Contents

Preface VII

Part I Classical Topics Revisited

1 **Sphere Packings** 3
- 1.1 Kissing Numbers of Spheres 3
- 1.2 One-Sided Kissing Numbers of Spheres 5
- 1.3 On the Contact Numbers of Finite Sphere Packings 6
- 1.4 Lower Bounds for the (Surface) Volume of Voronoi Cells in Sphere Packings 7
- 1.5 On the Density of Sphere Packings in Spherical Containers ... 12
- 1.6 Upper Bounds on Sphere Packings in High Dimensions 13
- 1.7 Uniform Stability of Sphere Packings 15

2 **Finite Packings by Translates of Convex Bodies** 17
- 2.1 Hadwiger Numbers of Convex Bodies 17
- 2.2 One-Sided Hadwiger Numbers of Convex Bodies 18
- 2.3 Touching Numbers of Convex Bodies 19
- 2.4 On the Number of Touching Pairs in Finite Packings 20

3 **Coverings by Homothetic Bodies - Illumination and Related Topics** 23
- 3.1 The Illumination Conjecture 23
- 3.2 Equivalent Formulations 24
- 3.3 The Illumination Conjecture in Dimension Three 24
- 3.4 The Illumination Conjecture in High Dimensions 25
- 3.5 On the X-Ray Number of Convex Bodies 28
- 3.6 The Successive Illumination Numbers of Convex Bodies 29
- 3.7 The Illumination and Covering Parameters of Convex Bodies . 31
- 3.8 On the Vertex Index of Convex Bodies 32

4 Coverings by Planks and Cylinders 35
4.1 Plank Theorems 35
4.2 Covering Convex Bodies by Cylinders 37
4.3 Covering Lattice Points by Hyperplanes 39
4.4 On Some Strengthenings of the Plank Theorems of Ball and Bang 41
4.5 On Partial Coverings by Planks: Bang's Theorem Revisited ... 43

5 On the Volume of Finite Arrangements of Spheres 47
5.1 The Conjecture of Kneser and Poulsen 47
5.2 The Kneser–Poulsen Conjecture for Continuous Contractions . 48
5.3 The Kneser–Poulsen Conjecture in the Plane 49
5.4 Non-Euclidean Kneser–Poulsen-Type Results 51
5.5 Alexander's Conjecture 53
5.6 Densest Finite Sphere Packings 54

6 Ball-Polyhedra as Intersections of Congruent Balls 57
6.1 *Disk-Polygons and Ball-Polyhedra* 57
6.2 Shortest Billiard Trajectories in Disk-Polygons 57
6.3 Blaschke–Lebesgue-Type Theorems for Disk-Polygons 59
6.4 On the Steinitz Problem for Ball-Polyhedra 61
6.5 On Global Rigidity of Ball-Polyhedra 62
6.6 Separation and Support for Spindle Convex Sets 63
6.7 Carathéodory- and Steinitz-Type Results 65
6.8 Illumination of Ball-Polyhedra 65
6.9 The Euler–Poincaré Formula for Ball-Polyhedra 67

Part II Selected Proofs

7 Selected Proofs on Sphere Packings 71
7.1 Proof of Theorem 1.3.5 71
7.1.1 A proof by estimating the surface area of unions of balls 71
7.1.2 On the densest packing of congruent spherical caps of special radius 73
7.2 Proof of Theorem 1.4.7 73
7.2.1 The Voronoi star of a Voronoi cell in unit ball packings 73
7.2.2 Estimating the volume of a Voronoi star from below ... 74
7.3 Proof of Theorem 1.4.8 75
7.3.1 Basic metric properties of Voronoi cells in unit ball packings 75
7.3.2 Wedges of types I, II, and III, and truncated wedges of types I, and II 76
7.3.3 The lemma of comparison and a characterization of regular polytopes 79

7.3.4 Volume formulas for (truncated) wedges 80
7.3.5 The integral representation of surface density in (truncated) wedges 81
7.3.6 Truncation of wedges increases the surface density 84
7.3.7 Maximum surface density in truncated wedges of type I 85
7.3.8 An upper bound for the surface density in truncated wedges of type II 86
7.3.9 The overall estimate of surface density in Voronoi cells . 88
7.4 Proof of Theorem 1.7.3 89
7.4.1 The signed volume of convex polytopes 89
7.4.2 The volume force of convex polytopes 90
7.4.3 Critical volume condition 91
7.4.4 Strictly locally volume expanding convex polytopes 92
7.4.5 From critical volume condition and infinitesimal rigidity to uniform stability of sphere packings 94

