CONTENTS

Preface xi

Acknowledgments xiii

Introduction xv

1 History of the Subject 1

1.1 History of the Idea 1

1.2 Deficiencies of the Riemann Integral 3

1.3 Motivation for the Lebesgue Integral 6

2 Fields, Borel Fields, and Measures 11

2.1 Fields, Monotone Classes, and Borel Fields 11

2.2 Additive Measures 18

2.3 Carathéodory Outer Measure 20

2.4 E. Hopf's Extension Theorem 24

2.4.1 Fields, σ-Fields, and Measures Inherited by a Subset 29

3 Lebesgue Measure 31

Richardson, Leonard F.
Measure and integration
2009

digitalisiert durch:
IDS Basel Bern

3.1 The Finite Interval $[-N, N)$ 31
3.2 Measurable Sets, Borel Sets, and the Real Line 34
3.2.1 Lebesgue Measure on $\mathbb{R}$ 36
3.3 Measure Spaces and Completions 38
3.3.1 Minimal Completion of a Measure Space 41
3.3.2 A Nonmeasurable Set 41
3.4 Semimetric Space of Measurable Sets 43
3.5 Lebesgue Measure in $\mathbb{R}^n$ 50
3.6 Jordan Measure in $\mathbb{R}^n$ 52

4 Measurable Functions 55

4.1 Measurable Functions 55
4.1.1 Baire Functions of Measurable Functions 56
4.2 Limits of Measurable Functions 58
4.3 Simple Functions and Egoroff's Theorem 61
4.3.1 Double Sequences 63
4.3.2 Convergence in Measure 65
4.4 Lusin's Theorem 66

5 The Integral 69

5.1 Special Simple Functions 69
5.2 Extending the Domain of the Integral 72
5.2.1 The Class $\mathcal{L}^+$ of Nonnegative Measurable Functions 74
5.2.2 The Class $\mathcal{L}$ of Lebesgue Integrable Functions 78
5.2.3 Convex Functions and Jensen's Inequality 81
5.3 Lebesgue Dominated Convergence Theorem 83
5.4 Monotone Convergence and Fatou's Theorem 89
5.5 Completeness of $L^1(X, \mathfrak{A}, \mu)$ and the Pointwise Convergence Lemma 92
5.6 Complex-Valued Functions 100

6 Product Measures and Fubini's Theorem 103

6.1 Product Measures 103
6.2 Fubini's Theorem 108
6.3 Comparison of Lebesgue and Riemann Integrals 117

7 Functions of a Real Variable 123

7.1 Functions of Bounded Variation 123
7.2 A Fundamental Theorem for the Lebesgue Integral 128
7.3 Lebesgue's Theorem and Vitali's Covering Theorem 131
7.4 Absolutely Continuous and Singular Functions 139

8 General Countably Additive Set Functions 151

8.1 Hahn Decomposition Theorem 152
8.2 Radon-Nikodym Theorem 156
8.3 Lebesgue Decomposition Theorem 161

9 Examples of Dual Spaces from Measure Theory 165

9.1 The Banach Space $L^p(X, \mathfrak{A}, \mu)$ 165
9.2 The Dual of a Banach Space 170
9.3 The Dual Space of $L^p(X, \mathfrak{A}, \mu)$ 174
9.4 Hilbert Space, Its Dual, and $L^2(X, \mathfrak{A}, \mu)$ 178
9.5 Riesz-Markov-Saks-Kakutani Theorem 185

10 Translation Invariance in Real Analysis 195

10.1 An Orthonormal Basis for $L^2(\mathbb{T})$ 196
10.2 Closed, Invariant Subspaces of $L^2(\mathbb{T})$ 203
10.2.1 Integration of Hilbert Space Valued Functions 204
10.2.2 Spectrum of a Subset of $L^2(\mathbb{T})$ 206
10.3 Schwartz Functions: Fourier Transform and Inversion 208
10.4 Closed, Invariant Subspaces of $L^2(\mathbb{R})$ 213
10.4.1 The Fourier Transform in $L^2(\mathbb{R})$ 213
10.4.2 Translation-Invariant Subspaces of $L^2(\mathbb{R})$ 216
10.4.3 The Fourier Transform and Direct Integrals 218
10.5 Irreducibility of $L^2(\mathbb{R})$ Under Translations and Rotations 219
10.5.1 Position and Momentum Operators 221
10.5.2 The Heisenberg Group 222

Appendix: The Banach-Tarski Theorem 225

A.1 The Limits to Countable Additivity 225

References 229

Index 231