

---

# Contents

<b>1</b>	<b>Sets and Numbers</b> .....	<b>1</b>
1.1	Sets .....	1
1.1.1	The concept of a set .....	1
1.1.2	Operations on sets .....	3
1.1.3	Relations and functions .....	6
1.2	Sets of numbers .....	11
1.2.1	Two examples .....	11
1.2.2	The real number system .....	12
1.2.3	Elements of algebra .....	20
1.2.4	Elements of topology on $\mathbb{R}$ .....	25
1.2.5	The extended real number system .....	30
1.2.6	The complex number system .....	31
1.3	Exercises .....	32
1.4	References and comments .....	40
<b>2</b>	<b>Vector Spaces and Metric Spaces</b> .....	<b>41</b>
2.1	Vector spaces .....	41
2.1.1	Finite-dimensional vector spaces .....	41
2.1.2	Vector spaces .....	44
2.1.3	Normed spaces .....	48
2.1.4	Hilbert spaces .....	49
2.1.5	Inequalities .....	51
2.2	Metric spaces .....	57
2.3	Compact spaces .....	65
2.4	Exercises .....	70
2.5	References and comments .....	70
<b>3</b>	<b>Sequences and Series</b> .....	<b>73</b>
3.1	Numerical sequences .....	73
3.1.1	Convergent sequences .....	73
3.1.2	Subsequences .....	76

3.1.3	Cauchy sequences	77
3.1.4	Monotonic sequences	80
3.1.5	Upper limits and lower limits	81
3.1.6	The big Oh and small oh notations	82
3.1.7	Stolz–Cesaro theorem and some of its consequences	85
3.1.8	Certain combinatorial numbers	88
3.1.9	Unimodal, log-convex, and Pólya-frequency sequences	94
3.1.10	Some special sequences	98
3.2	Sequences of functions	108
3.3	Numerical series	110
3.3.1	Series of nonnegative terms	113
3.3.2	The root and the ratio tests	123
3.3.3	Partial summation	125
3.3.4	Absolutely and conditionally convergent series	126
3.3.5	The $W$ – $Z$ method	130
3.4	Series of functions	134
3.4.1	Power series	137
3.4.2	Hypergeometric series	138
3.5	The Riemann Zeta function $\zeta(p)$	139
3.6	Exercises	139
3.7	References and comments	144
<b>4</b>	<b>Limits and Continuity</b>	<b>147</b>
4.1	Limits	147
4.1.1	The limit of a function	147
4.1.2	Right-hand side and left-hand side limits	152
4.2	Continuity	152
4.2.1	Continuity and compactness	155
4.2.2	Uniform continuous mappings	156
4.2.3	Continuity and connectedness	160
4.2.4	Discontinuities	161
4.2.5	Monotonic functions	161
4.3	Periodic functions	164
4.4	Darboux functions	165
4.5	Lipschitz functions	167
4.6	Convex functions	169
4.6.1	Convex functions	169
4.6.2	Jensen convex functions	173
4.7	Functions of bounded variations	177
4.8	Continuity of sequences of functions	183
4.9	Continuity of series of functions	184
4.10	Exercises	186
4.11	References and comments	189

