

CONTENTS

PREFACE	xxvii
TECHNICAL NOTE	xxxi
ACKNOWLEDGMENT	xxxii

CHAPTER I

FORMULATION OF THE ECONOMIC PROBLEM

1. THE MATHEMATICAL METHOD IN ECONOMICS	1
1.1. <i>Introductory remarks</i>	1
1.2. <i>Difficulties of the application of the mathematical method</i>	2
1.3. <i>Necessary limitations of the objectives</i>	6
1.4. <i>Concluding remarks</i>	7
2. QUALITATIVE DISCUSSION OF THE PROBLEM OF RATIONAL BEHAVIOR	8
2.1. <i>The problem of rational behavior</i>	8
2.2. <i>"Robinson Crusoe" economy and social exchange economy</i>	9
2.3. <i>The number of variables and the number of participants</i>	12
2.4. <i>The case of many participants: Free competition</i>	13
2.5. <i>The "Lausanne" theory</i>	15
3. THE NOTION OF UTILITY	15
3.1. <i>Preferences and utilities</i>	15
3.2. <i>Principles of measurement: Preliminaries</i>	16
3.3. <i>Probability and numerical utilities</i>	17
3.4. <i>Principles of measurement: Detailed discussion</i>	20
3.5. <i>Conceptual structure of the axiomatic treatment of numerical utilities</i>	24
3.6. <i>The axioms and their interpretation</i>	26
3.7. <i>General remarks concerning the axioms</i>	28
3.8. <i>The role of the concept of marginal utility</i>	29
4. STRUCTURE OF THE THEORY: SOLUTIONS AND STANDARDS OF BEHAVIOR	31
4.1. <i>The simplest concept of a solution for one participant</i>	31
4.2. <i>Extension to all participants</i>	33
4.3. <i>The solution as a set of imputations</i>	34
4.4. <i>The intransitive notion of "superiority" or "domination"</i>	37
4.5. <i>The precise definition of a solution</i>	39
4.6. <i>Interpretation of our definition in terms of "standards of behavior"</i>	40
4.7. <i>Games and social organizations</i>	43
4.8. <i>Concluding remarks</i>	43

CHAPTER II

GENERAL FORMAL DESCRIPTION OF GAMES OF STRATEGY

5. INTRODUCTION	46
5.1. <i>Shift of emphasis from economics to games</i>	46
5.2. <i>General principles of classification and of procedure</i>	46

6. THE SIMPLIFIED CONCEPT OF A GAME	48
6.1. Explanation of the <i>termini technici</i>	48
6.2. The elements of the game	49
6.3. Information and preliminary	51
6.4. Preliminarity, transitivity, and signaling	51
7. THE COMPLETE CONCEPT OF A GAME	55
7.1. Variability of the characteristics of each move	55
7.2. The general description	57
8. SETS AND PARTITIONS	60
8.1. Desirability of a set-theoretical description of a game	60
8.2. Sets, their properties, and their graphical representation	61
8.3. Partitions, their properties, and their graphical representation	63
8.4. Logistic interpretation of sets and partitions	66
*9. THE SET-THEORETICAL DESCRIPTION OF A GAME	67
*9.1. The partitions which describe a game	67
*9.2. Discussion of these partitions and their properties	71
*10. AXIOMATIC FORMULATION	73
*10.1. The axioms and their interpretations	73
*10.2. Logistic discussion of the axioms	76
*10.3. General remarks concerning the axioms	76
*10.4. Graphical representation	77
11. STRATEGIES AND THE FINAL SIMPLIFICATION OF THE DESCRIPTION OF A GAME	79
11.1. The concept of a strategy and its formalization	79
11.2. The final simplification of the description of a game	81
11.3. The role of strategies in the simplified form of a game	84
11.4. The meaning of the zero-sum restriction	84

