

Contents

1	Introduction	1
1.1	About This Monograph.....	1
1.1.1	Convergence of Laplacians on Graph-Like Spaces.....	1
1.1.2	Tools from Functional Analysis and Operator Theory.....	2
1.1.3	Outline of the Work.....	3
1.1.4	Related Topics Not Included in This Work.....	4
1.2	History, Results and Motivation.....	5
1.2.1	Convergence of Laplacians on Graph-Like Spaces: The Neumann Case.....	6
1.2.2	Convergence of Laplacians on Graph-Like Spaces: The Dirichlet Case.....	7
1.2.3	Convergence of Resonances.....	9
1.2.4	Mathematical Physics.....	11
1.2.5	Models from Mathematical Biology.....	15
1.2.6	Spectral Geometry and Spectral Invariants.....	15
1.2.7	Global Analysis.....	17
1.2.8	Convergence of Operators Acting in Different Spaces.....	19
1.2.9	Boundary Triples.....	22
1.3	Convergence of Operators and Spectra: A Brief Overview.....	25
1.3.1	Graph-Like Spaces.....	26
1.3.2	The Limit Hilbert Spaces Associated to the Graph Models.....	29
1.3.3	Convergence Results for Operators in Different Hilbert Spaces.....	32
1.3.4	Convergence Results for Graph-Like Spaces.....	33
1.4	Boundary Triples and Convergence of Resonances: A Brief Overview.....	35
1.4.1	Boundary Triples Associated with Quadratic Forms.....	35
1.4.2	Resonances and Complex Dilation.....	40
1.4.3	Convergence of Resonances on Graph-Like Spaces.....	44

1.5	Consequences of the Convergence Results	46
1.5.1	Spectral Gaps and Covering Manifolds	46
1.5.2	Eigenvalues in Gaps	48
1.5.3	Equilateral Graphs	49
1.5.4	Spectral Band Edges	52
1.5.5	The Decoupled Case	52
1.6	Outlook and Remarks	53
1.6.1	Geometric Perturbations	53
1.6.2	Scattering Properties	54
1.6.3	Convergence of Differential Forms and First Order Operators	54
1.6.4	Convergence of Boundary Triples Coupled via Graphs ...	55
1.6.5	Metric Structure on the Space of Operators	55
1.6.6	Dirichlet-to-Neumann Map and Inverse Problems	56
1.6.7	Fractal Metric Graphs	56
2	Graphs and Associated Laplacians	57
2.1	Discrete Graphs and Generalised Laplacians	58
2.1.1	Discrete Graphs and Vertex Spaces	58
2.1.2	Operators Associated with Vertex Spaces	64
2.2	Metric Graphs, Quantum Graphs and Associated Operators	68
2.2.1	Metric Graphs	69
2.2.2	Operators on Metric Graphs	70
2.2.3	Boundary Triples Associated with Quantum Graphs	76
2.3	Extended Quantum Graphs	81
2.4	Spectral Relations Between Discrete and Metric Graphs	84
2.4.1	Spectral Relation for Equilateral Graphs	84
2.4.2	Spectral Relation at the Bottom of the Spectrum	87
2.5	Some Trace Formulas on Metric and Discrete Graphs	91
3	Scales of Hilbert Space and Boundary Triples	97
3.1	Sesquilinear Forms, Associated Operators and Dense Subspaces	97
3.2	Scale of Hilbert Spaces Associated with a Non-negative Operator	102
3.3	Scale of Hilbert Spaces Associated with a Closed Operator	104
3.3.1	Scale of Hilbert Spaces of Second Order Associated with a Closed Operator	104
3.3.2	Scale of Hilbert Spaces of First Order Associated with a Closed Operator	107
3.4	Boundary Triples and Abstract Elliptic Theory	114
3.4.1	Boundary Triples Associated with Quadratic Forms	116
3.4.2	Elliptic Boundary Triples	125
3.4.3	Relation with Other Concepts of Boundary Triples and Examples	132
3.4.4	Krein's Resolvent Formulas and Spectral Relations	136

