Preface

In this book we consider (connected) smooth real hypersurfaces in the complex vector space $\mathbb{C}^{n+1}$ with $n \geq 1$. Specifically, we are interested in *tube hypersurfaces*, i.e. real hypersurfaces of the form

$$\Gamma + iV,$$

where Γ is a hypersurface in a totally real $(n+1)$-dimensional linear subspace $V \subset \mathbb{C}^{n+1}$. From now on we fix the subspace V and choose coordinates $z_0, \dots, z_n$ in $\mathbb{C}^{n+1}$ such that $V = \{\operatorname{Im} z_j = 0,\ j = 0, \dots, n\}$. Everywhere below V is identified with $\mathbb{R}^{n+1}$ by means of the coordinates $x_j := \operatorname{Re} z_j$, $j = 0, \dots, n$.

Tube hypersurfaces arise, for instance, as the boundaries of *tube domains*, that is, domains of the form

$$D + i\mathbb{R}^{n+1},$$

where D is a domain in $\mathbb{R}^{n+1}$. We refer to the hypersurface Γ and domain D as the *bases* of the above tubes. The study of tube domains is a classical subject in several complex variables and complex geometry, which goes back to the beginning of the 20th century. Indeed, already Siegel found it convenient to realize certain symmetric domains as tubes. For example (see Section 5.3 for details), the familiar unit ball in $\mathbb{C}^{n+1}$ is biholomorphically equivalent to the tube domain with the base given by the inequality

$$x_0 > \sum_{\alpha=1}^{n} x_\alpha^2. \tag{0.1}$$

Note that the boundary of the tube domain with base (0.1) is the tube hypersurface whose base is defined by the equation

$$x_0 = \sum_{\alpha=1}^{n} x_\alpha^2. \tag{0.2}$$

This tube hypersurface is equivalent to the $(2n+1)$-dimensional sphere in $\mathbb{C}^{n+1}$ with one point removed.

Although the definition of tube depends on the choice of the totally real subspace V, the structure of the direct product of a portion of V with all of iV is extremely useful. Indeed, the property that makes tube domains and hypersurfaces interesting from the complex-geometric point of view, is that they all possess an $(n+1)$-dimensional commutative group of holomorphic symmetries, namely the group of translations $\{Z \mapsto Z+ib\}$ with $b \in V$, $Z \in \mathbb{C}^{n+1}$. Furthermore, any affine automorphism of the base of a tube can be extended to a holomorphic affine automorphism of the whole tube (note, however, that in general – for example, for the tube domain with base (0.1) – there may be many more holomorphic automorphisms than affine ones). In the same way, any affine transformation between the bases of two tubes can be lifted to a holomorphic affine transformation between the tubes. This last observation, however simple, indicates an important link between complex and affine geometries. In this book we look at tube hypersurfaces from both the complex-geometric and affine-geometric points of view.

One can endow a tube hypersurface (in fact any real hypersurface in complex space) with a so-called *CR-structure*, which is the remnant of the complex structure on the ambient space $\mathbb{C}^{n+1}$ (see Section 1.1). We impose on the CR-structure the condition of *sphericity* (see Section 1.2). This is the condition for the hypersurface to be locally *CR-equivalent* (for example, locally biholomorphically equivalent – see Section 1.1) to the tube hypersurface with the base given by the equation

$$x_0 = \sum_{\alpha=1}^{k} x_\alpha^2 - \sum_{\alpha=k+1}^{n} x_\alpha^2$$

for some $1 \le k \le n$ with $n \le 2k$ (cf. equation (0.2)). For a given k the second fundamental form of the base of a locally closed spherical tube hypersurface is everywhere non-degenerate and has signature $(k, n-k)$ up to sign. Interestingly, the sphericity condition coincides with the condition of the vanishing of the *CR-curvature form* (see Section 1.1), thus spherical hypersurfaces are exactly those that are *flat* in the CR-geometric context (the reader should not be alarmed by the apparent linguistic inconsistency between "sphericity" and "flatness"). In this book we offer a comprehensive exposition of the theory of spherical tube hypersurfaces starting with the idea proposed in the pioneering work by Yang (1982) and ending with a new approach due to Fels and Kaup (2009).

Spherical tube hypersurfaces possess remarkable properties. For example, every such hypersurface is real-analytic (see Section 3.2) and extends to a real-analytic spherical (hence non-singular) tube hypersurface which is closed as a submanifold of $\mathbb{C}^{n+1}$ (see Section 4.5). Thus, it suffices to consider only closed spherical tube hypersurfaces, and the main goal of this book is to explicitly classify such hypersurfaces whenever possible. Note that while for a fixed k all spherical tube hypersurfaces are CR-equivalent locally, they may not be CR-equivalent globally. We, however, aim at obtaining not just a classification up to CR-equivalence but a much finer classification up to affine equivalence (that is, a classification up to the affine equivalence of their bases). In 1982 Yang [108] proposed to approach this problem for $k = n$ by means of utilizing the zero CR-curvature equations arising from the

Cartan-Tanaka-Chern-Moser invariant theory, and we follow this approach throughout most of the book.

We will now describe the book's structure. In Chapter 1 we give a detailed exposition of Chern's construction of a Cartan connection for a hypersurface satisfying a certain non-degeneracy condition (Levi non-degeneracy). For a locally closed tube hypersurface this condition is equivalent to the non-degeneracy of the second fundamental form of the base at every point. The curvature of the Cartan connection gives rise to the zero CR-curvature equations, which can be written in terms of any local defining function of the hypersurface (see Sections 1.3, 1.4). These equations involve partial derivatives of the defining function up to order 4 for $n > 1$ and up to order 6 for $n = 1$. In Chapter 3 we generalize the result of [108] from $k = n$ to any value of k by showing that the zero CR-curvature equations significantly simplify for tube hypersurfaces and lead to systems of partial differential equations of order 2 of a very special form (we call them *defining systems*). As an application of this result, we show in Section 3.2 that every spherical tube hypersurface is real-analytic. Our exposition in Chapter 3 is based on results of [52], [56], [58], [64].