CHAPTER III

ZERO-SUM TWO-PERSON GAMES: THEORY

12. PRELIMINARY SURVEY	85
12.1. General viewpoints	85
12.2. The one-person game	85
12.3. Chance and probability	87
12.4. The next objective	87
13. FUNCTIONAL CALCULUS	88
13.1. Basic definitions	88
13.2. The operations Max and Min	89
13.3. Commutativity questions	91
13.4. The mixed case. Saddle points	93
13.5. Proofs of the main facts	95
14. STRICTLY DETERMINED GAMES	98
14.1. Formulation of the problem	98
14.2. The minorant and the majorant games	100
14.3. Discussion of the auxiliary games	101

14.4.	Conclusions	105
14.5.	Analysis of strict determinateness	106
14.6.	The interchange of players. Symmetry	109
14.7.	Non strictly determined games	110
14.8.	Program of a detailed analysis of strict determinateness	111
*15.	GAMES WITH PERFECT INFORMATION	112
*15.1.	Statement of purpose. Induction	112
*15.2.	The exact condition (First step)	114
*15.3.	The exact condition (Entire induction)	116
*15.4.	Exact discussion of the inductive step	117
*15.5.	Exact discussion of the inductive step (Continuation)	120
*15.6.	The result in the case of perfect information	123
*15.7.	Application to Chess	124
*15.8.	The alternative, verbal discussion	126
16.	LINEARITY AND CONVEXITY	128
16.1.	Geometrical background	128
16.2.	Vector operations	129
16.3.	The theorem of the supporting hyperplanes	134
16.4.	The theorem of the alternative for matrices	138
17.	MIXED STRATEGIES. THE SOLUTION FOR ALL GAMES	143
17.1.	Discussion of two elementary examples	143
17.2.	Generalization of this viewpoint	145
17.3.	Justification of the procedure as applied to an individual play	146
17.4.	The minorant and the majorant games. (For mixed strategies)	149
17.5.	General strict determinateness	150
17.6.	Proof of the main theorem	153
17.7.	Comparison of the treatment by pure and by mixed strategies	155
17.8.	Analysis of general strict determinateness	158
17.9.	Further characteristics of good strategies	160
17.10.	Mistakes and their consequences. Permanent optimality	162
17.11.	The interchange of players. Symmetry	165
CHAPTER IV		
ZERO-SUM TWO-PERSON GAMES: EXAMPLES		
18.	SOME ELEMENTARY GAMES	169
18.1.	The simplest games	169
18.2.	Detailed quantitative discussion of these games	170
18.3.	Qualitative characterizations	173
18.4.	Discussion of some specific games. (Generalized forms of Matching Pennies)	175
18.5.	Discussion of some slightly more complicated games	178
18.6.	Chance and imperfect information	182
18.7.	Interpretation of this result	185
*19.	POKER AND BLUFFING	186
*19.1.	Description of Poker	186
*19.2.	Bluffing	188
*19.3.	Description of Poker (Continued)	189
*19.4.	Exact formulation of the rules	190

*19.5. Description of the strategy	191
*19.6. Statement of the problem	195
*19.7. Passage from the discrete to the continuous problem	196
*19.8. Mathematical determination of the solution	199
*19.9. Detailed analysis of the solution	202
*19.10. Interpretation of the solution	204
*19.11. More general forms of Poker	207
*19.12. Discrete hands	208
*19.13. m possible bids	209
*19.14. Alternate bidding	211
*19.15. Mathematical description of all solutions	216
*19.16. Interpretation of the solutions. Conclusions	218

CHAPTER V

ZERO-SUM THREE-PERSON GAMES

20. PRELIMINARY SURVEY	220
20.1. General viewpoints	220
20.2. Coalitions	221
21. THE SIMPLE MAJORITY GAME OF THREE PERSONS	222
21.1. Definition of the game	222
21.2. Analysis of the game: Necessity of "understandings"	223
21.3. Analysis of the game: Coalitions. The role of symmetry	224
22. FURTHER EXAMPLES	225
22.1. Unsymmetric distributions. Necessity of compensations	225
22.2. Coalitions of different strength. Discussion	227
22.3. An inequality. Formulae	229
23. THE GENERAL CASE	231
23.1. Detailed discussion. Inessential and essential games	231
23.2. Complete formulae	232
24. DISCUSSION OF AN OBJECTION	233
24.1. The case of perfect information and its significance	233
24.2. Detailed discussion. Necessity of compensations between three or more players	235

