a closed S_α-orbit in X^α. Namely Y^α is the diagonal of $X^\alpha\simeq\mathbb{P}^1\times\mathbb{P}^1$ in type a, a conic in $X^\alpha\simeq\mathbb{P}^2$ in type a', and a section of $X^\alpha\to\mathbb{P}^1$ in type b. Put $\delta_\lambda=\sum_{D\in\mathscr{D}^B}\langle D,\lambda\rangle D$. From the description of P_α-stable and unstable colors, we readily derive $\langle Y^\alpha,\delta_\lambda\rangle=\langle D_\alpha^+,\lambda\rangle+\langle D_\alpha^-,\lambda\rangle$, $2\langle D_\alpha,\lambda\rangle$, or $\langle D_\alpha,\lambda\rangle$, depending on the type of α.

On the other hand, $\delta_\lambda\sim-\sum\langle v_i,\lambda\rangle D_i$, where D_i runs over all G-stable prime divisors in X and $v_i\in\mathscr{V}$ is the corresponding G-valuation. Since $\mathscr{O}(D_i)|_{D_i}$ is the normal bundle to D_i, the fiber of $\mathscr{O}(D_i)$ at the B^--fixed point $z\in Y^\alpha$ is T_zX/T_zD_i for each $D_i\supseteq Y^\alpha$. Note that the T-weights λ_i of these fibers form the basis of $-\mathscr{C}_Y^\vee$ dual to the basis of $-\mathscr{C}_Y$ formed by the $-v_i$, where $Y=Gz$ is the closed G-orbit in X containing Y^α. Hence $\mathscr{O}(\delta_\lambda)|_{Y^\alpha}=\mathscr{L}(-\sum_{D_i\supseteq Y}\langle v_i,\lambda\rangle\lambda_i)=\mathscr{L}(\lambda)$ and $\langle Y^\alpha,\delta_\lambda\rangle=\deg\mathscr{L}(\lambda)=\langle\alpha^\vee,\lambda\rangle$. The lemma follows. □

30.11 Combinatorial Classification of Spherical Subgroups and Wonderful Varieties. These results show that $\mathscr{D}^{a'},\mathscr{D}^b$ as abstract sets and their representation in $\mathscr{E}(O)$ are determined by Π^p and $\Pi_O^{\min}$. The colors of type a, together with the weight lattice, the parabolic P, and the simple minimal roots, form a collection of combinatorial invariants supposed to identify O up to isomorphism. Namely $(\Lambda(O),\Pi^p,\Pi_O^{\min},\mathscr{D}^a)$ is a homogeneous spherical datum in the sense of the following

Definition 30.21 ([Lu6, §2]). A *homogeneous spherical datum* is a collection $(\Lambda,\Pi^p,\Sigma,\mathscr{D}^a)$, where Λ is a sublattice in $\mathfrak{X}(T)$, $\Pi^p\subseteq\Pi_G$, $\Sigma\subseteq\Sigma_G\cap\Lambda$ is a linearly independent set consisting of indivisible vectors in Λ, and $\mathscr{D}^a$ is a finite set equipped with a map $\varkappa:\mathscr{D}^a\to\Lambda^*$, which satisfies the following axioms:

(A1) $\langle\varkappa(D),\lambda\rangle\le 1$, $\forall D\in\mathscr{D}^a$, $\lambda\in\Sigma$, and the equality is reached if and only if $\lambda=\alpha\in\Sigma\cap\Pi$ and $D=D_\alpha^\pm$, where $D_\alpha^+,D_\alpha^-\in\mathscr{D}^a$ are two distinct elements depending on α.
(A2) $\varkappa(D_\alpha^+)+\varkappa(D_\alpha^-)=\alpha^\vee$ on Λ for any $\alpha\in\Sigma\cap\Pi$.
(A3) $\mathscr{D}^a=\{D_\alpha^\pm\mid\alpha\in\Sigma\cap\Pi\}$
(Σ1) If $\alpha\in\Pi\cap\frac{1}{2}\Sigma$, then $\langle\alpha^\vee,\Lambda\rangle\subseteq 2\mathbb{Z}$ and $\langle\alpha^\vee,\Sigma\setminus\{2\alpha\}\rangle\le 0$.

($\Sigma 2$) If $\alpha, \beta \in \Pi$, $\alpha \perp \beta$, and $\alpha + \beta \in \Sigma \sqcup 2\Sigma$, then $\alpha^\vee = \beta^\vee$ on Λ.
(S) $\langle \alpha^\vee, \Lambda \rangle = 0$, $\forall \alpha \in \Pi^p$, and the pair (λ, Π^p) comes from a wonderful variety of rank 1 for any $\lambda \in \Sigma$.

A *spherical system* is a triple $(\Pi^p, \Sigma, \mathscr{D}^a)$ satisfying the above axioms with $\Lambda = \mathbb{Z}\Sigma$.

The homogeneous spherical datum of the open orbit in a wonderful variety amounts to its spherical system. It is easy to see that there are finitely many spherical systems for given G. There is a transparent graphical representation of spherical systems by *spherical diagrams* [Lu6, 4.1], [BraL, 1.2.4], which are obtained from the Dynkin diagram of G by adding some supplementary data describing types of simple roots, colors, and spherical roots.

For the homogeneous spherical datum of O, most of the axioms (A1)–(A3), ($\Sigma 1$)–($\Sigma 2$), (S) are verified using the above results together with some additional general arguments. For instance, the inequality in ($\Sigma 1$) stems from the fact that $\Sigma = \Pi_O^{\min}$ is a base of a root system $\Delta_O^{\min}$. On the other hand, each axiom involves at most two simple or spherical roots, like the axioms of classical root systems. Thus the localizations at one or two simple or spherical roots reduce the verification to wonderful varieties of rank ≤ 2.

Actually the list of axioms was obtained by inspecting the classification of wonderful varieties of rank ≤ 2, which leads to the following conclusion: spherical systems (homogeneous data) with $|\Sigma| \leq 2$ bijectively correspond to wonderful varieties of rank ≤ 2 (resp. to spherical homogeneous spaces $O = G/H$ with $r(G/N_G(H)) \leq 2$). It is tempting to extend this combinatorial classification to arbitrary wonderful varieties and spherical spaces. Now this program is fulfilled after a decade of joint efforts of several researchers.

Theorem 30.22. *For any connected reductive group G, there are natural bijections:*

$$\left\{ \begin{array}{c} \textit{spherical homogeneous} \\ G\textit{-spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homogeneous spherical} \\ \textit{data for } G \end{array} \right\} \tag{30.2}$$

$$\{\textit{wonderful } G\textit{-varieties}\} \longleftrightarrow \{\textit{spherical systems for } G\} \tag{30.3}$$

Example 30.23. Solvable spherical subgroups or, more precisely, spherical subgroups contained in a Borel subgroup of G were classified by Luna [Lu3]. Spherical data arising here satisfy $\Sigma = \Pi^a$, $\Pi^{a'} = \Pi^p = \emptyset$, $D_\alpha^- \neq D_\beta^\pm$, and $\langle \varkappa(D_\alpha^+), \beta \rangle < 0 \implies \langle \varkappa(D_\beta^+), \alpha \rangle = 0$ ($\forall \alpha, \beta \in \Pi^a$, $\alpha \neq \beta$).