3.5	Half-Line Boundary Triples and Complex Dilation	140
3.5.1	Half-Line Boundary Triples	141
3.5.2	Complex Dilation.....	146
3.6	Coupled Boundary Triples and Dilation	152
3.6.1	Coupled Boundary Triples	153
3.6.2	Dilated Coupled Boundary Triples	158
3.7	Complexly Dilated Coupled Operators	164
3.7.1	Holomorphic Dependency	165
3.7.2	The Complexly Dilated Coupled Operator.....	167
3.7.3	The Complexly Dilated Coupled Operator on the First Order Spaces	170
3.8	Resonances	174
3.9	Boundary Maps and Triples Coupled via Graphs	180
4	Two Operators in Different Hilbert Spaces	187
4.1	Quasi-Unitary Identification Operators	188
4.2	Convergence of Self-Adjoint Operators in Different Hilbert Spaces	194
4.3	Spectral Convergence for Non-negative Operators	204
4.4	Convergence of Quadratic Forms in Different Hilbert Spaces	209
4.5	Convergence of Non-self-adjoint Operators	221
4.6	Spectral Convergence for Non-self-adjoint Operators	230
4.7	Convergence of Non-symmetric Forms	233
4.8	Closeness of Coupled Boundary Maps.....	242
4.9	Convergence of Resonances	246
5	Manifolds, Tubular Neighbourhoods and Their Perturbations	259
5.1	Manifolds	260
5.1.1	Manifolds with Boundary	260
5.1.2	Manifolds Constructed from Building Blocks	261
5.1.3	Laplacians and Quadratic Forms	262
5.1.4	Basic Estimates	264
5.1.5	Some Scaling Behaviour	266
5.2	Perturbations of the Metric	268
5.3	Tubular Neighbourhoods	273
5.3.1	Perturbations of the Product Structure	273
5.3.2	Shortened Edge Neighbourhoods.....	276
5.4	Embedded Tubular Neighbourhoods	278
5.5	Tubular Neighbourhoods with Neumann Boundary Conditions....	280
5.6	Tubular Neighbourhoods with Dirichlet Boundary Conditions....	284
6	Plumber's Shop: Estimates for Star Graphs and Related Spaces.....	291
6.1	The Graph Models for Neumann Boundary Conditions	292
6.1.1	Fast Decaying Vertex Volume	293
6.1.2	Slowly Decaying Vertex Volume	297
6.1.3	The Borderline Case.....	299

6.2	The Manifold Models	301
6.2.1	A Simple Graph-Like Manifold	301
6.2.2	Graph-Like Manifolds with Different Scalings	306
6.2.3	The Associated Quadratic Form, Operator and Boundary Triple	314
6.2.4	Manifolds with Infinite Ends	316
6.3	Some Vertex Neighbourhoods Estimates	316
6.4	Fast Decaying Vertex Neighbourhoods	324
6.5	Slowly Decaying Vertex Neighbourhoods	329
6.6	The Borderline Case: Reduction to the Graph Model	332
6.7	The Embedded Case	335
6.7.1	Embedded Graph-Like Spaces	335
6.7.2	Reduction to the Graph Model	338
6.8	Slowly Decaying and Borderline Case for Arbitrary Transversal Manifolds	342
6.8.1	The Enlarged Vertex Neighbourhood with Added Truncated Cones	342
6.8.2	Some More Vertex Neighbourhood Estimates	344
6.9	Slowly Decaying and Arbitrary Transversal Manifolds	348
6.10	The Borderline Case with Arbitrary Transversal Manifolds	352
6.11	Dirichlet Boundary Conditions: The Decoupling Case	354
6.11.1	The Graph and Manifold Models	354
6.11.2	Some Vertex and Edge Neighbourhoods Estimates	357
6.11.3	Reduction to the Graph Model	359
6.11.4	The Embedded Case	362
6.11.5	The Spectral Vertex Neighbourhood Condition	364
7	Global Convergence Results	367
7.1	Spectral Convergence for Graph-Like Spaces	368
7.1.1	Fast Decaying Vertex Volume	369
7.1.2	Slowly Decaying Vertex Volume	371
7.1.3	The Borderline Case	372
7.1.4	The Dirichlet Decoupled Case	374
7.1.5	The Embedded Case	375
7.2	Convergence of Resonances	377
7.2.1	Fast Decaying Vertex Volume	379
7.2.2	Slowly Decaying Vertex Volume	383
7.2.3	The Borderline Case	384
7.2.4	The Dirichlet Decoupled Case	385
A	Appendix	389
A.1	Convergence of Set Sequences	389
A.2	Estimates on Abstract Fibred Spaces	391
A.2.1	Vector-Valued Integrals	391

A.2.2	Fibred Spaces Over an Interval	393
A.2.3	Examples: Cones and Cylinders	398
References	407
Notation	419
Index	421