CHAPTER VI

FORMULATION OF THE GENERAL THEORY:
ZERO-SUM n -PERSON GAMES

25. THE CHARACTERISTIC FUNCTION	238
25.1. Motivation and definition	238
25.2. Discussion of the concept	240
25.3. Fundamental properties	241
25.4. Immediate mathematical consequences	242
26. CONSTRUCTION OF A GAME WITH A GIVEN CHARACTERISTIC FUNCTION	243
26.1. The construction	243
26.2. Summary	245

27. STRATEGIC EQUIVALENCE. INESSENTIAL AND ESSENTIAL GAMES	245
27.1. Strategic equivalence. The reduced form	245
27.2. Inequalities. The quantity γ	248
27.3. Inessentiality and essentiality	249
27.4. Various criteria. Non additive utilities	250
27.5. The inequalities in the essential case	252
27.6. Vector operations on characteristic functions	253
28. GROUPS, SYMMETRY AND FAIRNESS	255
28.1. Permutations, their groups and their effect on a game	255
28.2. Symmetry and fairness	258
29. RECONSIDERATION OF THE ZERO-SUM THREE-PERSON GAME	260
29.1. Qualitative discussion	260
29.2. Quantitative discussion	262
30. THE EXACT FORM OF THE GENERAL DEFINITIONS	263
30.1. The definitions	263
30.2. Discussion and recapitulation	265
*30.3. The concept of saturation	266
30.4. Three immediate objectives	271
31. FIRST CONSEQUENCES	272
31.1. Convexity, flatness, and some criteria for domination	272
31.2. The system of all imputations. One element solutions	277
31.3. The isomorphism which corresponds to strategic equivalence	281
32. DETERMINATION OF ALL SOLUTIONS OF THE ESSENTIAL ZERO-SUM THREE-PERSON GAME	282
32.1. Formulation of the mathematical problem. The graphical method	282
32.2. Determination of all solutions	285
33. CONCLUSIONS	288
33.1. The multiplicity of solutions. Discrimination and its meaning	288
33.2. Statics and dynamics	290

CHAPTER VII

ZERO-SUM FOUR-PERSON GAMES

34. PRELIMINARY SURVEY	291
34.1. General viewpoints	291
34.2. Formalism of the essential zero sum four person games	291
34.3. Permutations of the players	294
35. DISCUSSION OF SOME SPECIAL POINTS IN THE CUBE Q	295
35.1. The corner I . (and V ., VI ., VII .)	295
35.2. The corner $VIII$. (and II ., III ., IV .,). The three person game and a "Dummy"	299
35.3. Some remarks concerning the interior of Q	302
36. DISCUSSION OF THE MAIN DIAGONALS	304
36.1. The part adjacent to the corner $VIII$. Heuristic discussion	304
36.2. The part adjacent to the corner $VIII$.: Exact discussion	307
*36.3. Other parts of the main diagonals	312

37. THE CENTER AND ITS ENVIRONS	313
37.1. First orientation about the conditions around the center	313
37.2. The two alternatives and the role of symmetry	315
37.3. The first alternative at the center	316
37.4. The second alternative at the center	317
37.5. Comparison of the two central solutions	318
37.6. Unsymmetrical central solutions	319
*38. A FAMILY OF SOLUTIONS FOR A NEIGHBORHOOD OF THE CENTER	321
*38.1. Transformation of the solution belonging to the first alternative at the center	321
*38.2. Exact discussion	322
*38.3. Interpretation of the solutions	327

CHAPTER VIII

SOME REMARKS CONCERNING $n \geq 5$ PARTICIPANTS

39. THE NUMBER OF PARAMETERS IN VARIOUS CLASSES OF GAMES	330
39.1. The situation for $n = 3, 4$	330
39.2. The situation for all $n \geq 3$	330
40. THE SYMMETRIC FIVE PERSON GAME	332
40.1. Formalism of the symmetric five person game	332
40.2. The two extreme cases	332
40.3. Connection between the symmetric five person game and the 1, 2, 3-symmetric four person game	334