Indeed, by Example 15.13 a spherical subgroup $H \subset G$ is contained in a Borel subgroup if and only if there exists a subset $\mathscr{R}_0 \subset \mathscr{D}^B$ such that $\mathscr{V} \cup \varkappa(\mathscr{R}_0)$ generates $\mathscr{E}$ as a cone and $P[\mathscr{D}^B \setminus \mathscr{R}_0] = B$. This means that $\mathscr{R}_0 \subset \mathscr{D}^a$ contains a unique element, say D_α^+, from each pair $D_\alpha^\pm$, $\Sigma = \Pi^a$ (because $\langle \mathscr{V} \cup \varkappa(\mathscr{R}_0), \Sigma \setminus \Pi^a \rangle \leq 0$ by (A1)), and $\Pi^{a'} = \Pi^p = \emptyset$. Since $\mathscr{D}^B \setminus \mathscr{R}_0$ consists of the preimages of the Schubert divisors, each of these divisors is moved by a unique minimal parabolic, whence the condition on D_α^-. The condition on pairings holds, because otherwise $\langle \mathscr{V} \cup \varkappa(\mathscr{R}_0), \alpha + \beta \rangle \leq 0$ by (A1), a contradiction.

An explicit description of connected solvable spherical subgroups was recently obtained by Avdeev [Avd].

30.12 Proof of the Classification Theorem.

Stage 1. We may assume that G is of simply connected type. The bijection (30.2) was proved by Luna provided that $G/Z(G)$ satisfies (30.3) [Lu6, §7]. The basic idea is to replace $O = G/H$ by $\overline{O} = G/\overline{H}$. This passage preserves the types of simple roots and colors, and $\Pi_{\overline{O}}^{\min}$ is obtained from $\Pi_O^{\min}$ by a dilation: some $\lambda \in \Pi_O^{\min} \setminus (\Pi \sqcup 2\Pi)$ are replaced by 2λ. It is not hard to prove that spherical subgroups H with fixed very sober hull $\overline{H}$ bijectively correspond to homogeneous spherical data $(\Lambda, \Pi^p, \Sigma, \mathscr{D}^a)$ such that $(\Pi^p, \Pi_{\overline{O}}^{\min}, \mathscr{D}^a)$ is the spherical system of $\overline{O}$, $\Lambda \supset \Pi_{\overline{O}}^{\min}$, and Σ is obtained from $\Pi_{\overline{O}}^{\min}$ by replacing $\lambda \in \Pi_{\overline{O}}^{\min} \setminus (\Pi \sqcup 2\Pi)$ with $\lambda/2$ whenever $\lambda/2 \in \Lambda$ [Lu6, §6].

Indeed, let H_0 denote the common kernel of all characters in $\mathfrak{X}(\overline{H})$. In the notation of 30.5, $T_{\overline{O}} = \overline{H}/H_0$ is a diagonalizable group with $\mathfrak{X}(T_{\overline{O}}) = \mathfrak{X}(\overline{H})$ and $H_0 \subseteq H \subseteq \overline{H}$. (Note that H_0 itself may be not spherical and even if it is spherical, it may happen that $\overline{H_0} \neq \overline{H}$. Examples are: $G = \mathrm{SL}_2(\Bbbk)$, $\overline{H} = T$, $H_0 = \{e\}$, and $\overline{H} = N(T)$, $H_0 = T$, respectively.) So the problem is to classify intermediate spherical subgroups between H_0 and $\overline{H}$ with very sober hull $\overline{H}$. Intermediate subgroups H bijectively correspond to sublattices $\mathfrak{X}(\overline{H}/H) \subseteq \mathfrak{X}(T_{\overline{O}})$. It remains to determine which of them are spherical and have very sober hull $\overline{H}$.

We shall use the notation of 30.5. The equations $\eta_D \in \Bbbk[G]^{(B\times\overline{H})}$ of colors $D \subset \overline{O}$ can be chosen $Z(G)^0$-invariant, so that λ_D, χ_D vanish on $Z(G)^0$. There is an equivariant version of the commutative diagram (30.1):

$$
\begin{array}{ccccc}
\mathrm{Pic}_G \overline{O} & \longleftarrow & \mathrm{Pic}_G \overline{X} & \longrightarrow & \mathrm{Pic}_G \overline{Y} \\
\| & & \| & & \| \\
\mathfrak{X}(T_{\overline{O}}) & \longleftarrow & \mathbb{Z}\overline{\mathscr{D}}^B \oplus \mathfrak{X}(G) & \longrightarrow & \mathfrak{X}(P) \\
\chi_D + \mu & \longleftarrow & (D, \mu) & \longrightarrow & \lambda_D - \mu,
\end{array}
$$

where $\overline{\mathscr{D}}^B$ is the set of colors of $\overline{O}$ and $\overline{X} \hookleftarrow \overline{O}$ is the wonderful embedding with the closed G-orbit $\overline{Y}$. The weights λ_D are easy to determine.

Lemma 30.24 ([Fo]). *For any color D of a spherical homogeneous space $O = G/H$ one has*

$$
\lambda_D = \begin{cases} \sum_{D = D_{\alpha_i}^{\pm}} \omega_i, & D \in \mathscr{D}^a, \\ 2\omega_i, & D = D_{\alpha_i} \in \mathscr{D}^{a'}, \\ \sum_{D = D_{\alpha_i}} \omega_i \quad (\leq 2 \text{ summands}), & D \in \mathscr{D}^b, \end{cases}
$$

where $\omega_i \in \mathfrak{X}_+$ denote the fundamental weights corresponding to the simple roots $\alpha_i \in \Pi$.

Proof. Clearly, λ_D is a positive linear combination of the ω_i such that P_{α_i} moves D. In order to determine the coefficient at ω_i, it suffices to localize at α_i and consider the respective spherical SL_2-variety X^{α_i}. □

The subgroup H is not spherical if and only if there exists an eigenspace $\Bbbk[G]^{(B\times H)}_{(\lambda,\chi)}$ of dimension >1. Since $\overline{H}$ acts on $\mathfrak{X}(H)$ via a finite quotient group, we may multiply this eigenspace by $(B\times H)$-eigenfunctions whose H-weights run over the $\overline{H}$-orbit of χ (except χ itself) and thus assume that χ is $\overline{H}$-invariant, i.e., $\overline{H}$ acts on $\Bbbk[G]^{(B\times H)}_{(\lambda,\chi)}$ by right translations of an argument. Taking an $\overline{H}$-eigenbasis of $\Bbbk[G]^{(B\times H)}_{(\lambda,\chi)}$, we obtain at least two $(B\times\overline{H})$-eigenfunctions of distinct eigenweights (λ,χ_i) $(i=1,2)$ such that $\chi_i|_H=\chi$. This means that the preimage $\widehat{\Lambda}$ of $\mathfrak{X}(\overline{H}/H)$ in $\operatorname{Pic}_G(\overline{X})$ maps to $\mathfrak{X}(B)$ non-injectively. Note that the image of $\widehat{\Lambda}$ is nothing else but the weight lattice $\Lambda=\Lambda(O)$.