8 Selected Proofs on Finite Packings of Translates of Convex Bodies 95
8.1 Proof of Theorem 2.2.1 95
8.1.1 Monotonicity of a special integral function 95
8.1.2 A proof by slicing via the Brunn–Minkowski inequality . 96
8.2 Proof of Theorem 2.4.3 98

9 Selected Proofs on Illumination and Related Topics 101
9.1 Proof of Corollary 3.4.2 Using Rogers' Classical Theorem on Economical Coverings 101
9.2 Proof of Theorem 3.5.2 via the Gauss Map 102
9.3 Proof of Theorem 3.5.3 Using Antipodal Spherical Codes of Small Covering Radii 103
9.4 Proofs of Theorem 3.8.1 and Theorem 3.8.3 106
9.4.1 From the Banach–Mazur distance to the vertex index .. 106
9.4.2 Calculating the vertex index of Euclidean balls in dimensions 2 and 3 107
9.4.3 A lower bound for the vertex index using the Blaschke–Santaló inequality and an inequality of Ball and Pajor 112
9.4.4 An upper bound for the vertex index using a theorem of Rudelson 113

10 Selected Proofs on Coverings by Planks and Cylinders 115
10.1 Proof of Theorem 4.1.7 115
10.1.1 On coverings of convex bodies by two planks 115
10.1.2 A proof of the affine plank conjecture of Bang for non-overlapping cuts 116
10.2 Proof of Theorem 4.2.2 117

10.2.1 Covering ellipsoids by 1-codimensional cylinders 117
10.2.2 Covering convex bodies by cylinders of given codimension 118
10.3 Proof of Theorem 4.5.2 119
10.4 Proof of Theorem 4.5.8 119

11 Selected Proofs on the Kneser–Poulsen Conjecture 121
11.1 Proof of Theorem 5.3.2 on the Monotonicity of Weighted Surface Volume 121
11.2 Proof of Theorem 5.3.3 on Weighted Surface and Codimension Two Volumes 124
11.3 Proof of Theorem 5.3.4 - the Leapfrog Lemma 126
11.4 Proof of Theorem 5.4.1 127
11.4.1 The spherical leapfrog lemma 127
11.4.2 Smooth contractions via Schläfli's differential formula .. 127
11.4.3 Relating higher-dimensional spherical volumes to lower-dimensional ones 128
11.4.4 Putting pieces together 129
11.5 Proof of Theorem 5.4.6 130
11.5.1 Monotonicity of the volume of hyperbolic simplices 130
11.5.2 From Andreev's theorem to smooth one-parameter family of hyperbolic polyhedra 133

12 Selected Proofs on Ball-Polyhedra 135
12.1 Proof of Theorem 6.2.1 135
12.1.1 Finite sets that cannot be translated into the interior of a convex body 135
12.1.2 From generalized billiard trajectories to shortest ones .. 137
12.2 Proofs of Theorems 6.6.1, 6.6.3, and 6.6.4 138
12.2.1 Strict separation by spheres of radii at most one 138
12.2.2 Characterizing spindle convex sets 139
12.2.3 Separating spindle convex sets 139
12.3 Proof of Theorem 6.7.1 140
12.3.1 On the boundary of spindle convex hulls in terms of supporting spheres 140
12.3.2 From the spherical Carathéodory theorem to an analogue for spindle convex hulls 141
12.4 Proof of Theorem 6.8.3 142
12.4.1 On the boundary of spindle convex hulls in terms of normal images 142
12.4.2 On the Euclidean diameter of spindle convex hulls and normal images 143
12.4.3 An upper bound for the illumination number based on a probabilistic approach 144

12.4.4 Schramm's lower bound for the proper measure of polars of sets of given diameter in spherical space 145
12.4.5 An upper bound for the number of sets of given diameter that are needed to cover spherical space 147
12.4.6 The final upper bound for the illumination number 148
12.5 Proof of Theorem 6.9.1 148
12.5.1 The CW-decomposition of the boundary of a standard ball-polyhedron 148
12.5.2 On the number of generating balls of a standard ball-polyhedron 149
12.5.3 Basic properties of face lattices of standard ball-polyhedra 150

References 153