CHAPTER IX

COMPOSITION AND DECOMPOSITION OF GAMES

41. COMPOSITION AND DECOMPOSITION	339
41.1. Search for n -person games for which all solutions can be determined	339
41.2. The first type. Composition and decomposition	340
41.3. Exact definitions	341
41.4. Analysis of decomposability	343
41.5. Desirability of a modification	345
42. MODIFICATION OF THE THEORY	345
42.1. No complete abandonment of the zero sum restriction	345
42.2. Strategic equivalence. Constant sum games	346
42.3. The characteristic function in the new theory	348
42.4. Imputations, domination, solutions in the new theory	350
42.5. Essentiality, inessentiality and decomposability in the new theory	351
43. THE DECOMPOSITION PARTITION	353
43.1. Splitting sets. Constituents	353
43.2. Properties of the system of all splitting sets	353
43.3. Characterization of the system of all splitting sets. The decomposition partition	354
43.4. Properties of the decomposition partition	357
44. DECOMPOSABLE GAMES. FURTHER EXTENSION OF THE THEORY	358
44.1. Solutions of a (decomposable) game and solutions of its constituents	358
44.2. Composition and decomposition of imputations and of sets of imputations	359

44.3.	Composition and decomposition of solutions. The main possibilities and surmises	361
44.4.	Extension of the theory. Outside sources	363
44.5.	The excess	364
44.6.	Limitations of the excess. The non-isolated character of a game in the new setup	366
44.7.	Discussion of the new setup. $E(e_0), F(e_0)$	367
45.	LIMITATIONS OF THE EXCESS. STRUCTURE OF THE EXTENDED THEORY	368
45.1.	The lower limit of the excess	368
45.2.	The upper limit of the excess. Detached and fully detached imputations	369
45.3.	Discussion of the two limits, $ \Gamma _1, \Gamma _2$. Their ratio	372
45.4.	Detached imputations and various solutions. The theorem connecting $E(e_0), F(e_0)$	375
45.5.	Proof of the theorem	376
45.6.	Summary and conclusions	380
46.	DETERMINATION OF ALL SOLUTIONS OF A DECOMPOSABLE GAME	381
46.1.	Elementary properties of decompositions	381
46.2.	Decomposition and its relation to the solutions: First results concerning $F(e_0)$	384
46.3.	Continuation	386
46.4.	Continuation	388
46.5.	The complete result in $F(e_0)$	390
46.6.	The complete result in $E(e_0)$	393
46.7.	Graphical representation of a part of the result	394
46.8.	Interpretation: The normal zone. Heredity of various properties	396
46.9.	Dummies	397
46.10.	Imbedding of a game	398
46.11.	Significance of the normal zone	401
46.12.	First occurrence of the phenomenon of transfer: $n = 6$	402
47.	THE ESSENTIAL THREE-PERSON GAME IN THE NEW THEORY	403
47.1.	Need for this discussion	403
47.2.	Preparatory considerations	403
47.3.	The six cases of the discussion. Cases (I)-(III)	406
47.4.	Case (IV): First part	407
47.5.	Case (IV): Second part	409
47.6.	Case (V)	413
47.7.	Case (VI)	415
47.8.	Interpretation of the result: The curves (one dimensional parts) in the solution	416
47.9.	Continuation: The areas (two dimensional parts) in the solution	418

CHAPTER X

SIMPLE GAMES

48.	WINNING AND LOSING COALITIONS AND GAMES WHERE THEY OCCUR	420
48.1.	The second type of 41.1. Decision by coalitions	420
48.2.	Winning and Losing Coalitions	421

