Introduction

Groups entered mathematics as transformation groups. From the works of Cayley and Klein it became clear that any geometric theory studies the properties of geometric objects that are invariant under the respective transformation group. This viewpoint culminated in the celebrated Erlangen program [Kl]. An important feature of each one of the classical geometries—affine, projective, Euclidean, spherical, and hyperbolic—is that the respective transformation group is transitive on the underlying space. Another feature of these examples is that the transformation groups are linear algebraic and their action is regular. In this way algebraic homogeneous spaces arise in geometry.

Another source for algebraic homogeneous spaces are varieties of geometric figures or tensors of certain type. Examples are provided by Grassmannians, flag varieties, varieties of conics, of triangles, of matrices with fixed rank, etc. These homogeneous spaces are of great importance in algebraic geometry. They were explored intensively, starting with the works of Chasles, Schubert, Zeuthen et al, which gave rise to the enumerative geometry and intersection theory.

Homogeneous spaces play an important rôle in representation theory, since representations of linear algebraic groups are often realized in spaces of sections or cohomologies of line (or vector) bundles over homogeneous spaces. The geometry of a homogeneous space can be used to study representations of the respective group, and conversely. Shining examples are the Borel–Weil–Bott theorem [Dem3] and Demazure's proof of the Weyl character formula [Dem1].

In the study of an algebraic homogeneous space G/H, it is often useful by standard reasons of algebraic geometry to pass to a G-equivariant completion or, more generally, to an *embedding*, i.e., a G-variety X containing a dense open orbit isomorphic to G/H.

An example is provided by the following classical problem of enumerative algebraic geometry: compute the number of plane conics tangent to 5 given ones. Equivalently, one has to compute the intersection number of certain 5 divisors on the space of conics $\mathrm{PSL}_3/\mathrm{PSO}_3$, which is an open orbit in $\mathbb{P}^5 = \mathbb{P}(\mathrm{S}^2\mathbb{C}^3)$. To solve our enumerative problem, we pass to a good compactification of $\mathrm{PSL}_3/\mathrm{PSO}_3$. Namely, consider the closure X in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$ of the graph of a rational map sending a conic to the dual one. Points of X are called *complete conics*. It happens that our 5 divisors intersect the complement of the open orbit in X properly. Hence the sought number is just the intersection number of the 5 divisors in X, which is easier to compute, because X is compact.

Embeddings of homogeneous spaces arise naturally as orbit closures, when one studies arbitrary actions of algebraic groups. Such questions as normality of the orbit closure, the nature of singularities, adherence of orbits, the description of orbits in the closure of a given orbit, etc, are of importance.

Embeddings of homogeneous spaces of reductive algebraic groups are the subject of this survey. The reductivity assumption is natural for two reasons. First, reductive groups have a good structure and representation theory, and a deep theory of embeddings can be developed under this restriction. Secondly, most applications to algebraic geometry and representation theory deal with homogeneous spaces of reductive groups. However, homogeneous spaces of non-reductive groups and their embeddings are also considered. They arise naturally even in the study of reductive group actions as orbits of Borel and maximal unipotent subgroups and their closures. (An example: Schubert varieties.)

The main topics of our survey are:

- The description of all embeddings of a given homogeneous space.
- The study of geometric properties of embeddings: affinity, (quasi)projectivity, divisors and line bundles, intersection theory, singularities, etc.
- Application of homogeneous spaces and their embeddings to algebraic geometry, invariant theory, and representation theory.
- Determination of a "good" class of homogeneous spaces, for which the above problems have a good solution. Finding and studying natural invariants that distinguish this class.

Now we describe briefly the content of the survey.

In Chap. 1 we recall basic facts on algebraic homogeneous spaces and consider basic classes of homogeneous spaces: affine, quasiaffine, projective. We give group-theoretical conditions that distinguish these classes. Also bundles and fibrations over a homogeneous space G/H are considered. In particular, we compute $\mathrm{Pic}(G/H)$.

In Chap. 2 we introduce and explore two important numerical invariants of G/H—the complexity and the rank. The *complexity* of G/H is the codimension of a general B-orbit in G/H, where $B \subseteq G$ is a Borel subgroup. The *rank* of G/H is the rank of the lattice $\Lambda(G/H)$ of weights of rational B-eigenfunctions on G/H. These invariants are of great importance in the theory of embeddings. Homogeneous spaces of complexity ≤ 1 form a "good" class. It was noted by Howe [Ho] and Panyushev [Pan7] that a number of invariant-theoretic problems admitting a nice solution have a certain homogeneous space of complexity ≤ 1 in the background.