Thus H is spherical if and only if $\widehat{\Lambda}$ injects into $\mathfrak{X}(B)$. In other words, for each $\lambda\in\Lambda$ there must be a unique presentation $\lambda=\sum m_D\lambda_D-\mu$ such that $\sum m_D\chi_D+\mu\in\mathfrak{X}(\overline{H}/H))$. This allows a map $\varkappa:\overline{\mathscr{D}}^B\to\Lambda^*$ to be defined such that $\langle\varkappa(D),\lambda\rangle=m_D$. This map is well defined on colors of types a',b for any H, because $m_{D_\alpha}=\langle\alpha^\vee/2,\lambda\rangle$ or $\langle\alpha^\vee,\lambda\rangle$ for $\alpha\in\Pi^{a'}$ or Π^b, respectively, by Lemma 30.24, and so the existence of $\varkappa$ is essential only for colors of type a. Clearly, $\varkappa$ is compatible with the respective map for the spherical system of $\overline{O}$. Also by Lemma 30.24 the axioms (A2), (Σ1)–(Σ2), (S) hold. The group $\mathfrak{X}(\overline{H}/H)$ is recovered from $\Lambda,\varkappa$ as the set of all $\chi=\sum\langle\varkappa(D),\lambda\rangle\chi_D-\lambda|_{Z(G)^0}$ with $\lambda\in\Lambda$.

Now $\Sigma=\Pi_O^{\min}$ is obtained from $\Pi_{\overline{O}}^{\min}$ by replacing spherical roots with proportional indivisible vectors in Λ. The map $O\to\overline{O}$ may patch some colors together, namely $D_\alpha^\pm\subset O$ map onto $D_\alpha\subset\overline{O}$ whenever $\alpha\in\Pi\cap\Pi_O^{\min}$, $2\alpha\in\Pi_{\overline{O}}^{\min}$. So $\overline{H}$ is the very sober hull of H if and only if $\alpha\notin\Lambda$ whenever $2\alpha\in\Pi_{\overline{O}}^{\min}$. We conclude that spherical subgroups with very sober hull $\overline{H}$ bijectively correspond to homogeneous spherical data such that Σ is proportional to $\Pi_{\overline{O}}^{\min}$ and $\alpha\notin\Sigma$ whenever $2\alpha\in 2\Pi\cap\Pi_{\overline{O}}^{\min}$, which is our claim.

If (30.3) holds for the adjoint group of G, then $\Pi_{\overline{O}}^{\min}$ coincides with the set $\overline{\Sigma}$ obtained from $\Sigma=\Pi_O^{\min}$ by the "maximal possible" dilation: every $\lambda\in\Sigma\setminus(\Pi\sqcup 2\Pi)$ such that $2\lambda\in\Sigma_G$ and $(2\lambda,\Pi^p)$ corresponds to a wonderful variety of rank 1 is replaced by 2λ. The spherical system $(\Pi^p,\overline{\Sigma},\mathscr{D}^a)$ is said to be the *spherical closure* of $(\Pi^p,\Sigma,\mathscr{D}^a)$. Indeed, by (30.3) this spherical closure corresponds to a certain very sober subgroup $\overline{\overline{H}}\subseteq G$. By the above, the spherical system of $\overline{O}$ corresponds to a certain subgroup with very sober hull $\overline{\overline{H}}$. Again by (30.3), this subgroup is conjugate to $\overline{H}$, and hence coincides with $\overline{\overline{H}}$. It follows that the spherical homogeneous datum of O determines the spherical system of $\overline{O}$ in a pure combinatorial way. Conversely, this spherical system together with $\Lambda,\varkappa$ determines $\overline{O}$ and O by the above, which proves (30.2).

Stage 2. The proof of (30.3) for adjoint G is much more difficult. Luna proposed the following strategy. The first stage is to prove that certain geometric operations on wonderful varieties (localization, parabolic induction, direct product, etc) are expressed in a pure combinatorial language of spherical systems. Every spherical system is obtained by these combinatorial operations from a list of *primitive* systems.

The next stage is to classify primitive spherical systems. And finally, for primitive systems, the existence and uniqueness of a geometric realization is proved case by case. This strategy was implemented by Luna in the case where all simple factors of G are of type **A** [Lu6]. Later on, this approach was extended by Bravi and Pezzini to the groups with the simple factors of types **A** and **D** [Bra1], [BraP1] or **A** and **C** (with some technical restrictions) [Pez]. In [Bra2] Bravi settled the case of arbitrary G with simply laced Dynkin diagram, and the case $\mathbf{F}_4$ was considered in [BraL]. Recently the reduction to primitive spherical systems was justified in [BraP2] for any G and the complete list of primitive spherical systems was given in [Bra3]. Using this list, the case of G with classical factors was settled in [BraP2].

On the other hand, the uniqueness of a geometric realization was proved by Losev by a general argument [Los2]. It remained to prove the existence. A new conceptual approach was suggested by Cupit-Foutou, who completed the proof of the theorem in [C-F2]. We give an outline of her proof.

Instead of assuming that G is adjoint, it is more convenient to suppose that G is semisimple simply connected. An idea of how to reconstruct a wonderful G-variety X from its spherical system is inspired by Brion's description of the Cox ring $\mathscr{R}(X)$, see 30.5. By Theorem 30.12(3), $\mathscr{X} = \operatorname{Spec}\mathscr{R}(X)$ is the total space of a flat family π_G of $\widehat{G}$-varieties with categorical quotient by U isomorphic to $\mathbb{A}^d$, $d = |\mathscr{D}^B|$, where $\widehat{T}$ acts by weights $-\widehat{\lambda}_D$, $D \in \mathscr{D}^B$. Hence $\pi_G : \mathscr{X} \to \mathbb{A}^r$ is the pullback of the universal family $\mathscr{X}^{\text{univ}} \to \operatorname{Hilb}^{\widehat{G}}_{\mathbb{A}^d}$ along a T-equivariant map $\mathbb{A}^r \to \operatorname{Hilb}^{\widehat{G}}_{\mathbb{A}^d}$ (see Appendix E.3). It turns out that $\operatorname{Hilb}^{\widehat{G}}_{\mathbb{A}^d} \simeq \mathbb{A}^r$ as T-varieties if the spherical system of X is spherically closed (i.e., coincides with its spherical closure), see below. Moreover, the T-orbit in $\operatorname{Hilb}^{\widehat{G}}_{\mathbb{A}^d}$ of a typical fiber $G/\!\!/H_0$ of π_G is open by Theorem E.14(3), since $-\Pi_X^{\min}$, $-\Pi_{G/H_0}^{\min}$, and the tails of $G/\!\!/H_0$ generate one and the same cone. Therefore $\mathbb{A}^r$ is mapped to $\operatorname{Hilb}^{\widehat{G}}_{\mathbb{A}^d}$ dominantly, whence isomorphically. Thus $\mathscr{X} \simeq \mathscr{X}^{\text{univ}}$ depends (as a spherical $(G \times T_X)$-variety) only on the spherical system of X (see below). Now $\mathring{\mathscr{X}}$ is obtained from $\mathscr{X}$ by removing all $(G \times T_X)$-orbits contained in colors. Indeed, removing these orbits yields the regularity locus of the rational map $\mathscr{X} \dashrightarrow X$. This open set $\mathscr{X}'$ cannot be larger than $\mathring{\mathscr{X}}$, because $\mathring{\mathscr{X}} \to X$ is an affine morphism, whence $\operatorname{codim}(\mathscr{X}' \setminus \mathring{\mathscr{X}}) = 1$, while $\operatorname{codim}(\mathscr{X} \setminus \mathring{\mathscr{X}}) > 1$. Finally, $X = \mathring{\mathscr{X}}/T_X$.