Further, in Chapter 4 we reduce every defining system to a system of one of three types by applying suitable linear transformations and give a certain representation of the solution for a system of each type. These representations imply the result already mentioned above: every spherical tube hypersurface extends to a real-analytic closed spherical tube hypersurface in $\mathbb{C}^{n+1}$ (see Section 4.5). Our exposition in Chapter 4 is a refinement of that given in [56]. From Chapter 4 to the end of Chapter 8 we study only closed spherical tube hypersurfaces and concentrate on classifying such hypersurfaces up to affine equivalence. In Chapters 5–7 we consider the cases $k = n$, $k = n-1$, $k = n-2$. In each of these cases we use the representations of the solutions of defining systems found in Chapter 4. In Chapter 5 a complete classification for the case $k = n$ is obtained. This classification is due to Dadok and Yang (see [27]), but our arguments are simpler than the original proof. In Chapter 6 we derive a complete classification for $k = n-1$. This classification appeared in [64], but the present exposition is shorter and much more elegant. Finally, in Chapter 7 we give a complete classification for the case $k = n-2$. This classification was found by the author in 1989 and announced in article [53], where a proof was also briefly sketched. Full details were given in a very long preprint (see [54]). Because of the prohibitive length of the preprint the complete proof was never published in a journal article. In this book it appears in print for the first time.

One consequence of the results of Chapters 5–7 is the finiteness of the number of affine equivalence classes for every fixed n in each of the following cases: (a) $k = n$, (b) $k = n-1$, and (c) $k = n-2$ with $n \leq 6$. In Chapter 8 we show that this number is infinite (in fact uncountable) in the cases: (i) $k = n-2$ with $n \geq 7$, (ii) $k = n-3$ with $n \geq 7$, and (iii) $k \leq n-4$. This result was announced in [53] but has only recently appeared with complete proofs (see [59]). Further, the question about the number of affine equivalence classes in the only remaining case $k = 3$, $n = 6$ had been open since 1989 until Fels and Kaup resolved it in 2009 by constructing an example of a family of spherical tube hypersurfaces in $\mathbb{C}^7$ for $k = 3$ that contains uncountably many pairwise affinely non-equivalent elements. In Chapter 8 we present this family but deal with it by methods different from the original methods of Fels and Kaup. In particular, we use the j-invariant to show that this family indeed contains an uncountable subfamily of pairwise affinely non-equivalent hypersurfaces.

The example mentioned above naturally arises from the new analytic-algebraic approach to studying spherical tube hypersurfaces developed by Fels and Kaup in [42]. It is based on their earlier work [41] concerned with the question of describing *all* (local) tube realizations of a real-analytic CR-manifold (cf. [4]). Fels and Kaup recover the real-analyticity result of Section 3.2 and the globalization results of Section 4.5 by their methods. Further, their approach yields the affine classifications of spherical tube hypersurfaces for $k = n$ and $k = n-1$ contained in Chapters 5, 6. We outline the main ideas of [41], [42] in Section 9.2 of Chapter 9.

In Section 9.1 of Chapter 9 we consider tube hypersurfaces locally CR-equivalent to the tube hypersurface with the base given by the equation

$$x_0 = \sum_{\alpha=1}^{k} x_\alpha^2 - \sum_{\alpha=k+1}^{m} x_\alpha^2, \tag{0.3}$$

where $0 \le k \le m$, $m \le 2k$, $m < n$. Such hypersurfaces are no longer Levi non-degenerate (in the locally closed case the second fundamental forms of their bases are everywhere degenerate), thus the standard Cartan-Tanaka-Chern-Moser theory does not apply to them. As we explain in Section 9.1, for $m \ge 1$ every tube hypersurface of this kind is real-analytic and extends to a closed non-singular real-analytic tube hypersurface in $\mathbb{C}^{n+1}$ represented as the direct sum of a complex $(n-m)$-dimensional linear subspace of $\mathbb{C}^{n+1}$ and a closed spherical tube hypersurface lying in a complementary complex $(m+1)$-dimensional subspace. For $m = 0$ such a hypersurface is an open subset of a real affine hyperplane in $\mathbb{C}^{n+1}$. Thus, the study of tube hypersurfaces locally CR-equivalent to the tube with base (0.3) reduces to the study of spherical tube hypersurfaces. Our exposition in Section 9.1 is based on results of [56].

In addition, the book includes a short chapter on spherical *rigid* hypersurfaces (see Chapter 2). A locally closed real hypersurface M in a complex $(n+1)$-dimensional manifold N is called rigid if near its every point in some local coordinates $z_0 = x_0 + iy_0$, $z = (z_1, \dots, z_n)$ in N it can be given by an equation of the form $x_0 = F(z, \bar{z})$. Clearly, rigid hypersurfaces are much more general than tube ones, but it turns out that the zero CR-curvature equations significantly simplify already in the rigid case. One motivation for considering rigid hypersurfaces is that they naturally arise as a result of various scaling procedures (see references in Section 2.2). An application of the zero CR-curvature equations in the rigid case is given in Section 2.2. These equations serve as an intermediate step for obtaining defining systems in Chapter 3. Our exposition in Chapter 2 is an improvement of that given in [57].

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