Complexity and rank may be defined for any action $G : X$. We prove some semicontinuity results for complexity and rank of G-subvarieties in X. General methods for computing complexity and rank of X were developed by Knop and Panyushev, see [Kn1] and [Pan7, §§1–2]. We describe them in this chapter, paying special attention to the case $X = G/H$. The formulæ for complexity and rank are given in terms of the geometry of the cotangent bundle T^*X and of the doubled action $G : X \times X^*$.

The general theory of embeddings developed by Luna and Vust [LV] is the subject of Chap. 3. The basic idea of Luna and Vust is to patch all embeddings $X \hookleftarrow G/H$ together in a huge prevariety and consider particular embeddings as Noetherian separated open subsets determined by certain conditions. It appears, at least for normal embeddings, that X is determined by the collection of closed G-subvarieties $Y \subseteq X$, and each Y is determined by the collection of B-stable prime divisors containing Y. This leads to a "combinatorial" description of embeddings, which can be made really combinatorial in the case of complexity ≤ 1. In this case, embeddings are classified by certain collections of convex polyhedral cones, as in the theory of toric varieties [Ful2] (which is in fact a particular case of the Luna–Vust theory). The geometry of embeddings is also reflected in these combinatorial data, as in the toric case. In fact the Luna–Vust theory is developed here in more generality as a theory of G-varieties in a given birational class (not necessarily containing an open orbit).

G-invariant valuations of the function field of G/H correspond to G-stable divisors on embeddings of G/H. They play a fundamental rôle in the Luna–Vust theory as a key ingredient of the combinatorial data used in the classification of embeddings. In Chap. 4 we explore the structure of the set of invariant valuations, following Knop [Kn3], [Kn5]. This set can be identified with a certain collection of convex polyhedral cones patched together along their common face. This face consists of *central* valuations—those that are zero on B-invariant functions. It is a solid rational polyhedral cone in $\Lambda(G/H) \otimes \mathbb{Q}$ and a fundamental domain of a crystallographic reflection group $W_{G/H}$, which is called the *little Weyl group of* G/H. The cone of central valuations and the little Weyl group are linked with the geometry of the cotangent bundle.

Spaces of complexity 0 form the most remarkable subclass of homogeneous spaces. Their embeddings are called *spherical varieties*. They are studied in Chap. 5. Grassmannians, flag varieties, determinantal varieties, varieties of conics, of complexes, and algebraic symmetric spaces are examples of spherical varieties. We give several characterizations of spherical varieties from the viewpoint of algebraic transformation groups, representation theory, and symplectic geometry. We consider important classes of spherical varieties: symmetric spaces, reductive algebraic monoids, horospherical varieties, toroidal and wonderful varieties. The Luna–Vust theory is much more developed in the spherical case by Luna, Brion, Knop, et al. We consider the structure of the Picard group of a spherical variety, the intersection theory with applications to enumerative geometry, the cohomology of coherent sheaves, and a powerful technique of Frobenius splitting, which leads to deep conclusions on the geometry and cohomology of spherical varieties by reduction to positive characteristic. A classification of spherical homogeneous spaces and their embeddings, started by Krämer and Luna–Vust, respectively, was recently completed in pure combinatorial terms (like the classification of semisimple Lie algebras).

The theory of embeddings of homogeneous spaces is relatively new and far from being complete. This survey does not cover all developments and deeper interactions with other areas. Links for further reading may be found in the bibliography. We also recommend the surveys [Kn2], [Bri6], [Bri13] on spherical varieties and [Pan7] on complexity and rank in invariant theory. A short survey paper [Tim6] covers some of the topics of this survey in a concise manner.

The reader is supposed to be familiar with basic concepts of commutative algebra, algebraic geometry, algebraic groups, and invariant theory. Our basic sources in these areas are [Ma], [Sha] and [Har2], [Hum] and [Sp3], [PV] and [MFK], respectively. More special topics are covered by Appendices.