This argument also suggests a way to construct a wonderful variety from any given spherically closed spherical system $(\Pi^p, \Sigma, \mathscr{D}^a)$. Let $\mathscr{D} = \mathscr{D}^a \sqcup \mathscr{D}^{a'} \sqcup \mathscr{D}^b$ denote the set of colors of the spherical system obtained by adding to $\mathscr{D}^a$ the elements D_α, $\alpha \in \Pi^{a'} \sqcup \Pi^b$, with identifications as in 30.10. Consider a torus $T_X = (\Bbbk^\times)^{\mathscr{D}}$ and a subgroup $T_O \subseteq T_X$ defined by equations $\prod_{D \in \mathscr{D}} t_D^{\langle \varkappa(D), \lambda \rangle} = 1$, $\forall \lambda \in \Sigma$. (This notation is used for consistency, though there are no X and O at the moment.) Let $\varepsilon_D(t) = t_D$ denote the basic characters of T_X and $\chi_D = \varepsilon_D|_{T_O}$. Define the weights λ_D by the formulæ of Lemma 30.24. Note that $\sum \langle \varkappa(D), \lambda \rangle \lambda_D = \lambda$, $\forall \lambda \in \Sigma$, and the biweights $\widehat{\lambda}_D = (\lambda_D, \chi_D)$ are linearly independent. We shall freely use other notation from 30.5.

Consider the invariant Hilbert scheme $\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}}$ parameterizing affine $\widehat{G}$-varieties Z with $Z/\!/U \simeq \mathbb{A}^d$, $d = |\mathscr{D}|$, where $\widehat{T}$ acts by weights $-\widehat{\lambda}_D$, $D \in \mathscr{D}$. The isomorphism $Z/\!/U \xrightarrow{\sim} \mathbb{A}^d$ gives rise to a unique $\widehat{G}$-equivariant closed embedding $Z \hookrightarrow V = \bigoplus V(\widehat{\lambda}_D^*)$. It follows that $\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}}$ is an open subset of the invariant Hilbert scheme $\mathrm{Hilb}^{\widehat{G}}_m(V)$ parameterizing affine spherical $\widehat{G}$-subvarieties of V with rank semigroup $\widehat{\Lambda}_+ = \sum \mathbb{Z}_+ \widehat{\lambda}_D$ which are not contained in any proper $\widehat{G}$-submodule of V. (Here m is the indicator function of $\widehat{\Lambda}_+$.)

The $\widehat{T}$-action on $\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}}$, which is induced by $\widehat{T} : V$, where $\widehat{T}$ acts on $V(\widehat{\lambda}_D^*)$ by weight $-\widehat{\lambda}_D$, contracts $\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}}$ to the unique $\widehat{T}$-fixed point corresponding to an S-variety $Z_0 = \overline{\widehat{G}v}$, where $v = \sum v_{-\widehat{\lambda}_D}$ is the sum of lowest vectors, see Theorems E.14 and 28.3.

The tangent space and the obstruction space of $\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}}$ at $[Z_0]$ are given by Proposition E.11. It turns out that $T_{[Z_0]}\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}} \simeq \mathbb{A}^r$, $r = |\Sigma|$, where T acts with eigenweight set $-\Sigma$ and $T^2(Z_0)^{\widehat{G}} = 0$. The proof involves computation of cohomologies of non-reductive groups and Lie algebras; this is the most technical part of [C-F2]. It then follows that $\mathrm{Hilb}^{\widehat{G}}_{\widehat{\lambda}} \simeq \mathbb{A}^r$.

Let $\pi_G : \mathscr{X} \subseteq V \times \mathbb{A}^r \to \mathbb{A}^r$ be the universal family. The T_O-action on V extends to T_X by letting it act on each $V(\widehat{\lambda}_D^*)$ by weight $-\varepsilon_D$, and π_G is clearly T_X-equivariant.

Lemma 30.25. (1) *$\mathscr{X}$ is a factorial affine spherical $(G \times T_X)$-variety defined by a supported colored cone $(\mathscr{C}, \widehat{\mathscr{D}})$, where $\mathscr{C} = (\mathbb{Q}_+\Lambda_+(\mathscr{X}))^\vee$ and $\widehat{\mathscr{D}}$ is the set of colors of $\mathscr{X}$, which is identified with $\mathscr{D}$. The spherical roots and the types of simple roots and colors for $\mathscr{X}$ are the same as for the given spherical system.*
(2) *The algebra $\Bbbk[\mathscr{X}]^U$ (resp. the semigroup $\Lambda_+(\mathscr{X})$) is freely generated by the restrictions of linear $(B \times T_X)$-eigenfunctions on V (resp. by their biweights $(\lambda_D, \varepsilon_D)$) and by the coordinate functions on $\mathbb{A}^r$ (resp. by their biweights $(0, \sum\langle\varkappa(D), \lambda\rangle\varepsilon_D)$, $\lambda \in \Sigma$).*
(3) *These functions are the equations of the colors and of the $(G \times T_X)$-stable divisors on $\mathscr{X}$, respectively, or, equivalently, these divisors are mapped by $\varkappa$ to the basis dual to the basis of $\Lambda_+(X)$.*

Proof. $\mathscr{X}$ is spherical, because the fibers of π_G are spherical $\widehat{G}$-varieties and general fibers are transitively permuted by T_X. The description of the colored data of $\mathscr{X}$ stems from the results of 15.1 observing that $\mathscr{X}$ contains a fixed point 0, whose colored cone is solid by Proposition 15.14. The assertion (2) immediately follows from the definition of the invariant Hilbert scheme and the universal family.

By Theorem E.14(3), the tails of $\mathscr{X}$ or, equivalently, of a general fiber Z of π_G span a free semigroup with basis $-\Sigma$, whence, by (T_0), $\Pi^{\min}_{\mathscr{X}}$ is proportional to Σ. But it is easy to check that $\Lambda(\mathscr{X}) \cap \mathfrak{X}(T) = \mathbb{Z}\Sigma$, whence Σ consists of indivisible vectors in $\Lambda(\mathscr{X})$, i.e., $\Pi^{\min}_{\mathscr{X}} = \Sigma$.