49. CHARACTERIZATION OF THE SIMPLE GAMES	423
49.1. General concepts of winning and losing coalitions	423
49.2. The special role of one element sets	425
49.3. Characterization of the systems W, L of actual games	426
49.4. Exact definition of simplicity	428
49.5. Some elementary properties of simplicity	428
49.6. Simple games and their W, L . The Minimal winning coalitions: W^m	429
49.7. The solutions of simple games	430
50. THE MAJORITY GAMES AND THE MAIN SOLUTION	431
50.1. Examples of simple games: The majority games	431
50.2. Homogeneity	433
50.3. A more direct use of the concept of imputation in forming solutions	435
50.4. Discussion of this direct approach	436
50.5. Connections with the general theory. Exact formulation	438
50.6. Reformulation of the result	440
50.7. Interpretation of the result	442
50.8. Connection with the Homogeneous Majority game.	443
51. METHODS FOR THE ENUMERATION OF ALL SIMPLE GAMES	445
51.1. Preliminary Remarks	445
51.2. The saturation method: Enumeration by means of W	446
51.3. Reasons for passing from W to W^m . Difficulties of using W^m	448
51.4. Changed Approach: Enumeration by means of W^m	450
51.5. Simplicity and decomposition	452
51.6. Inessentiality, Simplicity and Composition. Treatment of the excess	454
51.7. A criterium of decomposability in terms of W^m	455
52. THE SIMPLE GAMES FOR SMALL n	457
52.1. Program. $n = 1, 2$ play no role. Disposal of $n = 3$	457
52.2. Procedure for $n \geq 4$: The two element sets and their role in classifying the W^m	458
52.3. Decomposability of cases C^*, C_{n-2}, C_{n-1}	459
52.4. The simple games other than $[1, \dots, 1, n-2]_k$ (with dummies): The Cases $C_k, k = 0, 1, \dots, n-3$	461
52.5. Disposal of $n = 4, 5$	462
53. THE NEW POSSIBILITIES OF SIMPLE GAMES FOR $n \geq 6$	463
53.1. The Regularities observed for $n \geq 6$	463
53.2. The six main counter examples (for $n = 6, 7$)	464
54. DETERMINATION OF ALL SOLUTIONS IN SUITABLE GAMES	470
54.1. Reasons to consider other solutions than the main solution in simple games	470
54.2. Enumeration of those games for which all solutions are known	471
54.3. Reasons to consider the simple game $[1, \dots, 1, n-2]_k$	472
*55. THE SIMPLE GAME $[1, \dots, 1, n-2]_k$	473
*55.1. Preliminary Remarks	473
*55.2. Domination. The chief player. Cases (I) and (II)	473
*55.3. Disposal of Case (I)	475
*55.4. Case (II): Determination of \forall	478
*55.5. Case (II): Determination of $\check{\forall}$	481
*55.6. Case (II): α and S_α	484

*55.7. Case (II') and (II''). Disposal of Case (II')	485
*55.8. Case (II''): α and V' . Domination	487
*55.9. Case (II''): Determination of V'	488
*55.10. Disposal of Case (II'')	494
*55.11. Reformulation of the complete result	497
*55.12. Interpretation of the result	499

CHAPTER XI

GENERAL NON-ZERO-SUM GAMES

56. EXTENSION OF THE THEORY	504
56.1. Formulation of the problem	504
56.2. The fictitious player. The zero sum extension Γ	505
56.3. Questions concerning the character of $\bar{\Gamma}$	506
56.4. Limitations of the use of $\bar{\Gamma}$	508
56.5. The two possible procedures	510
56.6. The discriminatory solutions	511
56.7. Alternative possibilities	512
56.8. The new setup	514
56.9. Reconsideration of the case when Γ is a zero sum game	516
56.10. Analysis of the concept of domination	520
56.11. Rigorous discussion	523
56.12. The new definition of a solution	526
57. THE CHARACTERISTIC FUNCTION AND RELATED TOPICS	527
57.1. The characteristic function: The extended and the restricted form	527
57.2. Fundamental properties	528
57.3. Determination of all characteristic functions	530
57.4. Removable sets of players	533
57.5. Strategic equivalence. Zero-sum and constant-sum games	535
58. INTERPRETATION OF THE CHARACTERISTIC FUNCTION	538
58.1. Analysis of the definition	538
58.2. The desire to make a gain vs. that to inflict a loss	539
58.3. Discussion	541
59. GENERAL CONSIDERATIONS	542
59.1. Discussion of the program	542
59.2. The reduced forms. The inequalities	543
59.3. Various topics	546
60. THE SOLUTIONS OF ALL GENERAL GAMES WITH $n \leq 3$	548
60.1. The case $n = 1$	548
60.2. The case $n = 2$	549
60.3. The case $n = 3$	550
60.4. Comparison with the zero sum games	554
61. ECONOMIC INTERPRETATION OF THE RESULTS FOR $n = 1, 2$	555
61.1. The case $n = 1$	555
61.2. The case $n = 2$. The two person market	555
61.3. Discussion of the two person market and its characteristic function	557
61.4. Justification of the standpoint of 58	559
61.5. Divisible goods. The "marginal pairs"	560
61.6. The price. Discussion	562