Structure of the Survey. The text is divided into chapters, chapters are subdivided into sections, and sections are subdivided into subsections. A link 1.2 refers to Subsection (or Theorem, Lemma, Definition, etc) 2 of Section 1. We try to give proofs, unless they are too long or technical.

Notation and Conventions. We work over an algebraically closed base field $\Bbbk$. A part of our results are valid over an arbitrary characteristic, but we impose the assumption $\operatorname{char}\Bbbk = 0$ whenever it simplifies formulations and proofs. More precisely, the characteristic is assumed to be zero in §§7 (most part of), 8–11, 22–23, 29, 30 (most part of), and subsections 4.3, 16.6, 17.2–17.5, 18.2–18.6, 25.3–25.5, 26.9–26.10, 27.4–27.6, 28.3, A.2, E.3. On the other hand, we assume $\operatorname{char}\Bbbk > 0$ in §31, unless otherwise specified. Let p denote the characteristic exponent of $\Bbbk$ ($= \operatorname{char}\Bbbk$, or 1 if $\operatorname{char}\Bbbk = 0$).

Algebraic varieties are assumed to be Noetherian and separated (not necessarily irreducible).

Algebraic groups are denoted by capital Latin letters, and their tangent Lie algebras by the respective lowercase Gothic letters. We consider only *linear* algebraic groups (although more general algebraic groups, e.g., Abelian varieties, are also of great importance in algebraic geometry).

Topological terms refer to the Zariski topology, unless otherwise specified.

By a *general point* of an algebraic variety we mean a point in a certain dense open subset (depending on the considered situation), in contrast with the *generic point*, which is the dense schematic point of an irreducible algebraic variety.

Throughout the paper, G denotes a reductive connected linear algebraic group, unless otherwise specified. We may always assume that G is of simply connected type, i.e., a direct product of a torus and a simply connected semisimple group. When we study the geometry of a given homogeneous space O and embeddings of O, we often fix a base point $o \in O$ and denote by $H = G_o$ its isotropy group, thus identifying O with G/H (at least set-theoretically).

We use the following general notation.

General:

$\sqcup$ denotes a union of pairwise disjoint sets.

$\subseteq$ denotes inclusion of sets, while $\subset$ stands for strict inclusion (excluding equality).

$\lhd$ denotes inclusion of normal subgroups or ideals.

$\mathbf{1}$ is the identity map of a set under consideration.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the sets of natural, integer, rational, real, and complex numbers, respectively. The sub(super)script "+" or "−" distinguishes the respective subset of non-negative (positive) or non-positive (negative) numbers, e.g., $\mathbb{Z}^+ = \mathbb{N}$, $\mathbb{Z}_+ = \mathbb{N} \sqcup \{0\}$.

$\mathbf{Z}_m = \mathbb{Z}/m\mathbb{Z}$ is the group (or ring) of residues $\mod m$.

$\mathbb{F}_{p^k}$ is the Galois field of cardinality p^k; $\mathbb{F}_{p^\infty}$ denotes its algebraic closure.

Linear algebra:

$\mathbb{A}^n$ denotes the n-dimensional coordinate affine space.

$e_1, \dots, e_n$ denote the standard basis of $\mathbb{k}^n$.

$x_1, \dots, x_n$ are the standard coordinates on $\mathbb{A}^n$ or $\mathbb{k}^n$.

$A^\top$ is the transpose of a matrix A.

$\mathrm{L}_n(\mathbb{k})$ is the algebra of $n \times n$ matrices over $\mathbb{k}$.

$\mathrm{GL}_n(\mathbb{k}), \mathrm{SL}_n(\mathbb{k}), \mathrm{O}_n(\mathbb{k}), \mathrm{SO}_n(\mathbb{k}), \mathrm{Sp}_n(\mathbb{k})$ are the classical matrix groups: of non-degenerate, unimodular, orthogonal, unimodular orthogonal, and symplectic $n \times n$ matrices over $\mathbb{k}$, respectively.

$\mathrm{L}(V)$ is the algebra of linear operators on a vector space V.

$\mathrm{GL}(V), \mathrm{SL}(V), \mathrm{O}(V), \mathrm{SO}(V), \mathrm{Sp}(V)$ are the classical linear groups acting on V: general linear, special linear, orthogonal, special orthogonal, and symplectic group, respectively.

$\langle S \rangle$ is the linear span of a subset $S \subseteq V$.