It follows from (2) that the simple roots α of type p for $\mathscr{X}$ are those satisfying $\langle\alpha^\vee,\lambda_D\rangle = 0$, $\forall D\in\mathscr{D}$, i.e., $\alpha\in\Pi^p$. Hence the types of simple roots with respect to $\mathscr{X}$ are the same as for our spherical system. In particular, $\widehat{\mathscr{D}}^{a'}$ and $\widehat{\mathscr{D}}^b$ are in bijection with $\mathscr{D}^{a'}$ and $\mathscr{D}^b$, respectively. Moreover, it stems from the definition of λ_D that for any $D\in\mathscr{D}^{a'}\sqcup\mathscr{D}^b$ the respective color $\widehat{D}\in\widehat{\mathscr{D}}^{a'}\sqcup\widehat{\mathscr{D}}^b$ is mapped by $\varkappa$ to the vector of the dual basis corresponding to the vector $(\lambda_D,\varepsilon_D)$ in the basis of $\Lambda_+(\mathscr{X})$.

As for colors of type a, we use an observation of R. Camus: for any $\alpha\in\Pi^a$, at least one of the two P_α-unstable colors $\widehat{D}_\alpha^\pm\in\widehat{\mathscr{D}}^a$ is mapped to an edge of $\mathscr{C}$. (Otherwise $-\alpha$ is non-negative on $\mathscr{C}$, because it is non-negative on $\mathscr{V}(\mathscr{X})\cup\varkappa(\widehat{\mathscr{D}}\setminus\{\widehat{D}_\alpha^\pm\})$ by Lemma 30.20 and axiom (A1), whence $-\alpha\in\mathbb{Z}_+\Lambda_+(\mathscr{X})$ is dominant, a contradiction.) Since $\varkappa(\widehat{D}_\alpha^+)+\varkappa(\widehat{D}_\alpha^-)=\overline{\alpha^\vee}$ takes the value 1 on $(\lambda_{D_\alpha^\pm},\varepsilon_{D_\alpha^\pm})$ and vanishes on the other vectors in the basis of $\Lambda_+(X)$, we deduce that $\varkappa(\widehat{D}_\alpha^\pm)$ are the vectors of the dual basis corresponding to $(\lambda_{D_\alpha^\pm},\varepsilon_{D_\alpha^\pm})$.

Thus $\widehat{\mathscr{D}}$ is in bijection with $\mathscr{D}$ preserving types of colors, and the colors of $\mathscr{X}$ are represented by the vectors of the dual basis of $\Lambda(\mathscr{X})^*$ corresponding to the vectors $(\lambda_D,\varepsilon_D)$ $(D\in\mathscr{D})$ in the basis of $\Lambda_+(\mathscr{X})$. Clearly, the remaining vectors in the dual basis (which spans $\mathscr{C}$) correspond to the $(G\times T_X)$-stable divisors, whence $\mathscr{X}$ is factorial. This completes the proof of (1) and (3). □

Let $\mathring{\mathscr{X}}\subseteq\mathscr{X}$ be an open subset obtained by removing all $(G\times T_X)$-orbits contained in a color. Alternatively, $\mathring{\mathscr{X}}$ can be described as the set of points having non-zero projection to each $V(\widehat{\lambda}_D^*)$ or the union of open $\widehat{G}$-orbits in the fibers of π_G. It is a smooth simple toroidal $(G\times T_X)$-variety defined by the face $\mathscr{C}'\subseteq\mathscr{C}$ spanned by the edges orthogonal to $(\lambda_D,\varepsilon_D)$ $(D\in\mathscr{D})$. Now $\mathring{\mathscr{X}}\subseteq\prod(V(\widehat{\lambda}_D^*)\setminus\{0\})\times\mathbb{A}^r$ admits a geometric quotient $X=\mathring{\mathscr{X}}/T_X$ by a free T_X-action, which is a smooth simple toroidal G-variety. It is easy to see that X is complete (because $\mathscr{C}'$ maps onto $\mathscr{V}(X)$), whence wonderful, and the spherical system of X coincides with $(\Pi^p,\Sigma,\mathscr{D}^a)$.

If $(\Pi^p,\Sigma,\mathscr{D}^a)$ is not spherically closed, then one may consider its spherical closure $(\Pi^p,\overline{\Sigma},\mathscr{D}^a)$ and construct the respective wonderful embedding $\overline{X}\hookleftarrow\overline{O}=G/\overline{H}$. A subgroup $H\subset\overline{H}$ such that $O=G/H$ has a wonderful embedding X with spherical system $(\Pi^p,\Sigma,\mathscr{D}^a)$ was constructed at Stage 1.

In fact, it is proved in [C-F2] that $\mathrm{Hilb}_{\widehat{\lambda}}^{\widehat{G}}$ is an r-dimensional vector space on which T acts with eigenweight set $-\overline{\Sigma}$, for any spherical system $(\Pi^p,\Sigma,\mathscr{D}^a)$. Arguing as above, we see that the Cox family $\pi_G:\mathscr{X}=\operatorname{Spec}\mathscr{R}(X)\to\mathbb{A}^r$ is pulled back from $\mathscr{X}^{\mathrm{univ}}$ along a unique T-equivariant finite map $\mathbb{A}^r\to\mathrm{Hilb}_{\widehat{\lambda}}^{\widehat{G}}$, and therefore X is uniquely determined by $(\Pi^p,\Sigma,\mathscr{D}^a)$. This concludes the proof of Theorem 30.22.

31 Frobenius Splitting

Frobenius splitting is a powerful tool of modern algebraic geometry which allows various geometric and cohomological results to be proved by reduction to positive characteristic. This notion was introduced by Mehta and Ramanathan [MRa] in their study of Schubert varieties.

31.1 Basic Properties. Let X be an algebraic variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The Frobenius endomorphism $f \mapsto f^p$ of $\mathscr{O}_X$ gives rise to the Frobenius morphism $F : X^{1/p} \to X$, where $X^{1/p} = X$ as ringed spaces but the $\Bbbk$-algebra structure on $\mathscr{O}_{X^{1/p}}$ is defined as $c * f = c^p f$, $\forall c \in \Bbbk$. (We emphasize here that F acts identically on points, but non-trivially on functions.)

If X is a subvariety in $\mathbb{A}^n$ or $\mathbb{P}^n$, then $X^{1/p}$ is, too. The defining equations of $X^{1/p}$ are obtained from those of X by replacing all coefficients with their p-th roots. The Frobenius morphism F is given by raising all coordinates to the power p.

The Frobenius endomorphism may be regarded as an injection of $\mathscr{O}_X$-modules $\mathscr{O}_X \hookrightarrow F_*\mathscr{O}_{X^{1/p}}$, where $F_*\mathscr{O}_{X^{1/p}} = \mathscr{O}_X$ is endowed with another $\mathscr{O}_X$-module structure: $f * h = f^p h$ for any local sections f of $\mathscr{O}_X$ and h of $F_*\mathscr{O}_{X^{1/p}}$.

Definition 31.1. The variety X is said to be *Frobenius split* if the Frobenius homomorphism has an $\mathscr{O}_X$-linear left inverse $\sigma : F_*\mathscr{O}_{X^{1/p}} \to \mathscr{O}_X$, called a *Frobenius splitting*. In other words, σ is a $\mathbf{Z}_p$-linear endomorphism of $\mathscr{O}_X$ such that $\sigma(1) = 1$ and $\sigma(f^p h) = f\sigma(h)$.