62. ECONOMIC INTERPRETATION OF THE RESULTS FOR $n = 3$: SPECIAL CASE	564
62.1. The case $n = 3$, special case. The three person market	564
62.2. Preliminary discussion	566
62.3. The solutions: First subcase	566
62.4. The solutions: General form	569
62.5. Algebraical form of the result	570
62.6. Discussion	571
63. ECONOMIC INTERPRETATION OF THE RESULTS FOR $n = 3$: GENERAL CASE	573
63.1. Divisible goods	573
63.2. Analysis of the inequalities	575
63.3. Preliminary discussion	577
63.4. The solutions	577
63.5. Algebraical form of the result	580
63.6. Discussion	581
64. THE GENERAL MARKET	583
64.1. Formulation of the problem	583
64.2. Some special properties. Monopoly and monopsony	584
CHAPTER XII	
EXTENSION OF THE CONCEPTS OF DOMINATION AND SOLUTION	
65. THE EXTENSION. SPECIAL CASES	587
65.1. Formulation of the problem	587
65.2. General remarks	588
65.3. Orderings, transitivity, acyclicity	589
65.4. The solutions: For a symmetric relation. For a complete ordering	591
65.5. The solutions: For a partial ordering	592
65.6. Acyclicity and strict acyclicity	594
65.7. The solutions: For an acyclic relation	597
65.8. Uniqueness of solutions, acyclicity and strict acyclicity	600
65.9. Application to games: Discreteness and continuity	602
66. GENERALIZATION OF THE CONCEPT OF UTILITY	603
66.1. The generalization. The two phases of the theoretical treatment	603
66.2. Discussion of the first phase	604
66.3. Discussion of the second phase	606
66.4. Desirability of unifying the two phases	607
67. DISCUSSION OF AN EXAMPLE	608
67.1. Description of the example	608
67.2. The solution and its interpretation	611
67.3. Generalization: Different discrete utility scales	614
67.4. Conclusions concerning bargaining	616
APPENDIX: THE AXIOMATIC TREATMENT OF UTILITY	617

Contents

<i>Introduction</i> , BY HAROLD W. KUHN	vii
Theory of Games and Economic Behavior, BY JOHN VON NEUMANN AND OSKAR MORGENSTERN	xv
<i>Afterword</i> , BY ARIEL RUBINSTEIN	633
REVIEWS	637
<i>The American Journal of Sociology</i> , BY HERBERT A. SIMON	637
<i>Bulletin of the American Mathematical Society</i> , BY ARTHUR H. COPELAND	640
<i>The American Economic Review</i> , BY LEONID HURWICZ	646
<i>Economica</i> , BY T. BARNA	664
<i>Psychometrika</i> , BY WALTER A. ROSENBLITH	667
Heads I Win, and Tails, You Lose, BY PAUL SAMUELSON	675
Big D, BY PAUL CRUME	678
Mathematics of Games and Economics, BY E. ROWLAND	680
Theory of Games, BY CLAUDE CHEVALLEY	683
Mathematical Theory of Poker Is Applied to Business Problems, BY WILL LISSNER	686
A Theory of Strategy, BY JOHN McDONALD	692
The Collaboration between Oskar Morgenstern and John von Neumann on the Theory of Games, BY OSKAR MORGENSTERN	712
<i>Index</i>	727
<i>Credits</i>	741