V^* is the vector space dual to V.

$\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the lattice dual to a lattice Λ.

$\langle \cdot, \cdot \rangle$ denotes the pairing between V and V^* or Λ and Λ^*.

$S^\perp \subseteq V^*$ (or Λ^*) is the annihilator of a subset $S \subseteq V$ (or Λ) with respect to this pairing.

$\mathbb{P}(V)$ denotes the projective space of all 1-subspaces in V.

$\mathbb{P}(X) \subseteq \mathbb{P}(V)$ is the projectivization of a subset $X \subseteq V$ stable under homotheties.

$[v] \in \mathbb{P}(V)$ is the point corresponding to a nonzero vector $v \in V$.

$\mathrm{Gr}_k(V)$ denotes the Grassmannian of k-dimensional subspaces in V.

$\mathrm{Fl}_{k_1,\dots,k_s}(V)$ is the variety of partial flags in V with subspace dimensions $k_1, \dots, k_s$.

Algebras and modules:

$A^\times$ is the unit group of an algebra A.

$\mathrm{Quot}\, A$ is the field of quotients of A.

$\mathbb{k}[S] \subseteq A$ is the $\mathbb{k}$-subalgebra generated by a subset $S \subseteq A$. In particular, the notation $\mathbb{k}[x_1, \dots, x_n]$ is used for the algebra of polynomials in the indeterminates $x_1, \dots, x_n$.

$\mathbb{k}[[t]]$ is the $\mathbb{k}$-algebra of formal power series in the indeterminate t.

$\mathbb{k}((t)) = \mathrm{Quot}\, \mathbb{k}[[t]]$ is the field of formal Laurent series in t.

$(S) \lhd A$ is the ideal generated by S.

$AS \subseteq M$ is the submodule of an A-module M generated by a subset $S \subseteq M$. This notation is also used for additive subsemigroups ($A = \mathbb{Z}_+$) and convex cones in a vector space over $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$ ($A = \mathbb{K}_+$).

$\operatorname{Ann} S \subseteq A$ is the annihilator of $S \subseteq M$ in A.

$\mathrm{S}^\bullet M = \bigoplus_{n=0}^\infty \mathrm{S}^n M$ is the symmetric algebra of a module (sheaf) M.

$\bigwedge^\bullet M = \bigoplus_{n=0}^\infty \bigwedge^n M$ is the exterior algebra of M.

$M^{(n)}$ denotes the n-th member of the filtration of a filtered object (algebra, module, sheaf) M.

$\operatorname{gr} M$ denotes the graded object associated with a filtered object M.

M_n is the n-th homogeneous part of a graded object M.

Algebraic geometry:

$\overline{Y}$ denotes the closure of a subset Y in an algebraic variety X, unless otherwise specified.

$\mathscr{O}_X$ is the structure sheaf of X.

$\mathscr{I}_Y \lhd \mathscr{O}_X$ is the ideal sheaf of a closed subvariety (subscheme) $Y \subset X$.

$\mathscr{O}_{X,Y}$ is the local ring of an irreducible subvariety (or a schematic point) $Y \subseteq X$.

$\mathfrak{m}_{X,Y} \lhd \mathscr{O}_{X,Y}$ is the maximal ideal.

$\Bbbk[X]$ is the algebra of regular functions on X.

$\mathscr{I}(Y) \lhd \Bbbk[X]$ is the ideal of functions on X vanishing on a closed subvariety (subscheme) $Y \subseteq X$.

$\Bbbk(X)$ is the field of rational functions on an irreducible variety X.

$\operatorname{Cl} X$ is the divisor class group of X.

$\operatorname{Pic} X$ is the Picard group of X.

$\mathscr{O}(\delta) = \mathscr{O}_X(\delta)$ is the line bundle corresponding to a Cartier divisor δ on X or, more generally, the reflexive sheaf corresponding to a Weil divisor δ.

$\operatorname{div}_0 \sigma$, $\operatorname{div}_\infty \sigma$, $\operatorname{div} \sigma = \operatorname{div}_0 \sigma - \operatorname{div}_\infty \sigma$ denote the divisor of zeroes, of poles, and the full divisor of a rational section σ of a line bundle on X (e.g., $\sigma \in \Bbbk(X)$), respectively.