For any closed subvariety $Y \subset X$ one has $\sigma(\mathscr{I}_Y) \supseteq \mathscr{I}_Y$, because $\mathscr{I}_Y \supseteq \mathscr{I}_Y^p$. The splitting σ is *compatible* with Y if $\sigma(\mathscr{I}_Y) = \mathscr{I}_Y$. Clearly, a compatible splitting induces a splitting of Y.

More generally, let δ be an effective Cartier divisor on X, with the canonical section $\eta_\delta \in \mathrm{H}^0(X, \mathscr{O}(\delta))$, $\operatorname{div} \eta_\delta = \delta$. We say that X is *Frobenius split relative to* δ if there exists an $\mathscr{O}_X$-module homomorphism, called a δ-*splitting*, $\sigma_\delta : F_*\mathscr{O}_{X^{1/p}}(\delta) \to \mathscr{O}_X$ such that $\sigma(h) = \sigma_\delta(h\eta_\delta)$ is a Frobenius splitting or, equivalently, $\sigma_\delta(\eta_\delta) = 1$ and $\sigma_\delta(f^p\eta) = f\sigma_\delta(\eta)$ for any local section η of $\mathscr{O}(\delta)$. The δ-splitting σ_δ is *compatible* with Y if the support of δ contains no component of Y (i.e., δ restricts to a divisor on Y) and σ is compatible with Y. Then σ_δ induces a $(\delta \cap Y)$-splitting of Y.

For a systematic treatment of Frobenius splitting and its applications, we refer to a monograph of Brion and Kumar [BKu]. Here we recall some of its most important properties.

Clearly, a Frobenius splitting of X (compatible with Y, relative to δ) restricts to a splitting of every open subvariety $U \subset X$ (compatible with $Y \cap U$, relative to $\delta \cap U$). Conversely, if X is normal and $\operatorname{codim}(X \setminus U) > 1$, then any splitting of U extends to X. In applications it is often helpful to consider $U = X^{\mathrm{reg}}$.

If $\varphi : X \to Z$ is a morphism such that $\varphi_*\mathscr{O}_X = \mathscr{O}_Z$, then a Frobenius splitting of X descends to a splitting of Z. If the splitting of X is compatible with $Y \subset X$, then the splitting of Z is compatible with $\overline{\varphi(Y)}$. For instance, one obtains a splitting of a normal variety X from that of a desingularization of X (if any exists).

It is not hard to prove that Frobenius split varieties are *weakly normal*, i.e., every bijective finite birational map onto a Frobenius split variety has to be an isomorphism [BKu, 1.2.5].

Proposition 31.2. (1) *Suppose that* X *is a Frobenius split projective variety; then* $\mathrm{H}^i(X,\mathscr{L})=0$ *for any ample line bundle* $\mathscr{L}$ *on* X *and all* $i>0$.
(2) *If* $Y\subset X$ *is a compatibly split subvariety, then the restriction map* $\mathrm{H}^0(X,\mathscr{L})\to\mathrm{H}^0(Y,\mathscr{L})$ *is surjective.*
(3) *If the splittings above are relative to an ample divisor* δ*, then the assertions of* (1)–(2) *hold for any numerically effective (e.g., globally generated) line bundle, i.e.,* $\mathscr{L}$ *such that* $\langle\mathscr{L},C\rangle\geq 0$ *for any closed curve* $C\subseteq X$.
(4) *There are relative versions of assertions* (1)–(3) *for a proper morphism* $\varphi: X\to Z$ *stating that* $\mathrm{R}^i\varphi_*\mathscr{L}=0$ *and* $\varphi_*\mathscr{L}\to\varphi_*(\iota_*\iota^*\mathscr{L})$ *is surjective under the same assumptions, with* $\iota: Y\hookrightarrow X$.

Proof. The idea is to embed the cohomology of $\mathscr{L}$ as a direct summand in the cohomology of a sufficiently big power of $\mathscr{L}$. Namely the canonical homomorphism $\mathscr{L}\to F_*F^*\mathscr{L}=\mathscr{L}\otimes_{\mathscr{O}_X}F_*\mathscr{O}_{X^{1/p}}$ has a left inverse $\mathbf{1}\otimes\sigma$, whence $\mathscr{L}$ is a direct summand in $F_*F^*\mathscr{L}$. Taking the cohomology yields a split injection

$$\mathrm{H}^i(X,\mathscr{L})\hookrightarrow\mathrm{H}^i(X,F_*F^*\mathscr{L})\simeq\mathrm{H}^i(X^{1/p},F^*\mathscr{L})\simeq\mathrm{H}^i(X,\mathscr{L}^{\otimes p}),\qquad\forall i\geq 0.$$

(The right isomorphism is only $\mathbf{Z}_p$-linear.) Iterating this procedure yields a split $\mathbf{Z}_p$-linear injection $\mathrm{H}^i(X,\mathscr{L})\hookrightarrow\mathrm{H}^i(X,\mathscr{L}^{\otimes p^k})$ compatible with the restriction to Y. Thus the assertions (1) and (2) are reduced to the case of the line bundle $\mathscr{L}^{\otimes p^k}$, $k\gg 0$, where the Serre theorem applies [Har2, III.5.3].

Similar reasoning applies to (3) making use of a split injection $\mathrm{H}^i(X,\mathscr{L})\hookrightarrow\mathrm{H}^i(X,\mathscr{L}^{\otimes p}\otimes\mathscr{O}(\delta))$ together with ampleness of $\mathscr{L}^{\otimes p}\otimes\mathscr{O}(\delta)$. The relative assertions are proved by the same arguments. □

Among other cohomology vanishing results for Frobenius split varieties we mention the extension of the Kodaira vanishing theorem [BKu, 1.2.10(i)]: if X is smooth projective and Frobenius split, then $\mathrm{H}^i(X,\mathscr{L}\otimes\omega_X)=0$ for ample $\mathscr{L}$ and $i>0$.

31.2 Splitting via Differential Forms. Now we reformulate the notion of Frobenius splitting for smooth varieties in terms of differential forms.

The de Rham derivation of $\Omega^\bullet_X$ may be considered as an $\mathscr{O}_X$-linear derivation of $F_*\Omega^\bullet_{X^{1/p}}$. Let $\mathscr{H}^k_X$ denote the respective cohomology sheaves. It is easy to check that $f\mapsto[f^{p-1}\mathrm{d}f]$ is a $\Bbbk$-derivation of $\mathscr{O}_X$ taking values in $\mathscr{H}^1_X$ (where $[\cdot]$ denotes the de Rham cohomology class). By the universal property of Kähler differentials, it induces a homomorphism of graded $\mathscr{O}_X$-algebras

$$c:\Omega^\bullet_X\to\mathscr{H}^\bullet_X,\qquad c(f_0\,\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k)=[f_0^p(f_1\cdots f_k)^{p-1}\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k],$$

called the *Cartier operator*. Cartier proved that c is an isomorphism for smooth X. (Using local coordinates, the proof is reduced to the case $X=\mathbb{A}^n$, where the verification is straightforward [BKu, 1.3.4].)