<b>5</b>	<b>Differential Calculus on <math>\mathbb{R}</math></b> .....	191
5.1	The derivative of a real function .....	191
5.2	Mean value theorems .....	197
5.2.1	Consequences of the mean value theorems .....	201
5.3	The continuity and the surjectivity of derivatives .....	206
5.4	L'Hospital theorem .....	207
5.5	Higher-order derivatives and the Taylor formula .....	208
5.6	Convex functions and differentiability .....	214
5.6.1	Inequalities .....	216
5.7	Differentiability of sequences and series of functions .....	217
5.8	Power series and Taylor series .....	219
5.8.1	Operations with power series .....	222
5.8.2	The Taylor expansion of some elementary functions .....	225
5.8.3	Bernoulli numbers and polynomials .....	228
5.9	Some elementary functions introduced by recurrences .....	230
5.9.1	The square root function .....	231
5.9.2	The logarithm function .....	231
5.9.3	The exponential function .....	235
5.9.4	The arctangent function .....	237
5.10	Functions with primitives .....	239
5.10.1	The concept of a primitive function .....	239
5.10.2	The existence of primitives for continuous functions .....	242
5.10.3	Operations with functions with primitives .....	244
5.11	Exercises .....	247
5.12	References and comments .....	249
<b>6</b>	<b>Integral Calculus on <math>\mathbb{R}</math></b> .....	251
6.1	The Darboux–Stieltjes integral .....	251
6.1.1	The Darboux integral .....	251
6.1.2	The Darboux–Stieltjes integral .....	252
6.2	Integrability of sequences and series of functions .....	262
6.3	Improper integrals .....	263
6.4	Euler integrals .....	271
6.4.1	Gamma function .....	271
6.4.2	Beta function .....	275
6.5	Polylogarithms .....	278
6.6	$e$ and $\pi$ are transcendental .....	280
6.7	The Grönwall inequality .....	283
6.8	Exercises .....	284
6.9	References and comments .....	287

<b>7</b>	<b>Differential Calculus on <math>\mathbb{R}^n</math></b>	289
7.1	Linear and bounded mappings	289
7.1.1	Multilinear mappings	293
7.1.2	Quadratic mappings	294
7.2	Differentiable functions	296
7.2.1	Variations	296
7.2.2	Gâteaux differential	297
7.2.3	Fréchet differential	298
7.2.4	Properties of the Fréchet differentiable functions	300
7.3	Partial derivatives	304
7.3.1	The inverse function theorem and the implicit function theorem	307
7.3.2	Directional derivatives and gradients	312
7.4	Higher-order differentials and partial derivatives	312
7.4.1	The case $X = \mathbb{R}^n$	314
7.5	Taylor formula	316
7.6	Problems of local extremes	317
7.6.1	First-order conditions	317
7.6.2	Second-order conditions	318
7.6.3	Constraint local extremes	319
7.7	Exercises	322
7.8	References and comments	322
<b>8</b>	<b>Double Integrals, Triple Integrals, and Line Integrals</b>	325
8.1	Double integrals	325
8.1.1	Double integrals on rectangles	325
8.1.2	Double integrals on simple domains	331
8.2	Triple integrals	333
8.2.1	Triple integrals on parallelepipeds	333
8.2.2	Triple integrals on simple domains	340
8.3	$n$ -fold integrals	341
8.3.1	$n$ -fold integrals on hyperrectangles	341
8.3.2	$n$ -fold integrals on simple domains	345
8.4	Line integrals	346
8.4.1	Line integrals with respect to arc length	346
8.4.2	Line integrals with respect to axis	347
8.4.3	Green formula	347
8.5	Integrals depending on parameters	349
8.6	Exercises	353
8.7	References and comments	354

<b>9</b>	<b>Constants</b> .....	355
9.1	Pythagoras's constant .....	355
9.1.1	Sequences approaching $\sqrt{2}$ .....	355
9.2	Archimedes' constant .....	356
9.2.1	Recurrence relation .....	356
9.2.2	Buffon needle problem .....	358
9.3	Arithmetic-geometric mean .....	358
9.4	BBP formulas .....	363
9.4.1	Computing the $n$ th binary or hexadecimal digit of $\pi$ ..	363
9.4.2	BBP formulas by binomial sums .....	368
9.5	Ramanujan formulas .....	372
9.6	Several natural ways to introduce number $e$ .....	374
9.7	Optimal stopping problem .....	377
9.8	References and comments .....	378
<b>10</b>	<b>Asymptotic and Combinatorial Estimates</b> .....	381
10.1	Asymptotic estimates .....	381
10.2	Algorithm analysis .....	384
10.3	Combinatorial estimates .....	390
10.3.1	Counting relations, topologies, and partial orders .....	394
10.3.2	Generalized Fubini numbers .....	396
10.3.3	The Catalan numbers and binary trees .....	401
10.4	References and comments .....	409
	<b>References</b> .....	411
	<b>List of Symbols</b> .....	419
	<b>Author Index</b> .....	423
	<b>Subject Index</b> .....	425