$X_\sigma = \{x \in X \mid \sigma(x) \neq 0\}$ is the non-vanishing locus of a section σ of a line bundle on X (including the case $\sigma \in \Bbbk[X]$).

φ^* denotes the pullback of functions, divisors, sheaves, etc, along a morphism $\varphi : X \to Y$.

φ_* is the pushforward along φ (whenever it exists).

$\mathrm{R}^i \varphi_* \mathscr{F}$ is the i-th higher direct image of a sheaf $\mathscr{F}$ on X.

$\mathrm{H}^i(X, \mathscr{F})$ denotes the i-th cohomology space of $\mathscr{F}$. In particular, $\mathrm{H}^0(X, \mathscr{F})$ is the space of global sections of $\mathscr{F}$.

X^{reg} is the regular (smooth) locus of X.

$T_x X$, $T_x^* X$ are the tangent, resp. cotangent, space to X at $x \in X$.

$\mathrm{d}_x\varphi : T_xX \to T_{\varphi(x)}Y$ is the differential of $\varphi : X \to Y$ at $x \in X$.

$\Omega_X^\bullet = \bigwedge^\bullet \Omega_X^1$ is the sheaf of differential forms on X.

$\omega_X = \Omega_X^{\dim X}$ is the canonical sheaf on X.

Groups, Lie algebras, and actions:

e is the unity element of a group G.

$Z(G)$ denotes the center of G.

$G' = [G,G]$ is the commutator subgroup of G.

G^0 is the unity component of an algebraic group G.

$\mathrm{R}(G)$ denotes the radical of G.

$\mathrm{R_u}(G)$ is the unipotent radical of G.

$\mathfrak{X}(G)$ is the character group of G, i.e., the group of homomorphisms $G \to \Bbbk^\times$ written additively.

$\mathfrak{X}^*(G)$ is the set of (multiplicative) one-parameter subgroups of G, i.e., homomorphisms $\Bbbk^\times \to G$.

$\mathrm{Ad} = \mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G.

$G : M$ denotes an action of a group G on a set M. As a rule, it is a regular action of an algebraic group on an algebraic variety.

G_x is the isotropy group (= stabilizer) in G of a point $x \in M$.

Gx is the G-orbit of x.

$N_G(S) = \{g \in G \mid gS = S\}$ is the normalizer of a subset $S \subseteq M$. In particular, this notation is used for the normalizer of a subgroup.

$Z_G(S) = \{g \in G \mid gx = x,\ \forall x \in S\}$ is the centralizer of S (e.g., of a subgroup).

M^G is the set of fixed elements under an action $G : M$.

$M^{(G)}$ is the set of all (nonzero) G-eigenvectors in a linear representation $G : M$.

$M_\chi = M_\chi^{(G)} \subseteq M$ is the subspace of G-eigenvectors of the weight $\chi \in \mathfrak{X}(G)$.

$\mathfrak{z}(\mathfrak{g})$ denotes the center of a Lie algebra $\mathfrak{g}$.

$\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is the commutator subalgebra.

$\mathfrak{n}_\mathfrak{g}(S) = \{\xi \in \mathfrak{g} \mid \xi S \subseteq S\}$ is the Lie algebra normalizer of a subspace S in a $\mathfrak{g}$-module M. In particular, this notation is used for the normalizer of a Lie subalgebra.

$\mathfrak{z}_\mathfrak{g}(S) = \{\xi \in \mathfrak{g} \mid \xi x = 0,\ \forall x \in S\}$ is the Lie algebra centralizer of a subset $S \subseteq M$ (e.g., of a Lie subalgebra).

ξx is the velocity vector of $\xi \in \mathfrak{g}$ at $x \in M$, i.e., the image of ξ under the differential of the orbit map $G \to Gx$, $g \mapsto gx$, where G is an algebraic group and M is an algebraic G-variety. In characteristic zero, $\mathfrak{g}x := \{\xi x \mid \xi \in \mathfrak{g}\} = T_xGx$.

$\mathrm{Aut}_G M$ denotes the group of G–equivariant automorphisms of a G-set (variety, module, algebra, ...) M.

$\mathrm{Hom}_G(M,N)$ is the space of G–equivariant homomorphisms of G-modules (or sheaves) $M \to N$.

Other notation is gradually introduced in the text, see the Notation Index.