Now suppose that X is smooth. Then we have the *trace map*

$$\tau : F_*\omega_{X^{1/p}} \to \omega_X, \qquad \tau(\omega) = c^{-1}[\omega].$$

In local coordinates $x_1,\dots,x_n$, the trace map can be characterized as the unique $\mathscr{O}_X$-linear map taking $(x_1\cdots x_n)^{p-1}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n \mapsto \mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$ and $x_1^{k_1}\cdots x_n^{k_n}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n \mapsto 0$ unless $k_1\equiv\cdots\equiv k_n\equiv p-1 \pmod p$.

Using the trace map, it is easy to establish an isomorphism

$$\mathscr{H}om(F_*\mathscr{O}_{X^{1/p}},\mathscr{O}_X) \simeq F_*\omega_{X^{1/p}}^{1-p}, \qquad \sigma \leftrightarrow \widehat{\sigma},$$

such that $\sigma(h)\omega = \tau(h\omega^{\otimes p}\otimes\widehat{\sigma})$ for any local sections h of $F_*\mathscr{O}_{X^{1/p}}$ and ω of ω_X. Similarly, for any divisor δ on X we have

$$\mathscr{H}om(F_*\mathscr{O}_{X^{1/p}}(\delta),\mathscr{O}_X) \simeq F_*\omega_{X^{1/p}}^{1-p}(-\delta).$$

This leads to the following conclusion.

Proposition 31.3 ([BKu, 1.3.8, 1.4.10]). *Suppose that X is smooth and irreducible. Then $\sigma \in \mathrm{Hom}(F_*\mathscr{O}_{X^{1/p}},\mathscr{O}_X)$ is a splitting of X if and only if the Taylor expansion of $\widehat{\sigma}$ at some (hence any) $x\in X$ has the form*

$$\Big((x_1\cdots x_n)^{p-1} + \sum c_{k_1,\dots,k_n} x_1^{k_1}\cdots x_n^{k_n}\Big)(\partial_1\wedge\cdots\wedge\partial_n)^{\otimes(p-1)},$$

where the sum is taken over all multiindices $(k_1,\dots,k_n)$ such that $\exists k_i \not\equiv p-1 \pmod p$. (Here x_i denote local coordinates and ∂_i the vector fields dual to $\mathrm{d}x_i$.) If X is complete, then it suffices to have

$$\widehat{\sigma} = ((x_1\cdots x_n)^{p-1}+\cdots)(\partial_1\wedge\cdots\wedge\partial_n)^{\otimes(p-1)}.$$

The splitting σ is relative to any effective divisor $\delta \le \operatorname{div}\widehat{\sigma}$.

By abuse of language, we shall say that $\widehat{\sigma}$ splits X if σ does. Also, X is said to be *split by a $(p-1)$-th power* if $\alpha^{\otimes(p-1)}$ splits X for some $\alpha\in \mathrm{H}^0(X,\omega_X^{-1})$. This splitting is compatible with the zero set of α. For instance, a smooth complete variety X is split by the $(p-1)$-th power of α if the divisor of α in a neighborhood of some $x\in X$ is a union of $n=\dim X$ smooth prime divisors intersecting transversally at x.

Example 31.4. Every smooth toric variety X is Frobenius split by a $(p-1)$-th power compatibly with ∂X. For complete X, this stems from the structure of its canonical divisor, given by Proposition 30.8 (which extends to positive characteristic in the toric case). The general case follows by passing to a smooth toric completion. Now toric resolution of singularities readily implies that all normal toric varieties are Frobenius split compatibly with their invariant subvarieties.

Example 31.5 ([Ram], [BKu, Ch. 2–3]). Generalized flag varieties are Frobenius split by a $(p-1)$-th power. For $X=G/B$, $\omega_X^{-1}=\mathscr{L}(-2\rho)$ and the splitting is provided by $\alpha = f_\rho \cdot f_{-\rho} \in V^*(2\rho)$, where $f_{\pm\rho} \in V^*(\rho)$ are T-weight vectors of weights $\pm\rho$.

Moreover, this splitting is compatible with all Schubert subvarieties $S_w = \overline{B(wo)} \subset X$, $w \in W$. Using the weak normality of S_w and the Bott–Samelson resolution of singularities

$$\varphi : \check{S} = \check{S}_{\alpha_1,\dots,\alpha_l} := P_{\alpha_1} *_B \cdots *_B P_{\alpha_l}/B \to S_w,$$
$$w = r_{\alpha_1} \cdots r_{\alpha_l}, \quad \alpha_i \in \Pi, \quad l = \dim S_w,$$

with connected fibers and $\mathrm{R}^i\varphi_*\mathscr{O}_{\check{S}} = 0$, $\forall i > 0$, one deduces that S_w are normal (Demazure, Seshadri) and have rational resolution of singularities (Andersen, Ramanathan). These properties descend to Schubert subvarieties in G/P, $\forall P \supset B$.

Splitting by a $(p-1)$-th power has further important consequences. For instance, the Grauert–Riemenschneider theorem extends to this situation, due to Mehta–van der Kallen [MK]:

> If $\varphi : X \to Y$ is a proper birational morphism, X is smooth and split by $\alpha^{\otimes(p-1)}$ such that φ is an isomorphism on X_α, then $\mathrm{R}^i\varphi_*\omega_X = 0$, $\forall i > 0$.

31.3 Extension to Characteristic Zero. Although the concept of Frobenius splitting is defined in characteristic $p > 0$, it successfully applies to algebraic varieties in characteristic zero via reduction $\bmod p$.

Namely let X be an algebraic variety over an algebraically closed field $\Bbbk$ of characteristic 0. One can find a finitely generated subring $R \subset \Bbbk$ such that X is defined over R, i.e., is obtained from an R-scheme $\mathscr{X}$ by extension of scalars. One may assume that $\mathscr{X}$ is flat over R, after replacing R by a localization. For any maximal ideal $\mathfrak{p} \lhd R$ we have $R/\mathfrak{p} \simeq \mathbb{F}_{p^k}$. The variety $X_\mathfrak{p}$ obtained from the fiber $\mathscr{X}_\mathfrak{p}$ of $\mathscr{X} \to \operatorname{Spec} R$ over $\mathfrak{p}$ by an extension of scalars $\mathbb{F}_{p^k} \to \mathbb{F}_{p^\infty}$ is called a *reduction* $\bmod p$ of X and sometimes denoted simply by X_p (by abuse of notation).

Reductions $\bmod p$ exist and share geometric properties of X (affinity, projectivity, completeness, smoothness, normality, rational resolution of singularities, etc) for all sufficiently large p. Conversely, a local geometric property of open type (e.g., smoothness, normality, rational resolution of singularities) holds for X if it holds for X_p whenever $p \gg 0$. Replacing R by an appropriate localization, one may always assume that a given finite collection of algebraic and geometric objects on X (subvarieties, line bundles, coherent sheaves, morphisms, etc) is defined over R, and hence specializes to X_p for $p \gg 0$; coherent sheaves may be supposed to be flat over R.

Cohomological applications of reduction $\bmod p$ are based on the semicontinuity theorem [Har2, III.12.8], which may be reformulated in our setup as follows:

> If X is complete and $\mathscr{F}$ is a coherent sheaf on X, then $\dim \mathrm{H}^i(X,\mathscr{F}) = \dim \mathrm{H}^i(X_p,\mathscr{F}_p)$ for all $p \gg 0$.

This implies, for instance, that the assertions of Proposition 31.2 hold in characteristic zero provided that X_p are Frobenius split for $p \gg 0$. This is the case, e.g., for Fano varieties. Another case, which is important in the scope of this chapter, are spherical varieties.

31.4 Spherical Case.

Theorem 31.6 ([BI]). *If X is a spherical G-variety in characteristic 0, then X_p is Frobenius split by a $(p-1)$-th power compatibly with all G-subvarieties and relative to any given B-stable effective divisor, for $p \gg 0$.*

Proof. Using an equivariant completion of X and a toroidal desingularization of this completion, we may assume that X is smooth, complete, and toroidal. Consider the natural morphism $\varphi : X \to X(\mathfrak{h})$, where $\mathfrak{h}$ is a generic isotropy subalgebra for $G : X$. By Proposition 30.8, $\omega_X^{-1} = \mathscr{O}(\partial X + \varphi^*\mathscr{H})$, where $\mathscr{H}$ is a hyperplane section of $X(\mathfrak{h})$.

The restriction of $\mathscr{O}(\partial X)$ to a closed G-orbit $Y \subset X$ is the top exterior power of the normal bundle to Y, whence $\omega_Y^{-1} = \omega_X^{-1}|_Y \otimes \mathscr{O}(-\partial X)|_Y = \mathscr{O}(\varphi^*\mathscr{H})|_Y$. Since Y is a generalized flag variety, Y_p is split by the $(p-1)$-th power of (the reduction mod p of) some $\alpha_Y \in \mathrm{H}^0(Y, \omega_Y^{-1})$. The G-module $\mathrm{H}^0(Y, \omega_Y^{-1})$ being irreducible and $\mathscr{O}(\varphi^*\mathscr{H})$ globally generated, the restriction map $\mathrm{H}^0(X, \mathscr{O}(\varphi^*\mathscr{H})) \to \mathrm{H}^0(Y, \omega_Y^{-1})$ is surjective and α_Y extends to $\alpha_0 \in \mathrm{H}^0(X, \mathscr{O}(\varphi^*\mathscr{H}))$.

We have $\partial X = D_1 \cup \dots \cup D_k$, where D_i runs over all G-stable prime divisors of Y. It is easy to see from Proposition 31.3 that $\alpha = \alpha_0 \otimes \alpha_1 \otimes \dots \otimes \alpha_k$ provides a splitting for X_p, where $\alpha_i \in \mathrm{H}^0(X, \mathscr{O}(D_i))$, $\operatorname{div} \alpha_i = D_i$. Moreover, this splitting is compatible with all $(D_i)_p$ and therefore with all G-subvarieties in X_p, because the latter are unions of transversal intersections of some $(D_i)_p$.

Finally, for any B-stable effective divisor δ we have $\delta \le (1-p)K_X$ for $p \gg 0$, by Proposition 30.8. Hence the splitting is relative to δ_p by Proposition 31.3. □

It is worth noting that not all spherical varieties in positive characteristic are Frobenius split. Counterexamples are provided by some complete homogeneous spaces with non-reduced isotropy group subschemes [La].

Frobenius splitting of spherical varieties provides short and conceptual proofs for a number of important geometric and cohomological properties. In particular, Theorem 15.20 can be deduced in the following way.

Consider a resolution of singularities $\psi : X' \to X$, where X' is toroidal and quasiprojective. Choose an ample B-stable effective divisor δ on X'; then X'_p is split relative to δ_p for $p \gg 0$. By semicontinuity and Proposition 31.2(4) applied to the trivial line bundle over X'_p, $\mathrm{R}^i\psi_*\mathscr{O}_{X'} = 0$ for $i > 0$, whence X has rational singularities. By the same reason, $\mathscr{O}_X = \psi_*\mathscr{O}_{X'}$ surjects onto $\psi_*\mathscr{O}_{Y'}$ for any irreducible closed G-subvariety $Y' \subset X'$, whence $\psi_*\mathscr{O}_{Y'} = \mathscr{O}_Y$ for $Y = \psi(Y')$. Since Y' is smooth, Y is normal and has rational singularities by the above.

For any line bundle $\mathscr{L}$ on X denote $\mathscr{L}' = \psi^*\mathscr{L}$. The Leray spectral sequence

$$\mathrm{H}^{i+j}(X', \mathscr{L}') \Longleftarrow \mathrm{H}^i(X, \mathrm{R}^j\psi_*\mathscr{L}') = \mathrm{H}^i(X, \mathscr{L} \otimes \mathrm{R}^j\psi_*\mathscr{O}_{X'})$$

degenerates to $\mathrm{H}^i(X', \mathscr{L}') = \mathrm{H}^i(X, \mathscr{L})$, $\forall i \ge 0$. The same holds for direct images instead of cohomology. Together with Proposition 31.2, applied to X'_p and $\mathscr{L}'_p$, this proves the following

Corollary 31.7. *Suppose that* $\operatorname{char}\Bbbk = 0$*. If X is a complete spherical G-variety, $Y \subset X$ a G-subvariety, and $\mathscr{L}$ a numerically effective line bundle on X, then* $\mathrm{H}^i(X,\mathscr{L}) = 0$*, $\forall i > 0$, and the restriction map* $\mathrm{H}^0(X,\mathscr{L}) \to \mathrm{H}^0(Y,\mathscr{L})$ *is surjective. More generally, if X is spherical and $\varphi : X \to Z$ is a proper morphism, then* $\mathrm{R}^i\varphi_*\mathscr{L} = 0$*, $\forall i > 0$, and $\varphi_*\mathscr{L} \to \varphi_*(\iota_*\iota^*\mathscr{L})$ is surjective, where $\iota : Y \hookrightarrow X$ is a closed G-embedding.*

See [Bri7], [Bri12] for other proofs.

More precise results on Frobenius splitting of spherical varieties and their subvarieties (usually G- or B-orbit closures) are obtained in special cases.

As noted above, generalized flag varieties are Frobenius split compatibly with their Schubert subvarieties, and the latter have rational resolution of singularities in positive, hence any (by semicontinuity), characteristic.

Equivariant normal embeddings of G (see §27) are Frobenius split compatibly with their $(G \times G)$-subvarieties, in all positive characteristics. For wonderful completions of adjoint semisimple groups, this was established by Strickland [Str1]. The general case is due to Rittatore [Rit2], see also [BKu, Ch. 6]. This implies that normal reductive group embeddings have rational resolution of singularities (in particular, they are Cohen–Macaulay) and that the coordinate algebras of normal reductive monoids have "good" filtration [Rit2, §4], [BKu, 6.2.13].

Brion and Polo proved that the closures of the Bruhat double cosets in wonderful completions of adjoint semisimple groups (called *large Schubert varieties*) are compatibly split and deduced that they are normal and Cohen–Macaulay [BPo].

De Concini and Springer proved that wonderful embeddings of symmetric spaces for adjoint G are Frobenius split compatibly with their G-subvarieties in odd characteristics [CS, 5.9]. However this splitting is not always compatible with B-orbit closures; in fact, the latter may be neither normal nor Cohen–Macaulay [Bri16].

See [Bri16] for a detailed study of B-orbits in spherical varieties and their closures. This is an area of active current research, with many open questions.